Gaussian Factor Models - Futures and Forward Prices

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March 10, 2007

Abstract

We completely characterise the futures price and forward price of a risky asset (commodity) paying a stochastic dividend yield (convenience yield). The asset (commodity) price is modelled as an exponential affine function of a Gaussian factors process while the interest rate and dividend yield are affine functions of the factors process. The characterisation we provide is based on the method of stochastic flows. We believe this method leads to simpler and more clear-cut derivations of the futures price and forward price formulae than alternative methods. Hedging a long term forward contract with shorter term futures contracts and bonds is also examined.

Keywords: futures price; forward price; stochastic flows; factor models; Gaussian state variables

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The author would like to thank Andrew Heunis and Robert Elliott for helpful discussions.

The author would like to acknowledge the financial support of MITACS through the "Prediction in Interacting Systems (PINTS)" and "Modelling Trading and Risk in the Market" projects and the financial support of the Institut de finance mathématique de Montréal (IFM^2).

1 Introduction

Gaussian factor models of asset prices have been extensively used in financial modelling. Gaussian models remain popular due to their analytic tractability and well established statistical methodology. Continuous-time models for futures and forward prices have been studied by Gibson and Schwartz (1990), Schwartz (1997, 1998), Cortazar and Schwartz (1994), Miltersen and Schwartz (1998), Schwartz and Smith (2000), and Manoliu and Tompaidis (2002) among others. The work of Miltersen and Schwartz (1998) is notable in that it develops an analogue of the Heath, Jarrow and Morton (1992) model in the context of futures markets. Other general studies on futures and forward prices include Schroder (1999), Björk and Landén (2002), and the references contained therein.

In this paper we completely characterise the futures price and forward price of a risky asset (commodity) paying a stochastic dividend yield (convenience yield). The interest rate and dividend yield are modelled as affine functions of a Gaussian factors process. We also assume that the asset price is modelled as an exponential affine function of the Gaussian factors process. Our analysis is based on the method of stochastic flows introduced to the term structure literature by Elliott and van der Hoek (2001). The stochastic flows method presented in this paper includes the Gaussian factors models which have appeared in the literature as a special case and has been generalized to include models which include non-Gaussian (square-root or Cox, Ingersoll and Ross (1985) type) factors. One of the main contributions of this paper is a unified framework under which to study a wide class of models in a systematic and clear-cut way.

Most of the derivations of futures and forward prices which have appeared in the literature, with the exception of Björk and Landén (2002), have been model specific and do not address the entire class of Gaussian factor models. Derivations of futures and forward prices that have appeared in the literature have usually been based on solving a partial differential equation (PDE) or calculating a conditional expectation using the distributional properties of the factors process. However, in the case of PDE derivations the exponential affine form of the solution is guessed and then substituted into the PDE to reduce the problem to the solution of ordinary differential equations (ODEs). Our method shows why the solution is exponential affine by characterizing the futures and forward price as the solution to a linear ODE. Derivations of the prices based on distributional properties of the factors process, for example Bjerksund (1991), are often complicated when some factors are correlated. Further, our method has been generalised to study other models, outside the assumptions of this paper, where the functional form of the solution to the PDE for the contingent claim is not easily guessed.

Due to certain shortcomings of Gaussian models, or the desire to better model asset price movements, more complicated models which often incorporate non-Gaussian components and jumps have become widely used. For example Yan (2002), Björk and Landén (2002), and Ribeiro and Hodges (2004) employ combinations of Gaussian, non-Gaussian, and jump factors. For non-Gaussian factor models certain approximations and assumptions, often left unjustified, are made to allow for a practical implementation of the model or to calibrate the model to historical data. For models containing a combination of Gaussian and Cox, Ingersoll and Ross (1985) type factors the validity of applying the Feynman-Kac formula to derive derivative prices has only been recently justified by Levendorskiĭ (2004) despite the previous appearance of such models and derivations in the literature.

