A forward-backward SDE approach to affine models

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Abstract

We consider factor models for interest rates and asset prices where the risk-neutral dynamics of the factors process is modelled by an affine diffusion. We characterize the factors process and bond price in terms of forward-backward stochastic differential equations, prove an existence and uniqueness theorem which gives the solution explicitly, and characterize the bond price as an exponential affine function of the factors in a new way. Our approach unifies the results, based on stochastic flows, of Elliott and van der Hoek [14] with the approach, based on the Feynman-Kac formula, of Duffie and Kan [11], and addresses a mistake in the approach of Elliott and van der Hoek [14]. We extend our results on the bond price to consider the futures and forward price of a risky asset or commodity.

Keywords: Affine models, forward-backward stochastic differential equations, stochastic flows, bond price, futures price, forward price.

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1 Introduction

Affine term structure models (ATSMs) in interest rate theory have been the focus of a great deal of study. The popularity of ATSMs is due to their analytic tractability and empirical properties. Examples of ATSMs include the models of Vašíček [39], Cox, Ingersoll, and Ross [7], Hull and White [20], Longstaff and Schwartz [29], Chen and Scott [4], and many others. The term ATSM was introduced by Duffie and Kan [11] who carried out a general study of ATSMs and provided a characterization of the bond price when the underlying factors process is an affine diffusion. The general ATSM of Duffie and Kan [11] included as special cases most of the popular term structure models in the literature. ATSMs have been further generalized to include a jump component in the factors process (see, for example, Björk et al. [1] or Chacko and Das [3] and the references therein). The study of ATSMs in finance has also lead to very general studies of a class of processes, the regular affine processes, by Duffie et al. [10].

Elliott and van der Hoek [14] studied ATSMs in the context of stochastic flows and the forward measure to provide an alternative proof that the bond price is an exponential affine function of the factors. A contribution to ATSMs of the flows methodology offered by Elliott and van der Hoek [14] is that it avoids the necessity, as is in Duffie and Kan [11], of solving Riccati equations to determine the bond price. Instead, the flows method involves solving a nonlinear integral equation with two parameters and then integrating the solution with respect to one of the parameters. A key result upon which the flows methodology is based is an approximation lemma which states that the conditional expectation under the forward measure of the Jacobian of the stochastic flow is deterministic and is equivalent to the two parameter integral equation.

While the flows methodology as presented by Elliott and van der Hoek [14] is conceptually very interesting we have not been able to verify, in the case of the general ATSM, the approximation lemma on which the method is based; there is a definite mistake in the proof of the approximation lemma. Part of this paper completes and clarifies the results of Elliott and van der Hoek [14], to the extent possible, by the introduction of forward-backward stochastic differential equations (FBSDEs) to the stochastic flows approach. The introduction of FBSDEs avoids the technical difficulties with the original approach of Elliott and van der Hoek [14] but it does come at a cost. The solvability of the Riccati equation which appears in Duffie and Kan [11] is a sufficient condition for the FBSDE approach to recover the main results of Elliott and van der Hoek [14] and thus their claim that "Riccati equations are not needed" is weakened.

The application of FBSDEs to the particular case of the one-dimensional (CIR) model has already been treated in Hyndman [21]. There are several important differences between the present paper and

[21] beyond the obvious difference in dimensions of the model. Elliott and van der Hoek [14] employed particular properties of the one-dimensional model, namely the semi-group property, to prove that the Jacobian of the stochastic flow was deterministic and did not rely on the flawed approximation lemma. The approach employed by Elliott and van der Hoek [14] for the CIR model cannot be directly applied to multidimensional ATSM hence the authors' use of the approximation lemma. Therefore, in contrast to [21], the present paper addresses the results of Elliott and van der Hoek [14] in the case of the general multidimensional ATSM.

The exposition of the FBSDE method presented in [21], while closer to the way we originally derived our results, cannot be directly generalized to the multidimensional ATSM without some careful modifications. A further difference between this paper and [21] is that the derivation of the backward stochastic differential equation (BSDE) for the bond price presented in this paper is entirely different from the approach presented in [21]. Where the approach presented in [21] requires the use of a result on the representation of one component of the FBSDE in terms of the derivatives of the other two components [21, equation (17)] a similar result is not used in this paper. In fact the result in question emerges as Corollary 4.2 of the present paper. Therefore, the approach presented in this paper is the probabilistic approach we state would be preferable in [21].

Grasselli and Tebaldi [16, 17] study the relationship between the flows approach, Riccati equations, and interest rate risk-management by algebraic methods. The flows approach must necessarily be equivalent to solving the Riccati equation in cases where the ATSM is well-posed in the sense of admissibility defined by Dai and Singleton [8] and consistency defined by Levendorskiĭ [27, 28]. Grasselli and Tebaldi [16] take as a starting point for certain calculations that the conditional expectation of the Jacobian of the stochastic flow is deterministic as claimed Elliott and van der Hoek [14]. However, Grasselli and Tebaldi [16] assume the admissibility conditions of Dai and Singleton [8] and, as we shall show, these conditions are sufficient to ensure that the conditional expectation of the Jacobian of the stochastic flow is deterministic. Therefore, the results of Grasselli and Tebaldi [16] are not in question. Nevertheless, one of these results, Grasselli and Tebaldi [16, Proposition 5], follows from the proof of Corollary 4.4 of this paper.

The main result of this paper is the proof of an existence and uniqueness theorem for a coupled nonlinear FBSDE, under the forward measure, associated with the bond price. Apart from the financial applications the result is of independent interest since few explicit existence and uniqueness results are available for coupled nonlinear FBSDEs and it represents a partial generalization of results proved by Yong [40] for linear FBSDEs. From the existence and uniqueness theorem the characterization of the ATSM and other results of Elliott and van der Hoek [14] and Grasselli and Tebaldi [16] follow as corollaries.

The remainder of the paper considers affine price models (APMs) which have been used extensively in financial modelling. The spot price of a risky asset (or a commodity) is specified as an exponential affine function of a factors process. These models include Gaussian models as special cases. Examples of Gaussian factor models include those of Gibson and Schwartz [15], Schwartz [36, 37], Cortazar and Schwartz [5], Miltersen and Schwartz [32], Schwartz and Smith [38], and Manoliu and Tompaidis [31] among others. General APMs offer further flexibility by also incorporating square-root or Cox, Ingersoll, and Ross [7]-type factors and jump components. The general APM, in the context of futures and forward contracts was studied by Björk and Landén [2].

We extend the FBSDE approach for the bond to consider the futures and forward prices of a risky asset (or commodity) paying a stochastic dividend yield (or convenience yield). The interest rate and dividend yield are modelled as affine functions of the factors process. We also assume that the asset price is modelled as an exponential affine function of the factors process. Similar to the case of the bond we are able to completely characterize the futures price and forward price as exponential affine functions of the factors process. We also indicate how the stochastic flows approach can be applied to futures and forward prices, generalizing the results presented in Hyndman [22] for the Gaussian case.

The remainder of this paper is organized as follows. In Section 2 we provide the set-up for the ATSM, review some of the existing results, and deal with some technical preliminaries. In Section 3 we introduce FBSDEs associated with the ATSM, and demonstrate the main results of the paper. In Section 4 we examine the flows approach of Elliott and van der Hoek [14] and show the relation to the FBSDE approach. Section 5 considers the generalization of the FBSDE method to APMs, futures and forward prices, and stochastic flows. Section 6 concludes.

2 Preliminaries and Notation

As is done in much of the literature on ATSMs, a notable exception being Duffee [9], we shall begin our analysis on the risk neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, Q)$ for $0 \leq t \leq T^*$ where T^* is the investment horizon and $\mathcal{F}_t$ is a right-continuous and complete filtration, and Q is the risk-neutral (martingale) measure. The price of the zero-coupon bond is then given by

$$P(t,T) = E_Q[\exp(-\int_t^T r_u du)|\mathcal{F}_t] \tag{1}$$

at time t for maturity $T \leq T^*$. There are numerous methods for calculating this conditional expectation. However, before any method can be applied some description of the risk-neutral dynamics of the riskless interest rate, (r_u), must be proposed.