The methods of this paper have been extended to ‘affine term structure models’ and ‘affine price models’ in Hyndman (2005) where the factors process follows an affine diffusion. This extension includes as special cases the continuous factors model of Duffie and Kan (1996) for bond prices and Björk and Landén (2002) for futures and forward prices. Additional tools, namely changes of measure and forward-backward stochastic differential equations, are required to derive the bond, futures, and forward prices for general affine factor models which are not required in the Gaussian case. Therefore, we shall concentrate exclusively on the Gaussian case in this paper.

The remainder of this paper is organised as follows. Section 2 discusses the market model and sets some notation. Section 3 studies the futures price, Section 4 reviews some results on the bond price which are necessary for our discussion of the forward price in Section 5, and Section 6 examines hedging a long term forward commitment using shorter term futures contracts and bonds. Section 7 briefly discusses implementation and the estimation of model parameters, Section 8 shows how our methods apply to three different commodity market models that have appeared in the literature, and Section 9 concludes.

2 Market Model

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which there is given a standard, m-dimensional Brownian motion $W_t = (W_t^{(1)}, \ldots, W_t^{(m)})'$ with $'$ denoting the transpose. For a fixed, positive, and finite time horizon T^* define $\mathcal{F}_t^W = \sigma(W_u : 0 \leq u \leq t)$ for all $t \in [0, T^*]$. Let $\mathcal{N}$ denote the P-null subsets of $\mathcal{F}_{T^*}^W$. The filtration generated by W and augmented by the null sets, $\mathcal{F}_t = \sigma(\mathcal{F}_t^W \cup \mathcal{N})$, models the information available to investors and should be regarded as fixed. That is, we cannot choose to work with a different filtration. The modelling framework is the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, P)$. To avoid repeating the entire development of classical mathematical finance we take the following as given.

Assumption 2.1 *Throughout we shall assume a* ***complete*** *and* ***standard*** *financial market in the sense of Definitions 1.5.1 and 1.6.1 of Karatzas and Shreve (1998).*

Under Assumption 2.1 there exists a unique equivalent martingale measure Q on $\mathcal{F}_{T^*}$. The practical advantage of working with the measure Q is that the prices of derivative securities can be expressed as conditional expectations.

On $(\Omega, \mathcal{F}, Q)$ consider a factors process X taking values in $\mathbf{R}^n$ given by

$$dX_t = (A_t X_t + \gamma_t)dt + \sigma_t dB_t \tag{1}$$

where B is any m-dimensional Brownian motion with respect to the fixed filtration $\{\mathcal{F}_t,\ t \geq 0\}$ and the risk-neutral measure Q. In particular the usual Brownian motion constructed using Girsanov's Theorem as in Karatzas and Shreve (1998, p. 17) can be used. Also, A_t is a time-varying (deterministic) $(n \times n)$-matrix, γ_t is a time-varying (deterministic) $(n \times 1)$-column vector, and σ_t is a time-varying (deterministic) $(n \times m)$-matrix. Further, we require the functions $\gamma : [0,T] \to \mathbf{R}^n$, $A : [0,T] \to \mathbf{R}^{n\times n}$, and $\sigma : [0,T] \to \mathbf{R}^{n\times m}$ to be Borel measurable, integrable, and globally Lipschitz so that the SDE (1) has a unique strong solution.

Assumption 2.2 *We assume that the riskless interest rate, the asset (spot) price, and the dividend (convenience) yield are functions of the factors process. That is,*

$$r_t = r(X_t), \quad S_t = S(X_t), \quad \textit{and } \delta_t = \delta(X_t).$$

where $r : \mathbf{R}^n \to \mathbf{R}$, $\delta : \mathbf{R}^n \to \mathbf{R}$, *and* $S : \mathbf{R}^n \to \mathbf{R}_{++}$ *are specified by:*

(i) $r_t = r(X_t)$ *where, for* $x \in \mathbf{R}^n$, $r(x) = R^{'}x + k$, R *is an* $(n \times 1)$*-column vector, and* k *is a scalar;*

(ii) $S_t = S(X_t)$ *where, for* $x \in \mathbf{R}^n$, $S(x) = \exp(M^{'}x + h)$, M *is an* $(n \times 1)$*-column vector, and* h *is a scalar; and*

(iii) $\delta_t = \delta(X_t)$ *where, for* $x \in \mathbf{R}^n$, $\delta(x) = N^{'}x + l$, N *is an* $(n \times 1)$*-column vector, and* l *is a scalar.*

The process r_t given by this setup can take negative values with positive probability. However, the Vašíček (1977) and Hull and White (1994) interest rate models are special cases of the general Gaussian factors model that are widely used despite the limitation of allowing negative rates. Assumption 2.2 also includes as special cases the commodity price models of Gibson and Schwartz (1990), Schwartz (1997, 1998), Schwartz and Smith (2000), and Manoliu and Tompaidis (2002).