We shall follow the methodology of Duffie and Kan [11] and assume that the riskless interest rate is a function of an $\mathbf{R}^n$-valued, $\{\mathcal{F}_t\}$-adapted state process X_t defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, Q)$ for $0 \leq t \leq T^*$. That is, $r_t = r(X_t)$, for some function $r : \mathbf{R}^n \to \mathbf{R}$ which will be specified shortly. As in Duffie and Kan [11] and Elliott and van der Hoek [14] we study a factors process given by an affine diffusion

$$dX_t = (AX_t + \tilde{B})dt + Sdiag\left(\sqrt{\alpha_i + \beta_i X_t}\right)dW_t \tag{2}$$

where W is an n-dimensional $\mathcal{F}_t$-Brownian motion (with respect to Q), A is an $(n \times n)$-matrix of scalars, $\tilde{B}$ is an $(n \times 1)$-vector of scalars, for each $i \in \{1, \ldots, n\}$ the α_i are scalars, for each $i \in \{1, \ldots, n\}$ the $\beta_i = (\beta_{i1}, \ldots, \beta_{in})$ are $(1 \times n)$-vectors taking values in $\mathbf{R}^n$, and S is a non-singular $(n \times n)$-matrix.

In order to ensure nonnegative volatilities Duffie and Kan [11] consider solutions to equation (2) taking values in the open set

$$D := \{x \in \mathbf{R}^n : \alpha_i + \beta_i x > 0, \quad i \in \{1, \ldots, n\}\}.$$

Further, Duffie and Kan [11] show that if, for all i, the conditions:

(A-I) for all x such that $\alpha_i + \beta_i x = 0$, $\beta_i(Ax + \tilde{B}) > \beta_i SS' \beta_i'/2$;

(A-II) for all j, if $(\beta_i S)_j \neq 0$, then $\alpha_i + \beta_i x = \alpha_j + \beta_j x$

are satisfied then there exists a unique strong solution X_t to the SDE (2) that takes values in D. Further, for all i, $\alpha_i + \beta_i X_t$ is strictly positive for all t almost surely.

Assumption 2.1 *Throughout we shall assume that conditions (A-I) and (A-II) hold.*

As remarked by Duffie and Kan [11], the set D is open and convex since it is the intersection of open half-spaces. Therefore, the separating hyperplane theorem can be applied to prove the existence of a strictly positive non-constant interest rate process $r_t = r(X_t)$ which is an affine transformation of X_t. That is, we have:

Assumption 2.2 *the short rate process is given by $r_t = r(X_t)$ where, for $x \in D$,*

$$r(x) = R'x + k > 0.$$

Where R is an $(n \times 1)$-column vector and k is a scalar.

For example, as in Duffie and Kan [11], we may set $r(x) = \sum_{i=1}^{n} \gamma_i(\alpha_i + \beta_i x)$ for scalars $\gamma_i \geq 0$ not all equal zero.

3 Connection between ATSMs and FBSDEs

In this section we explore connections between the bond price, the forward measure, and forward-backward stochastic differential equations (FBSDEs). Basic results on FBSDEs can be found in [33], [13], and [30]. The derivation of the BSDE for the bond price presented here differs from that previously presented in [21].

Define $H_s = \exp(-\int_0^s r(X_v)dv)$ and $V_s = E_Q[\exp\left(-\int_0^T r(X_v)dv\right)|\mathcal{F}_s]$ for all $s \in [0,T]$. Note that H_s is of finite variation and satisfies

$$dH_s = -r(X_s)H_s ds. \tag{3}$$

Since V_s is a martingale there exists a progressively measurable process, J, taking values in $\mathbf{R}^n$ and written as a $(1 \times n)$-row vector $J = (J^{(1)}, \dots, J^{(n)})$ such that

$$V_s = V_0 + \int_0^s J_u dW_u. \tag{4}$$

Define $Y_s = V_s/H_s$. Then clearly $P(s,T) = Y_s$ and by Itô's formula we have that Y_s satisfies

$$Y_s = Y_0 + \int_0^s Y_u r(X_u)du + \int_0^s \frac{J_u}{H_u} dW_u.$$

Define $Z_u = J_u/H_u$ to find

$$Y_s - Y_T = -\int_s^T r(X_u)Y_u du - \int_s^T Z_u dW_u.$$

Since $Y_T = 1$ we have that the factors process and the bond price satisfy the decoupled FBSDE

$$X_s = X_t + \int_t^s (AX_v + \tilde{B})dv + \int_t^s S diag\left(\sqrt{\alpha_i + \beta_i X_v}\right) dW_v \tag{5}$$

$$Y_s = 1 - \int_s^T \left(R' X_v + k\right) Y_v dv - \int_s^T Z_v dW_v \tag{6}$$

under the Q measure, for $s \in [t,T]$. Recall the definition of the forward measure:

Definition 3.1 *The forward measure, Q^T, is defined on $\mathcal{F}_T$ by*

$$Q^T(A) := E_Q[\Lambda_T 1_A]$$

where

$$\Lambda_T = \left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_T} := \{P(0,T)\}^{-1} \exp(-\int_0^T r(X_u)du). \tag{7}$$

Define $\Lambda_t = E[\Lambda_T|\mathcal{F}_t]$. Note that $\Lambda_t = V_t/V_0$ so that, from equation (4), Λ_t satisfies

$$\Lambda_t = 1 + \int_0^t \frac{J_u}{V_0} dW_u = 1 + \int_0^t \frac{J_u H_u V_u}{V_0 H_u V_u} dW_u = 1 + \int_0^t \frac{Z_u}{Y_u} \Lambda_u dW_u.$$

Hence, by Girsanov's theorem

$$W_t^T = W_t - \int_0^t \frac{Z_u'}{Y_u} du \tag{8}$$

is an $\mathcal{F}_t$-Brownian motion under the forward measure. We may then write the dynamics of the FBSDE (5)-(6) under the forward measure:

$$X_s = X_t + \int_t^s \{AX_v + \tilde{B} + Sdiag\left(\sqrt{\alpha_i + \beta_i X_v}\right) \frac{(Z_v)'}{Y_v}\} dv + \int_t^s Sdiag\left(\sqrt{\alpha_i + \beta_i X_v}\right) dW_v^T \tag{9}$$

$$Y_s = 1 - \int_s^T \left\{ (R' X_v + k) Y_v + \frac{Z_v (Z_v)'}{Y_v} \right\} dv - \int_s^T Z_v dW_v^T. \tag{10}$$

Note that Itô's formula, from s to T, applied to equation (10) gives

$$\log Y_s = -\int_s^T \left\{ \left(R' X_v + k\right) + \frac{1}{2} \frac{Z_v Z_v'}{Y_v^2} \right\} dv - \int_s^T \frac{Z_v}{Y_v} dW_v^T. \tag{11}$$

The preceding arguments, employing the martingale representation theorem and Itô's formula, exhibit the existence of a solution of the FBSDE (5)-(6) which characterizes the joint dynamics of the factors process and the bond price. By changing to the forward measure we are also able to construct a solution of the FBSDE (9)-(10). This nonlinear and coupled FBSDE appears much more complicated and if we are interested in providing an explicit solution and proving uniqueness we might initially attempt to study the FBSDE under the risk neutral measure. However, as is often the case, the forward measure simplifies things despite the rather more complicated appearance of the FBSDE (9)-(10). Indeed, we next prove an existence and uniqueness result, independent of the construction already presented, for the nonlinear FBSDE (9)-(10), by adapting a technique for linear FBSDEs from Yong [40], which gives the solution explicitly. A corollary completes the characterization of the bond price as an exponential affine function of the factors process. The following notation will be needed to state the main result.

Since the diffusion matrix of the square root affine SDE (2) is an affine function of the state variables, adopting the notation of Björk and Landén [2], we may write

$$Sdiag(\alpha_i + \beta_i x) S' = k_0 + \sum_{j=1}^n k_j x_j$$

for symmetric $(n \times n)$ matrices k_j, where x_j is the j-th element of a vector $x \in D$. Define the $(n^2 \times n)$ matrix K and, given a $(1 \times n)$ row vector $\vec{y}$, the $(n \times n^2)$ matrix $\beta(\vec{y})$ by

$$K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \quad \text{and} \quad \beta(\vec{y}) = \begin{bmatrix} \vec{y} & 0_{1\times n} & \cdots & 0_{1\times n} \\ 0_{1\times n} & \vec{y} & & \\ \vdots & & \ddots & \vdots \\ 0_{1\times n} & \cdots & & \vec{y} \end{bmatrix}$$

respectively.

Theorem 3.2 *If the Riccati equation*

$$\begin{aligned} &\dot{U}(u) + U(u)A + \tfrac{1}{2}U(u)K^{'}\left[\beta\left(U(u)\right)\right]^{'} - R^{'} = 0, \qquad u \in [0,T] \\ &U(T) = 0. \end{aligned} \tag{12}$$

admits a unique solution $U(\cdot)$ *over the interval* $[0,T]$ *then the FBSDE (9)-(10) admits a unique adapted solution (X,Y,Z) with explicit representation given by*

$$dX_s = \left(AX_s + \tilde{B} + k_0\left[U(s)\right]^{'} + K^{'}\left[\beta\left(U(s)\right)\right]^{'}X_s\right)ds + Sdiag\left(\sqrt{\alpha_i + \beta_i X_s}\right)dW_s^T, \tag{13}$$

$$\log Y_s = U(s)X_s + p(s), \text{ and} \tag{14}$$

$$Z_s = U(s)Sdiag\left(\sqrt{\alpha_i + \beta_i X_s}\right)Y_s, \tag{15}$$

where, for all $s \in [0,T]$,

$$p(s) = -\int_s^T \left(k - \frac{1}{2}U(u)k_0\left[U(u)\right]^{'} - U(u)\tilde{B}\right)du. \tag{16}$$

Proof: Applying Itô's formula from s to T to $f(s,x) = \exp\left(U(s)x + p(s)\right)$, when X_s is given by the SDE (13) and $p(s)$ satisfies equation (16), gives that $Y_s = f(s,X_s)$ satisfies

$$\begin{aligned} Y_s = 1 \quad &- \int_s^T \left\{\dot{U}(u)X_u - U(u)\tilde{B} - \frac{1}{2}U(u)k_0\left[U(u)\right]^{'} + k\right\}Y_u du \\ &- \int_s^T \left\{U(u)\left[(AX_u + \tilde{B}) + k_0\left[U(u)\right]^{'} + K^{'}\left[\beta\left(U(u)\right)\right]^{'}X_u\right]\right\}Y_u du \\ &- \int_s^T U(u)Sdiag\left(\sqrt{\alpha_i + \beta_i X_u}\right)Y_u dW_u^T \\ &- \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n\int_s^T U_i(u)U_j(u)\left[k_0 + \sum_{l=1}^n k_l X_u^{(l)}\right]_{ij} Y_u du \end{aligned} \tag{17}$$

where $U_j(u)$ is the j-th component of $U(u)$, $X_u^{(l)}$ is the l-th component of X_u, and $[A]_{ij}$ is the (i,j) component of a matrix A for $i,j = 1,\ldots,n$. Note,

$$\sum_{i=1}^n\sum_{j=1}^n U_i(u)U_j(u)\left[k_0 + \sum_{l=1}^n k_l X_u^{(l)}\right]_{ij} = U(u)k_0\left[U(u)\right]^{'} + U(u)K^{'}\left[\beta\left(U(u)\right)\right]^{'}X_u$$

so that after some simplification equation (17) becomes

$$\begin{aligned} Y_s = 1 \quad &- \int_s^T \left\{\dot{U}(u) + U(u)A + \frac{1}{2}U(u)K^{'}\left[\beta\left(U(u)\right)\right]^{'} - R^{'}\right\}X_u Y_u du \\ &- \int_s^T \left\{(R^{'}X_u + k) + U(u)k_0\left[U(u)\right]^{'} + U(u)K^{'}\left[\beta\left(U(u)\right)\right]^{'}X_u\right\}Y_u du \\ &- \int_s^T U(u)Sdiag\left(\sqrt{\alpha_i + \beta_i X_u}\right)Y_u dW_u^T. \end{aligned} \tag{18}$$

Applying equations (12) and (15) to (18) gives that Y_s satisfies equation (10). Substituting equation (15) into equation (13) gives that X_s satisfies equation (9). Therefore, the process (X,Y,Z) that is determined by equations (13), (14), (15) and (16) is an adapted solution of the FBSDE (9)-(10).

To prove uniqueness let (X,Y,Z) be any adapted solution of the FBSDE (9)-(10). Set

$$\begin{aligned} \log \bar{Y}_s &= U(s)X_s + p(s) \\ \bar{Z}_s &= U(s)Sdiag\Big(\sqrt{\alpha_i+\beta_i X_s}\Big)e^{U(s)X_s+p(s)}. \end{aligned} \tag{19}$$

Applying Itô's formula from s to T to the function $f(s,x)=U(s)x+p(s)$ when X_s is given by the SDE (9) gives that $f(s,X_s)=\log \bar{Y}_s$ satisfies

$$\begin{aligned} \log \bar{Y}_s = & - \int_s^T \left\{ \dot{U}(u)U(u)A + \frac{1}{2}U(u)K^{'}\left[\beta\left(U(u)\right)\right]^{'} - R^{'} \right\} X_u du \\ & - \int_s^T \left\{ (R^{'}X_u + k) - \frac{1}{2}U(u)k_0\left[U(u)\right]^{'} - \frac{1}{2}U(u)K^{'}\left[\beta\left(U(u)\right)\right]^{'} X_u \right. \\ & + \left. U(u)Sdiag\Big(\sqrt{\alpha_i+\beta_i X_u}\Big)\frac{Z_u^{'}}{Y_u} \right\} du - \int_s^T U(u)Sdiag\Big(\sqrt{\alpha_i+\beta_i X_u}\Big)dW_u^T \\ = & - \int_s^T \left\{ (R^{'}X_u + k) - \frac{1}{2}\frac{\bar{Z}_u\bar{Z}_u^{'}}{\bar{Y}_u^2} + \frac{\bar{Z}_u}{\bar{Y}_u}\frac{Z_u^{'}}{Y_u} \right\} du - \int_s^T \frac{\bar{Z}_u}{\bar{Y}_u} dW_u^T. \end{aligned}$$

Therefore, by equation (11), we have

$$\log Y_s - \log \bar{Y}_s = -\int_s^T \left\{ \frac{1}{2}\frac{Z_u Z_u^{'}}{Y_u^2} - \frac{\bar{Z}_u}{\bar{Y}_u}\frac{Z_u^{'}}{Y_u} + \frac{1}{2}\frac{\bar{Z}_u\bar{Z}_u^{'}}{\bar{Y}_u^2} \right\} du - \int_s^T \left\{ \frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u} \right\} dW_u^T.$$

Define $\hat{Y}_s := (\log Y_s - \log \bar{Y}_s)$ and $\tilde{Z}_u := (Z_u/Y_u - \bar{Z}_u/\bar{Y}_u)$ to obtain the equivalent BSDE

$$\hat{Y}_s = -\int_s^T \frac{1}{2}\tilde{Z}_u\tilde{Z}_u^{'}du - \int_s^T \tilde{Z}_u dW_u^T. \tag{20}$$

By the results of Kobylanski [26] the BSDE (20) admits a unique adapted solution $(\hat{Y},\tilde{Z}) = 0_{1\times(n+1)}$. This means that any adapted solution (X,Y,Z) of the FBSDE (9)-(10) must satisfy (14)-(15). Then, X given by (9)-(10) must also satisfy the equation (13). Hence, we obtain uniqueness from the SDE (13).