Write $X_u^{t,x}$ for the solution of equation (1) started from the point $x \in \mathbf{R}^n$ at time $t \geq 0$. That is, $X_u^{t,x}$ is the solution of equation (1) such that $X_t^{t,x} = x$ and

$$X_u^{t,x} = x + \int_t^u (A_v X_v^{t,x} + \gamma_v) dv + \int_t^u \sigma_v dB_v.$$

We refer to $X_u^{t,x}$ as the **stochastic flow** associated with the factors process. The map $x \to X_u^{t,x}$ is almost surely differentiable and the Jacobian matrix of partial derivatives satisfies the equation

$$\partial_x X_u^{t,x} = I + \int_t^u A_v \partial_x X_v^{t,x} dv, \quad u \geq t$$

Protter (1990, Theorem 39, pp. 250).

Consider the $n \times n$-matrix

$$\Psi(t) = \left(\begin{array}{ccc} \vec{x}^{(1)}(t) & \dots & \vec{x}^{(n)}(t) \end{array} \right)$$

whose columns are the vector functions which form a fundamental set of solutions for the system

$$\vec{x}'(t) = A_t \vec{x}(t). \tag{2}$$

Since the columns of $\Psi(t)$ are linearly independent the matrix is invertible. Define

$$D_{tu} \stackrel{\triangle}{=} \Psi(u)\Psi^{-1}(t), \qquad 0 \leq t \leq u \leq T.$$

Then D_{tu} satisfies the integral equation

$$D_{tu} = I + \int_t^u A_v D_{tv} dv. \tag{3}$$

Therefore, $\partial_x X_u^{t,x}$ is the deterministic matrix D_{tu} which does not depend on x for all $0 \leq t \leq u \leq T$. That is, $D_{tu} = \partial_x X_u^{t,x}$. If $A_t = A$ is independent of t, then we have that D_{tu} is the matrix exponential

$$D_{tu} = e^{A(u-t)}, \quad u \geq t \geq 0, \quad x \in \mathbf{R}^n. \tag{4}$$

3 Futures Price

A futures contract is an agreement to deliver some quantity of the underlying asset in the future for a price agreed upon at the initiation of the contract. The delivery price which makes the value to both parties of the contract zero at all times is called the futures price. By the mechanism of marking to market, where changes in the value of the futures contract are settled daily, in accordance with changes in the futures price the risk of default by one party is transfered to the exchange. Basic information on futures contracts and market mechanics can be found in Hull (2002).

In this section we shall characterise the futures price as an exponential affine function of the factors process, whose dynamics are given by (1), when the market model satisfies Assumptions 2.1 and 2.2. By Assumption 2.1 the futures price of the risky asset S is given by

$$G(t,T) = E_Q[S_T|\mathcal{F}_t] \tag{5}$$

at time t for maturity T (see Karatzas and Shreve (1998, Theorem 3.7, pp. 45-46) for a proof). The result upon which our methods and results are based is the following version of the Markov property (see Friedman (1975) for a more general result and proof).

Proposition 3.1 *For* $0 \leq t \leq T$

$$G(t,T) = G(t,T,X_t)$$

$Q-a.s$*, where for* $x \in \mathbf{R}^n$ *we define*

$$G(t,T,x) \stackrel{\triangle}{=} E_Q[S(X_T^{t,x})]. \tag{6}$$

The notation $G(t,T,x)$ introduced in Proposition 3.1 can be confused with the notation for the futures price. However, $G(t,T,x)$ is not a price *per se*, rather, it is a function expressing the dependence of the futures price on the factors of the economy at time t. If we can completely understand the functional dependence of $G(t,T,x)$ on (t,T,x) then Proposition 3.1 allows us to understand the dependence of the futures price on the factors and sensitivities to changes in the factors.