■

Remark 3.3 *In a more general setting Duffie et al. [10] prove that admissibility of the model parameters is a necessary and sufficient condition for the associated Riccati equations to have a solution over the interval $[0,T]$. In the special case of the model considered in this paper Assumption 2.1 (see also the admissibility conditions of Dai and Singleton [8]) give that the Riccati equation (12) has a unique solution on $[0,T]$ by [10, Theorem 6.1]. Therefore, the remainder of our results shall not explicitly mention the solvability of the Riccati equation (12) as this is guaranteed by Assumption 2.1.*[1]

The complete characterization of bond prices as exponential affine functions of the factors process follows as a corollary to Theorem 3.2, in particular the explicit representation of the solution given by equation (14).

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[1]We thank an anonymous referee for pointing out this fact.

Corollary 3.4 *Under Assumption 2.1 the bond price has exponential affine form,*

$$P(t,T) = e^{U(t)X_t + p(t)},$$

where $U(t)$ and $p(t)$ solve equations (12) and (16) respectively.

Proof: Since $P(t,T) = Y_t$ and, by Assumption 2.1, the Riccati equation (12) has a unique solution over $[0,T]$ the result follows from equation (14) of Theorem 3.2. ■

In the next section we discuss some results of Elliott and van der Hoek [14] and their relationship to our FBSDE approach.

4 Stochastic Flows and the Forward Measure

We shall consider, as in Elliott and van der Hoek [14], a stochastic flow associated with the factors process. For $0 \le t \le s < T$ write $X_s^{t,x}$ for the flow associated with the solution of equation (2) such that $X_t^{t,x} = x$. That is, consider

$$X_s^{t,x} = x + \int_t^s (AX_v^{t,x} + \tilde{B})dv + \int_t^s Sdiag(\,dB_v)vec(\sqrt{\alpha_i + \beta_i X_v^{t,x}}) \tag{21}$$

where $vec(\sqrt{\alpha_i + \beta_i X_v^{t,x}}) = (\sqrt{\alpha_1 + \beta_1 X_v^{t,x}}, \ldots, \sqrt{\alpha_n + \beta_n X_v^{t,x}})'$. For $x \in D$ let $\zeta(x,\omega)$ be the explosion time of the SDE (21) as in [34, pp. 247-248]. As pointed out by Grasselli and Tebaldi [16] the admissibility conditions of Dai and Singleton [8] imply, by Duffie et al. [10, Theorem 2.7], the existence of a solution to the SDE (21) for all $s \ge t$. Further, since the coefficient of the stochastic integral in (21) is locally Lipschitz with respect to $X_v^{t,x}$ and $\zeta(x,\omega) = \infty$, for all $x \in D$, we have (see Protter [34, Theorem 39]) that for $x \in D$, the map $x \to X_u^{t,x}$ is almost surely differentiable and the Jacobian matrix of partial derivatives with respect to x satisfies the equation

$$(\partial_x X_u^{t,x}) = I + \int_t^u A(\partial_x X_v^{t,x})dv + \frac{1}{2}\int_t^u Sdiag(dW_v)diag\Big((\alpha_i + \beta_i X_v^{t,x})^{-\frac{1}{2}}\Big)C(\partial_x X_v^{t,x}) \tag{22}$$

where C is the $(n \times n)$-matrix whose rows are the vectors $\beta_1, \ldots, \beta_n$.

For $0 \le t \le T$, since X_t is a Markov process, it follows that

$$P(t,T) = P(t,T,X_t) \tag{23}$$

$Q-a.s$, where for $x \in D$ we define

$$P(t,T,x) := E_Q[\exp(-\int_t^T r(X_u^{t,x})du)]. \tag{24}$$

By differentiating $P(t,T,x)$ with respect to x, where we write $\partial_x P(t,T,x)$ for the vector of partial derivatives, we obtain, subject to regularity conditions that allow the exchange of expectation and differentiation

$$\partial_x P(t,T,x) \quad = \quad E_Q[\exp(-\int_t^T r(X_u^{t,x})du)L(t,T,x)] \tag{25}$$

where

$$L(t,T,x) := -\int_t^T (\partial_x X_u^{t,x})' R du. \tag{26}$$

We briefly consider the purely Gaussian dynamics where $\beta_i = 0$ for $i = 1, \dots, n$. In this case $C = 0$ and equation (22) reduces to

$$(\partial_x X_u^{t,x}) = I + \int_t^u A(\partial_x X_v^{t,x})dv$$

which has solution $D_{tu} := (\partial_x X_u^{t,x}) = \exp(A[u-t])$ not depending on x. With

$$B(t,T) := \int_t^T D_{tu}' R du$$

equation (25) reduces to

$$\partial_x P(t,T,x) = -B(t,T)P(t,T,x)$$

and can be solved to obtain $P(t,T,x) = \exp(-[B(t,T)]' x + A(t,T))$. All that remains is to identify $A(t,T)$ (which can be done using the Feynman-Kac theorem as in Elliott and van der Hoek [14]) and substitute $x = X_t$ to complete the characterization of the bond price as an exponential affine function of the factors in the Gaussian case.

For $C \neq 0$, in contrast to the Gaussian case, $L(t,T,x)$ cannot be brought outside of the expectation in equation (25). Instead, by applying a general form of Bayes' theorem with the forward measure it can be shown (as in Elliott and van der Hoek [14]) that

$$\partial_x P(t,T,X_t) = P(t,T,X_t) E_T[L(t,T,X_t)|\mathcal{F}_t]. \tag{27}$$

Provided $E_T[L(t,T,X_t)|\mathcal{F}_t]$ does not depend on X_t for all $t \in [0,T]$ equation (27) can be solved to obtain an exponential affine form for the bond price. In order to explore this possibility it is necessary to examine the dynamics of $(\partial_x X_u^{t,x})$ under the forward measure. As a first step Elliott and van der Hoek [14] employ Girsanov's Theorem to construct a Brownian motion with respect to the forward measure.

Theorem 4.1 (Elliott and van der Hoek [14]) *The process* $(W_\cdot^T)$ *defined by*

$$W_t^T := W_t - \int_0^t diag\left(\sqrt{\alpha_i + \beta_i X_u}\right) S' E_T[L(t,T,X_u)|\mathcal{F}_u]du \tag{28}$$

is a standard Brownian motion with respect to $(Q^T, \mathcal{F}_t)$.

Using the notation

$$\hat{D}_{tu} \quad := E_T\left[\,(\partial_x X_u^{t,x})\big|_{x=X_t}\,|\mathcal{F}_t\right], \quad \text{for } \ 0 \le t \le u \le T,$$

the dynamics for the i-th component, W_t^{iT}, of W^T can be written in differential form as

$$dW_t^{iT} = dW_t^i + \sqrt{\alpha_i + \beta_i X_t} R^{'} \left(\int_t^T \hat{D}_{tv} dv \right) Se_i dt \tag{29}$$

where e_i denotes the unit vector in $\mathbf{R}^n$ with 1 in the i-th position. Equation (29) can then be used to write the dynamics of the Jacobian matrix of partial derivatives of the stochastic flow, evaluated at $x = X_t$, under the forward measure.

Taking the Q^T conditional expectation of these dynamics with respect to $\mathcal{F}_t$ gives that $\hat{D}_{tu}$ satisfies

$$\hat{D}_{tu} = I + \int_t^u A\hat{D}_{tv} dv - \frac{1}{2} \sum_{i=1}^n \int_t^u Sdiag(e_i) CE_T [R^{'} (\int_v^T \hat{D}_{vv_1} dv_1) Se_i \left(\partial_x X_v^{t,x}\right)\big|_{x=X_t} |\mathcal{F}_t] dv \tag{30}$$

almost surely. At this point it should be obvious that the key to proving the exponential affine form of the bond price from equation (27) is a complete understanding of $\hat{D}_{tv}$ for $0 \le t \le v \le T$.

Based on equation (30) it is stated by Elliott and van der Hoek [14, Lemma 4.3] that $\hat{D}_{tu}$ is deterministic for $0 \le t \le u \le T$. The proof given by Elliott and van der Hoek [14] proceeds by constructing a sequence of deterministic processes which are supposed to converge to $\hat{D}_{tu}$ represented by equation (30). However, there is a mistake in the proof using the proposed approximation since an upper bound which is assumed, by Elliott and van der Hoek [14], to be constant actually grows from iteration to iteration and the application of Gronwall's inequality is ineffective. It should be noted that for the particular case of the (one-dimensional) CIR model [7] Elliott and van der Hoek [14] prove, directly, that $\hat{D}_{tu}$ is deterministic by using the semi-group property of the stochastic flow and properties of expectation to solve equation (30) explicitly. However, these techniques are not generalizable to the multidimensional case unless all the square-root factors are independent.

It is possible to prove a local version of the approximation lemma (see Hyndman [24]) by carefully modifying the original proof of Elliott and van der Hoek [14] which, while of independent interest, is not strong enough for our purposes. In fact, we shall prove that, under Assumption 2.1, $\hat{D}_{tu}$ is deterministic for all $0 \le t \le u \le T$. This result emerges as a simple corollary to our existence and uniqueness result, Theorem 3.2, for the FBSDE (9)-(10).

By adapting the results of Section 3 to include dependence on the parameters (t,x), through a combination of a translation argument similar to that of [13, Proposition 4.1] with the derivation of the BSDE (6), we may consider the BSDE associated with $P(t,T,x)$

$$Y_s^{t,x} = 1 - \int_s^T (R^{'} X_u^{t,x} + k) Y_u^{t,x} du - \int_s^T Z_u^{t,x} dW_u. \tag{31}$$

where $X_s^{t,x}$ is the solution to equation (21). For $0 \leq t \leq s < T$ define

$$\mathcal{F}_s^t = \sigma(W(u) - W(t) : t \leq u \leq s) \vee \mathcal{N}$$

where $\mathcal{N}$ denotes the Q-null subsets of $\mathcal{F}_{T^*}^W$. Then $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ is the unique $\mathcal{F}_s^t$-adapted solution to the FBSDE defined by equations (21) and (31). Further, we find that $Y_t^{t,x}$ is deterministic, $Y_t^{t,x} = P(t,T,x)$, and comparing with equation (27)

$$\left.\frac{\frac{\partial}{\partial x} Y_t^{t,x}}{Y_t^{t,x}}\right|_{x=X_t} = E_T[L(t,T,X_t)|\mathcal{F}_t]. \tag{32}$$

The derivation of forward measure, Brownian motion (8), and the FBSDE (9)-(10) may also be carried through with a dependence on the initial conditions (t,x) so that we may consider the version of the FBSDE depending on (t,x). Let $Q^{T,t,x}$ and $W^{T,t,x}$ denote the parameterized forward measure and parameterized $\mathcal{F}_s^t$-Brownian motion respectively. Then, given Assumption 2.1, the Riccati equation (12) has a unique solution over $[0,T]$ and Theorem 3.2 gives that $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ satisfy (13)-(15) with respect to $W^{T,t,x}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_s^t\}_{s \geq t}, Q^{T,t,x})$.

An immediate corollary is a partial generalization of Pardoux and Peng [33, Lemma 2.5] and El Karoui et al. [13, Proposition 5.9] adapted to the specific setting of ATSMs which was employed in [21]

Corollary 4.2 *Under Assumption 2.1, for any $t \in [0,t]$, $s \in [t,T]$, and $x \in D$*

$$Z_s^{t,x} = (\partial_x Y_s^{t,x})' (\partial_x X_s^{t,x})^{-1} \sigma(X_s^{t,x}),$$

where $\sigma(x) = S diag\left(\sqrt{\alpha_i + \beta_i x}\right)$. In particular $Z_t^{t,x} = (\partial_x Y_t^{t,x})' \sigma(x)$.

Note that as the function $\sigma(x)$ is only locally Lipschitz and the generator of the BSDE does not satisfy the required differentiability hypotheses the results of Pardoux and Peng [33], El Karoui et al. [13] do not apply.

By comparing the integrand in equation (28) with our alternative derivation of the dynamics (8) of the Brownian motion, the existence and uniqueness result, and equation (15) one can formally observe the following completion, in a weaker form, of the results of Elliott and van der Hoek [14].

Corollary 4.3 *Under Assumption 2.1*

$$E_T[L(t,T,X_t)|\mathcal{F}_t] = [U(t)]'$$

for all $t \in [0,T]$.

Proof: If we evaluate equations (21) and (31) at $x = X_t$ then, under the forward measure, the triple $(X_s^{t,X_t}, Y_s^{t,X_t}, Z_s^{t,X_t})$ satisfies the FBSDE (9)-(10). Since, by Assumption 2.1, the Riccati equation (12) is solvable Theorem 3.2 implies that

$$\log Y_s^{t,x} = U(s)X_s^{t,x} + p(s), \qquad s \in [t,T].$$

From the initial conditions $X_t^{t,x} = x$ so, for $s = t$, we obtain $\log Y_t^{t,x} = U(t)x + p(t)$. Therefore, by equation (32)

$$E_T[L(t,T,X_t)|\mathcal{F}_t] = \left(\frac{\frac{\partial}{\partial x} Y_t^{t,x}}{Y_t^{t,x}} \right)\Bigg|_{x=X_t} = \left(\frac{\partial}{\partial x} \log Y_t^{t,x} \right)\Bigg|_{x=X_t} = [U(t)]'.$$

■

The final result of this section completes the results of Elliott and van der Hoek [14] by providing a sufficient condition for the conditional expectation, under the forward measure, of the Jacobian of the stochastic flow to be deterministic.

Corollary 4.4 *Under Assumption 2.1 the matrix $\hat{D}_{tu}$ is deterministic for $0 \le t \le u \le T$ and satisfies the integral equation*

$$\Phi(u,t) = I + \int_t^u \left\{ A + \frac{1}{2} \sum_{i=1}^n S diag(e_i) C U(v) S e_i \right\} \Phi(v,t) dv. \tag{33}$$

Proof: Assumption 2.1 and Corollary 4.3 give that $[U(v_1)]' = E_T[L(v_1,T,X_{v_1})|\mathcal{F}_{v_1}]$ for $v_1 \in [t,u]$. Therefore, by equation (30), we have that $\hat{D}_{tu}$ satisfies

$$\begin{aligned}
\hat{D}_{tu} &= I + \int_t^u A \hat{D}_{tv_1} dv_1 + \frac{1}{2} \sum_{i=1}^n \int_t^u S diag(e_i) C U(v_1) S e_i E_T[(\partial_x X_{v_1}^{t,X_t})|\mathcal{F}_t] dv_1 \\
&= I + \int_t^u \left\{ A + \frac{1}{2} \sum_{i=1}^n S diag(e_i) C U(v_1) S e_i \right\} \hat{D}_{tv_1} dv_1 \qquad (34)
\end{aligned}$$

almost surely. Equation (34) is equivalent to a deterministic linear system of ordinary differential equations. Consider the $(n \times n)$ matrix $\Psi(t)$ whose columns are the vectors $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$, which form a fundamental set of solutions for the system

$$\vec{x}'(t) = \left\{ A + \frac{1}{2} \sum_{i=1}^n S diag(e_i) C U(t) S e_i \right\} \vec{x}(t). \tag{35}$$

Since the columns of the fundamental matrix, $\Psi(t)$, for the system (35) are linearly independent $\Psi(t)$ is invertible. Define

$$\Phi(u,t) := \Psi(u)\Psi^{-1}(t), \qquad 0 \le t \le u \le T.$$

Then $\Phi(u,t)$ satisfies the integral equation (33). Therefore, we have that $\hat{D}_{tu}$ is the deterministic matrix $\Phi(u,t)$ for all $0 \le t \le u \le T$. ■

Note that equation (33) is analogous to Proposition 4.5 of Grasselli and Tebaldi [16] after reconciling notation.