By differentiating $G(t,T,x)$ with respect to x (denoted by $\partial_x G(t,T,x)$) we obtain, subject to regularity conditions that allow the exchange of expectation and differentiation, that

$$\partial_x G(t,T,x) = E_Q[S(X_T^{t,x})D_{tT}^{'}M]. \tag{7}$$

Since $D_{tT}^{'}M$ is deterministic it can be brought outside of the expectation and we obtain the following result which shall be used to characterise the futures price as an exponential affine function of the factors.

Theorem 3.2 *For* $0 \leq t \leq T$

$$\partial_x G(t,T,x) = D_{tT}^{'}MG(t,T,x) \tag{8}$$

for all $x \in \mathbf{R}^n$.

The solution of the (system of) linear ODE (8) is an exponential affine function of x and, from equation (6), the terminal condition is $G(T,T,x) = S(x)$.

Corollary 3.3 *For* $0 \leq t \leq T$

$$G(t,T,x) = \exp\left(M^{'} D_{tT}\ x + L(t,T)\right). \tag{9}$$

for all $x \in \mathbf{R}^n$ *and some non-random function* $L(t,T)$ *such that* $L(T,T) = h$*, where* h *is defined in Assumption 2.2 (ii).*

Applying Proposition 3.1 to equation (9) gives

Theorem 3.4 *For* $0 \leq t \leq T$

$$G(t,T) = \exp\left(M^{'} D_{tT}\ X_t + L(t,T)\right)$$

$Q - a.s.$

We now turn our attention to representing $L(t,T)$ as the solution of an ODE. Note that for any vector b we write b_i for the i-th element and for any matrix Σ we write Σ_{ij} for the (i,j)-th element.

Theorem 3.5 $G(t,T,x)$ *satisfies the PDE*

$$0 = \frac{\partial G(t,T,x)}{\partial t} + \frac{\partial G^{'}(t,T,x)}{\partial x}\left[A_t x + \gamma_t\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 G(t,T,x)}{\partial x_i \partial x_j}[\sigma_t \sigma_t^{'}]_{ij} \tag{10}$$

for all $(t,x) \in [0,T] \times \mathbf{R}^n$ *with* $G(T,T,x) = S(x)$.

Proof: For all $(t,x) \in [0,T] \times \mathbf{R}^n$ define $b(t,x) = [A_t x + \gamma_t]$, $\sigma(t,x) = \sigma_t$, and $a(t,x) = \sigma(t,x)\sigma^{'}(t,x)$ so that

$$a_{ik}(t,x) = \textstyle\sum_{j=1}^{m} \sigma_{ij}(t,x)\sigma_{kj}(t,x) \quad 1 \leq i,k \leq n.$$

Consider the operator $\mathcal{A}_t$ defined, for $f \in C^2(\mathbf{R}^n)$, by

$$(\mathcal{A}_t f)(x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(t,x)\frac{\partial f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial f(x)}{\partial x_i}.$$

By the Feynman-Kac formula, Karatzas and Shreve (1991, pp. 366-367), there exists $v(t,x) : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ such that $v \in C^{1,2}([0,T] \times \mathbf{R}^n)$, satisfies the Cauchy problem

$$-\frac{\partial v}{\partial t} = \mathcal{A}_t v \quad ; \quad \text{in } [0,T) \times \mathbf{R}^n,$$
$$v(T,x) = S(x) \quad ; \quad x \in \mathbf{R}^n,$$

and

$$v(t,x) = E_Q[S(X_T^{t,x})]$$

is the unique solution. Therefore, by equation (6) we have

$$v(t,x) = G(t,T,x) \quad \forall (t,x) \in [0,T] \times \mathbf{R}^n.$$

Hence, $G(t,T,x)$ satisfies the PDE (10). ■

Corollary 3.6 *$L(t,T)$ satisfies the ODE*

$$0 = \frac{\partial}{\partial t} L(t,T) + M' D_{tT} \gamma_t + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} [D'_{tT} M M' D_{tT}]_{ij} (\sigma_t \sigma'_t)_{ij} \tag{11}$$

for all $t \in [0,T]$, with terminal condition $L(T,T) = h$ and h is defined in Assumption 2.2 (ii).