In the next section we consider the relationship between FBSDEs and affine price models for a risky asset. We show how the method developed for the bond can be adapted to characterize the futures and forward prices.

5 Connection between APMs and FBSDEs

Suppose the factors process given by (2) is driving not only the short interest rate but also the price of a risky asset (or commodity) and the dividend yield of the asset (or convenience yield of the commodity). We shall retain Assumption 2.1, Assumption 2.2, and make the following assumptions about the functional form of the dividend (convenience) yield and risky asset price similar to Björk and Landén [2].

Assumption 5.1 *We assume that the the asset (spot) price, and the dividend (convenience) yield are functions of the factors process. That is,*

$$S_t = S(t, X_t), \quad \textit{and} \quad \delta_t = \delta(X_t),$$

where $\delta : \mathbf{R}^n \to \mathbf{R}$, *and* $S : [0,T] \times \mathbf{R}^n \to \mathbf{R}_{++}$ *are specified by*

(i) $S_t = S(t, X_t)$, *where for* $(t,x) \in [0,T] \times \mathbf{R}^n$, $S(t,x) = \exp\left(M(t)^{'} x + h(t)\right)$, $M : [0,T] \to \mathbf{R}^n$, $M(t)$ *is an* $(n \times 1)$*-column vector,* $h(t) : [0,T] \to \mathbf{R}$ *and both* $M(t)$ *and* $h(t)$ *are differentiable functions of* t,

(ii) $\delta_t = \delta(X_t)$, *where for* $x \in \mathbf{R}^n$, $\delta(x) = N^{'} x + l$, N *is an* $(n \times 1)$*-column vector and* l *is a scalar.*

We have included time dependence in the specification of the asset price so that we may consider futures and forward contracts on zero coupon bonds with exponential affine forms, that is bond prices resulting from affine term structure models, as underlying assets. Models satisfying Assumptions2.1-5.1 have been referred to as affine price models (APMs). Within this framework a number of interest rate and commodity price models which have appeared in the literature are included as special cases. In the special case of Gaussian factor models ($\alpha_i = 1, \beta_i = 0$, $i = 1, \ldots, n$) futures and forward prices were considered separately in Hyndman [22] where we employed a version of the flows method. Examples of models covered by Assumptions 2.1-5.1 which incorporate factors that are not all Gaussian include the models of Ribeiro and Hodges [35] and Heston [18] as well as the continuous version of Björk and Landén [2].

5.1 The Futures Price

We next develop connections between the futures price, the risk-neutral measure for the futures price reinvested in the bank account as numéraire, and forward-backward stochastic differential equations. We shall also characterize the futures price as an exponential affine function of the factors process, whose dynamics are given by (2), when the market model satisfies Assumptions 2.1-5.1.

A futures contract is an agreement to deliver some quantity of the underlying asset in the future for a price agreed upon at the initiation of the contract. The delivery price which makes the value to both parties of the contract zero at all times is called the futures price. By the mechanism of marking to market, where changes in the value of the futures contract are settled daily, in accordance with changes in the futures price the risk of default by one party is transfered to the exchange. Basic information on futures contracts and market mechanics can be found in Hull [19]. The futures price of the risky asset S is given by

$$G(t,T) = E_Q[S(T,X_T)|\mathcal{F}_t] \tag{36}$$

at time t for maturity T (see Karatzas and Shreve [25, Theorem 3.7, pp. 45-46] for a proof).

We next derive the FBSDE for the factors process and the futures price which is similar to the case the bond but actually simpler. We shall use similar notation as in Section 3. In particular, define $Y_s = G(s,T)$ for all $s \in [0,T]$. Since Y_s is a martingale there exists a progressively measurable process, Z, taking values in $\mathbf{R}^n$ and written as a $(1 \times n)$-row vector $Z = (Z^{(1)}, \dots, Z^{(n)})$ such that

$$Y_s = Y_0 + \int_0^s Z_u dW_u. \tag{37}$$

Since $Y_T = S(T,X_T)$ we have that the futures price satisfies the BSDE

$$Y_s = S(T,X_T) - \int_s^T Z_v dW_v \tag{38}$$

for $s \in [t,T]$ and, taken together, equations (5) and (38) constitute a decoupled FBSDE for the factors process and the futures price.

We may define the following risk-neutral measure for the futures price reinvested in the bank account as numéraire. This is the natural measure change that will allow us to characterize the futures price in terms of a linear ordinary differential equation similar to equation (27) and, ultimately, as an exponential affine function of the square root affine factors.

Definition 5.2 *The risk-neutral measure for the futures price invested in the bank account as numéraire,* $\exp(\int_0^{\cdot} r(X_u)du)G(\cdot,T)$*, is defined on* $\mathcal{F}_T$ *by*

$$Q^G(A) := E_Q[\Xi_T 1_A]$$

for all $A \in \mathcal{F}_T$, where

$$\Xi_T = \left.\frac{dQ^G}{dQ}\right|_{\mathcal{F}_T} := \frac{G(T,T)}{G(0,T)} = \frac{S(T,X_T)}{G(0,T)}.$$

For $0 \le t \le T$ define $\Xi_t := E_Q[\Xi_T|\mathcal{F}_t]$. Note that $\Xi_t = Y_t/Y_0$ so that, from equation (37), Ξ_t satisfies

$$\Xi_t = 1 + \int_0^t Z_u dW_u = 1 + \int_0^t \frac{Z_u}{Y_u}\frac{Y_u}{Y_0} dW_u = 1 + \int_0^t \frac{Z_u}{Y_u}\Xi_u dW_u.$$

Hence, by Girsanov's theorem,

$$W_t^G = W_t - \int_0^t \frac{Z_u}{Y_u} du$$

is an $\mathcal{F}_t$-Brownian motion under the risk-neutral measure Q^G for the futures price reinvested in the bank account as numéraire. We may then write the dynamics of the FBSDE (5),(38) under the measure Q^G as

$$X_s = X_t + \int_t^s \{AX_v + \tilde{B} + Sdiag\left(\sqrt{\alpha_i + \beta_i X_v}\right)\frac{(Z_v)^{'}}{Y_v}\}dv + \int_t^s Sdiag\left(\sqrt{\alpha_i + \beta_i X_v}\right)dW_v^G \tag{39}$$

$$Y_s = S(T,X_T) - \int_s^T \frac{Z_v (Z_v)^{'}}{Y_v} dv - \int_s^T Z_v dW_v^G. \tag{40}$$

As in Section 3 we are able to prove an existence and uniqueness result, independent of the financial application, for the nonlinear FBSDE (39)-(40). We omit the proof in the case of the futures price as it is similar to, and simpler than, the proof of Theorem 3.2. A corollary completes the characterization of the futures price as an exponential affine function of the factors process.

Theorem 5.3 *If the Riccati equation*

$$\begin{aligned} &\dot{U}_G(u) + U_G(u)A + \tfrac{1}{2}U_G(u)K^{'}\left[\beta\left(U_G(u)\right)\right]^{'} = 0, \quad u \in [0,T] \\ &U_G(T) = M(T)^{'}. \end{aligned} \tag{41}$$

admits a unique solution $U_G(\cdot)$ over the interval $[0,T]$ then the FBSDE (39)-(40) admits a unique adapted solution (X,Y,Z) given by

$$dX_s = \left(AX_s + \tilde{B} + k_0\left[U_G(s)\right]^{'} + K^{'}\left[\beta\left(U_G(s)\right)\right]^{'}X_s\right)dt + Sdiag\left(\sqrt{\alpha_i + \beta_i X_s}\right)dW_t^G, \tag{42}$$

$$\log Y_s = U_G(s)X_s + p_G(s), \text{ and} \tag{43}$$

$$Z_s = U_G(s)Sdiag\left(\sqrt{\alpha_i + \beta_i X_s}\right)Y_s, \tag{44}$$

where, for all $s \in [0,T]$,

$$p_G(s) = h(T) - \int_s^T \left(-\frac{1}{2}U_G(u)k_0\left[U_G(u)\right]^{'} - U_G(u)\tilde{B}\right)du. \tag{45}$$

As in Section 3, Assumption 2.1 guarantees that the Riccati equation (41) has a unique solution over $[0,T]$ (see Remark 3.3). Therefore, we obtain complete characterization of futures prices as exponential affine functions of the factors process as a corollary to Theorem 5.3.

Corollary 5.4 *Under Assumption 2.1 the futures price has exponential affine form,*

$$G(t,T) = e^{U_G(t)X_t + p_G(t)},$$

where $U_G(t)$ and $p_G(t)$ solve equations (41) and (45) respectively.

Example 5.5 [Bond Futures] With the dynamics (2) as the model for the factors of the economy and Assumptions 2.1-5.1 in force, Theorem 3.2 gives that the bond price has exponential affine form. Therefore, we may consider a zero coupon bond with maturity T_B as the asset underlying a futures contract with maturity T_G where $T_G < T_B$. Set $M(t) = U(t)$ and $h(t) = p(t)$, where $U(t)$ and $p(t)$ solve equations (12) and (16) respectively with T replaced by T_B. It then follows from Assumptions 2.1-5.1 and Corollary 5.4 that, with the terminal condition $U_G(T_G) = U(T_G)$ in equation (41), for all $t \in [0, T_G]$ the futures price with a bond as underlying asset is an exponential affine function of the factors.

5.2 Stochastic flows and the measure Q^G

As in Section 4 we may consider the stochastic flow (21) associated with the factors process (2). In the Gaussian case futures and forward prices were considered in [22] where, due to the Gaussian dynamics, a change of measure was not necessary. For $0 \le t \le T$ we may write

$$G(t,T) = G(t,T,X_t)$$

$Q - a.s$, where for $x \in \mathbf{R}^n$ we define

$$G(t,T,x) \stackrel{\triangle}{=} E_Q[S(T, X_T^{t,x})]. \tag{46}$$

Similar to Section 4 we may consider the vector of partial derivatives, $\partial_x G(t,T,x)$, of $G(t,T,x)$ with respect to x

$$\partial_x G(t,T,x) = E_Q[(\partial_x X_T^{t,x})' M(T) S(T, X_T^{t,x})]$$

where $(\partial_x X_T^{t,x})$ satisfies equation (22). By applying a general form of Bayes' theorem with the measure Q^G it can be shown that

$$\partial_x G(t,T,X_t) = G(t,T,X_t) E_G[L_G(t,T,X_t)|\mathcal{F}_t]$$

where $L_G(t,T,x) = (\partial_x X_T^{t,x})' M(t)$. Define, for $0 \le u \le v \le T$, $\check{D}_{uv} := E_G[\partial_x X_v^{u,x}|\mathcal{F}_u]$. Then, similar to Elliott and van der Hoek [14] and Section 4, we may express the dynamics of the Jacobian of the stochastic flow, equation (22), under the measure Q^G and take the conditional expectation with respect to $\mathcal{F}_t$ to show that $\check{D}_{uv}$ satisfies

$$\check{D}_{tu} = I + \int_t^u A\check{D}_{tv} dv + \frac{1}{2}\sum_{i=1}^n \int_t^u S diag(e_i) C E_G[M(T)' \check{D}_{vT} S e_i (\partial_x X_v^{t,X_t})|\mathcal{F}_t] dv. \tag{47}$$

almost surely. Generalizing the derivation of the FBSDE (39)-(40) to include dependence on the initial conditions (t,x), as outlined in Section 4 in the case of the bond, we obtain the following corollary to Theorem 5.3.

Corollary 5.6 *Under Assumption 2.1*

$$E_G[L_G(t,T,X_t)|\mathcal{F}_t] = [U_G(t)]^{'}$$

for $t \in [0,T]$.

Then, similar to Corollary 4.4, combining Corollary 5.6 with equation (47) we find that $\check{D}_{tu}$ is deterministic for $0 \le t \le u \le T$ and satisfies the integral equation (33) with $U(v)$ replaced by $U_G(v)$.

We next consider the forward price of the risky asset.

5.3 The Forward Price

Similar to a futures contract a forward contract is an agreement to deliver some quantity of the underlying asset in the future for a price agreed upon at the initiation of the contract. The delivery price which makes the value to both parties of the contract zero at the time of initiation is called the forward price. A forward contract, in contrast to a futures contract, is not marked to market so the value of the contract may differ from zero beyond the initiation date. The forward price of the risky asset S is given by

$$F(t,T) = \frac{E_Q[\exp(-\int_t^T r_u du)S_T|\mathcal{F}_t]}{P(t,T)} \tag{48}$$

at time t for maturity T, where $P(t,T)$ is the zero coupon bond price at time t for maturity T (see Karatzas and Shreve [25, Sec 2.3, pp. 43-45] for details).

In the absence of a stochastic dividend (or convenience) yield the numerator of equation (48) reduces to the current spot price S_t by the fact that Q is a martingale measure. In the case of deterministic interest rates the discount factor in the conditional expectation of equation (48) can be brought outside and cancels the denominator. That is, in the case of deterministic interest rates the forward price (48) of the risky asset is equal to the futures price (36) as noted by Cox et al. [6]. Therefore, in both cases the results on the bond from Section 3 and futures price from Section 5.1 may be used to prove that the forward price is an exponential affine function of the factors. As such, we shall only consider models which include stochastic interest rates and a stochastic dividend (or convenience) yield given by Assumption 5.1.

Define

$$V_s = E_Q[\exp(-\int_0^T r(X_u)du)S(T,X_T)|\mathcal{F}_s] \tag{49}$$

and note that V_s is a martingale and apply the martingale representation theorem as in Section 3. Define $Y_s = V_s/H_s$ where H_s is given by equation (3). Then clearly $F(s,T)P(s,T) = Y_s$ and $Y_T = S(T,X_T)$. Similar to Section 3 we find that Y_s satisfies the BSDE

$$Y_s = S(T,X_T) - \int_s^T \left(R^{'}X_v + k\right) Y_v dv - \int_s^T Z_v dW_v \tag{50}$$

for $s \in [t,T]$ and taken together equations (5) and (50) constitute a decoupled FBSDE for the factors process and $F(s,T)P(s,T)$.

We may define the following risk-neutral measure for the forward price reinvested in the zero-coupon bond as numéraire.

Definition 5.7 *The risk-neutral measure for the numéraire $F(\cdot,T)P(\cdot,T)$ is defined on $\mathcal{F}_T$ by*

$$Q^F(A) = E_Q[\Gamma_T 1_A]$$

for all $A \in \mathcal{F}_T$, where

$$\Gamma_T = \left.\frac{dQ^F}{dQ}\right|_{\mathcal{F}_T} := \frac{F(T,T)}{F(0,T)P(0,T)} \exp(-\int_0^T r(X_u))du).$$

Define $\Gamma_t := E_Q[\Gamma_T|\mathcal{F}_t]$. Note that $\Gamma_t = V_t/V_0$ so that, similar to Section 3, Γ_t satisfies

$$\Gamma_t = 1 + \int_0^t \frac{Z_u}{Y_u}\Gamma_u dW_u.$$

Hence, by Girsanov's theorem, the process W^F defined by

$$W_t^F = W_t - \int_0^t \frac{Z_u}{Y_u} du.$$

is an $\mathcal{F}_t$-Brownian motion under the measure Q^F. We may then write the dynamics of the FBSDE (5),(50) under the measure Q^F as

$$X_s = X_t + \int_t^s \{AX_v + \tilde{B} + Sdiag\left(\sqrt{\alpha_i + \beta_i X_v}\right)\frac{(Z_v)^{'}}{Y_v}\}dv + \int_t^s Sdiag\left(\sqrt{\alpha_i + \beta_i X_v}\right)dW_v^F \tag{51}$$

$$Y_s = S(T,X_T) - \int_s^T \left\{(R^{'}X_v + k)Y_v + \frac{Z_v(Z_v)^{'}}{Y_v}\right\}dv - \int_s^T Z_v dW_v^F. \tag{52}$$

We next give an existence and uniqueness result, independent of the construction already presented, for the coupled nonlinear FBSDE (51)-(52), by adapting a technique for linear FBSDEs from Yong [40], which gives the solution explicitly. A corollary completes the characterization of the forward price as an exponential affine function of the factors process.

Theorem 5.8 *If the Riccati equation*

$$\begin{aligned} &\dot{U}_F(u) + U_F(u)A + \tfrac{1}{2}U_F(u)K^{'}\left[\beta\left(U_F(u)\right)\right]^{'} - R^{'} = 0, \quad u \in [0,T] \\ &U_F(T) = M(T)^{'}. \end{aligned} \tag{53}$$

admits a unique solution $U_F(\cdot)$ over $[0,T]$ then the FBSDE (51)-(52) admits a unique adapted solution (X,Y,Z) given by

$$dX_s = \left(AX_s + \tilde{B} + k_0\left[U_F(u)\right]' + K'\left[\beta\left(U_F(u)\right)\right]' X_s\right)dt + Sdiag\left(\sqrt{\alpha_i + \beta_i X_s}\right)dW_t^F, \tag{54}$$

$$\log Y_s = U_F(s)X_s + p_F(s), \text{ and} \tag{55}$$

$$Z_s = U_F(s)Sdiag\left(\sqrt{\alpha_i + \beta_i X_s}\right)Y_s, \tag{56}$$

where, for all $s \in [0,T]$,

$$p_F(s) = h(T) - \int_s^T \left(k - \frac{1}{2}U_F(u)k_0\left[U_F(u)\right]' - U_F(u)\tilde{B}\right)du. \tag{57}$$

As in Section 3, Assumption 2.1 guarantees that the Riccati equation (53) has a unique solution over $[0,T]$ (see Remark 3.3). Therefore, we obtain complete characterization of forward prices as exponential affine functions of the factors process as a corollary to Theorem 5.8.

Corollary 5.9 *Under Assumption 2.1 the forward price has exponential affine form,*

$$F(t,T) = e^{([U_F(t) - U(t)]X_t + [p_F(t) - p(t)])},$$

where $U(t)$, $p(t)$, $U_F(t)$, and $p_F(t)$ solve (12), (16), (53), and (57) respectively.

5.4 Stochastic flows and the measure Q^F

Again, as in Section 4, we may consider the stochastic flow (21) associated with the factors process (2). For $0 \le t \le T$ we may write

$$F(t,T) = F(t,T,X_t)$$

Q-almost surely, where for $x \in \mathbf{R}^n$ we define

$$F(t,T,x) \triangleq \frac{E_Q[\exp(-\int_t^T r(X_u^{t,x})du)S(T,X_T^{t,x})]}{P(t,T,x)} \tag{58}$$

and $P(t,T,x)$ is as in equation (24). Differentiating (58) with respect to x gives

$$\partial_x F(t,T,x) = \frac{E_Q[\exp(-\int_t^T r(X_u^{t,x})du)S(T,X_T^{t,x})L_F(t,T,x)] - F(t,T,x)\partial_x P(t,T,x)}{P(t,T,x)} \tag{59}$$

where $L_F(t,T,x) = (-\int_t^T (\partial_x X_u^{t,x})' R du + (\partial_x X_T^{t,x})' M(T))$. Applying a general form of Bayes' theorem with the measure Q^F and using equation (27) it can be shown that

$$\partial_x F(t,T,X_t) = F(t,T,X_t)\left(E_F[L_F(t,T,X_t)|\mathcal{F}_t] - E_T[L(t,T,X_t)|\mathcal{F}_t]\right) \tag{60}$$

Define, for $0 \le u \le v \le T$, $\tilde{D}_{uv} = E_F[\partial_x X_v^{u,x} | \mathcal{F}_u]$. Then, similar to Elliott and van der Hoek [14] and Section 4, we can write the dynamics (22) under the measure Q^F and take the conditional expectation with respect to $\mathcal{F}_t$ to show that $\tilde{D}_{tu}$ satisfies

$$\tilde{D}_{tu} = I + \int_t^u A\tilde{D}_{tv}dv - \frac{1}{2}\sum_{i=1}^n \int_t^u Sdiag(e_i)CE_F[\left(R' \int_v^T \tilde{D}_{vv_1}dv_1 \; - \; M(T)'\tilde{D}_{vT}\right) Se_i(\partial_x X_v^{t,X_t})|\mathcal{F}_t]dv \tag{61}$$

almost surely. Generalizing the derivation of the FBSDE (51)-(52) to include dependence on the initial conditions (t,x), as outlined in Section 4 in the case of the bond, we obtain the following corollary to Theorem 5.8.

Corollary 5.10 *Under Assumption 2.1*

$$E_F[L_F(t,T,X_t)|\mathcal{F}_t] = [U_F(t)]'$$

for $t \in [0,T]$.

Then, similar to Corollary 4.4, combining Corollary 5.10 with equation (61) we find that $\tilde{D}_{tu}$ is deterministic for $0 \le t \le u \le T$ and satisfies the integral equation (33) with $U(v)$ replaced by $U_F(v)$.

6 Summary

In this paper we have considered a factor model whose risk-neutral dynamics are given by an affine diffusion. The short interest rate is supposed to be an affine function of the factors process. We provided a characterization of the joint dynamics of the interest rate and the zero-coupon bond price in terms of a forward-backward stochastic differential equation (FBSDE) which is, after a change of measure, coupled and nonlinear and to which the usual existence and uniqueness theorems for FBSDEs do not apply. The main result of the paper is to prove that provided certain Riccati equations are solvable, a condition guaranteed by the assumptions of the model, the nonlinear FBSDE associated with the bond price under the forward measure has a unique solution. The solution of the nonlinear FBSDE is given explicitly and is determined by the solution of the Riccati equation, the solution of a deterministic terminal value problem, the solution of an SDE, and a pair of equations expressing the backward components of the FBSDE in terms if the solution of the SDE.

The first corollary to the existence and uniqueness result provides the characterization of the bond price as an exponential affine function of the factors process. Further corollaries provide sufficient conditions such that the conditional expectation (under the forward measure) associated with the linear ordinary differential equation for the bond price is deterministic. In fact this conditional expectation is equal to the solution of the Riccati equation. This result unifies the approach of Elliott and van der Hoek [14] with the approach of Duffie and Kan [11]. A final corollary to the existence and uniqueness theorem shows that, if the Riccati equation is solvable, the conditional expectation under the forward measure of the Jacobian of the stochastic flow is deterministic. This result addresses a mistake in the proof of the approximation lemma of Elliott and van der Hoek [14].

The methods presented for the bond price were also applied to characterize futures and forward prices. We assume that the underling asset price is given by an exponential affine function of the factors and the dividend yield (or convenience yield in the case of a commodity) is an affine function of the factors process. The characterization of futures and forward prices in terms of FBSDEs proceeds in much the same way, apart from the measure changes, as the characterization of the bond price. Our approach, based on FBSDEs, can be extended to the affine jump-diffusion models for the factors studied by Duffie et al. [12], Duffie et al. [10], Björk and Landén [2], Chacko and Das [3], Björk et al. [1], and Levendorskiĭ [28]. The approach has been further generalized to consider the transforms of an affine diffusion introduced by Duffie et al. [12]. The inclusion of jumps and the consideration of transforms allows for the consideration of other financial derivatives and is considered in Hyndman [23].

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