Proof: Equation (11) follows by first calculating the partial derivatives of $G(t,T,x)$ using equation (9). Substituting these partial derivatives into equation (10), dividing by the positive quantity $G(t,T,x)$, and setting $x = 0$ gives equation (11).

The terminal condition follows by setting $t = T$ in equation (6) to find $G(T,T,x) = S(x)$ for all $x \in \mathbf{R}^n$. Then, with $t = T$, comparing equation (9) with Assumption 2.2 *(ii)* we must have

$$M' D_{TT} x + L(T,T) = M' x + h$$

for all $x \in \mathbf{R}^n$ where h is from Assumption 2.2 *(ii)*. However, from equation (3), $D_{TT} = I$, the $n \times n$ identity matrix, for all $x \in \mathbf{R}^n$. Therefore, we must have $L(T,T) = h$. ■

We shall show, in Section 8, that the general methodology presented in this section can be easily applied to various examples of commodity market models found in the literature. We delay the examples until after our discussion of the forward price so that we may consider the futures and forward price of a given model simultaneously. In the next section we briefly review some results on the zero-coupon bond price that are necessary for our discussion of the forward price.

4 Bond Price

The results of this section are based on results presented for affine term structure models and appeared in Elliott and van der Hoek (2001). For ease of reference and unity of notation we recall here the results necessary for our analysis of the forward price of the risky asset.

By Assumption 2.1 the price of the zero-coupon bond is given by

$$P(t,T) = E_Q[\exp(-\int_t^T r_u du)|\mathcal{F}_t] \tag{12}$$

at time t for maturity T. For $0 \leq t \leq T$, since X_t is a Markov process (Friedman, 1975), it follows that

$$P(t,T) = P(t,T,X_t) \tag{13}$$

$Q - a.s$, where for $x \in \mathbf{R}^n$ we define

$$P(t,T,x) \stackrel{\triangle}{=} E_Q[\exp(-\int_t^T r(X_u^{t,x})du)]. \tag{14}$$

Therefore, if we characterise the dependence of $P(t,T,x)$ on (t,T,x), the Markov property allows us to understand the dependence of the bond price on the factors. Differentiating $P(t,T,x)$ with respect to the initial condition (denoted by $\partial_x P(t,T,x)$), subject to regularity conditions that allow the exchange of expectation and differentiation, we find

$$\partial_x P(t,T,x) \quad = \quad E_Q[\exp(-\int_t^T r(X_u^{t,x})du)\left(-\int_t^T D_{tu}^{'}Rdu\right)]. \tag{15}$$

We define

$$B(t,T) \stackrel{\triangle}{=} \int_t^T D_{tu}^{'}Rdu. \tag{16}$$

Since $B(t,T)$ is deterministic it may be brought outside of the expectation to obtain the following result which is used to characterise the bond price as an exponential affine function of the factors.

Theorem 4.1 *For* $0 \leq t \leq T$

$$\partial_x P(t,T,x) = -B(t,T)P(t,T,x) \tag{17}$$

for all $x \in \mathbf{R}^n$.

The solution of the (system of) ODE (17) is an exponential affine function of x and, from equation (14), the terminal condition is $P(T,T,x) = 1$.

Corollary 4.2 *For* $0 \leq t \leq T$ *and for all* $x \in \mathbf{R}^n$

$$P(t,T,x) = \exp(A(t,T) - B^{'}(t,T)x) \tag{18}$$

where $B(t,T)$ *is from equation (16) with* $B(T,T) = 0$*, and some non-random function* $A(t,T)$ *such that* $A(T,T) = 0$.

Applying the Markov property, equation (13), to equation (18) gives:

Theorem 4.3 *For* $0 \leq t \leq T$

$$P(t,T) = \exp\left(A(t,T) - B^{'}(t,T)X_t\right) \tag{19}$$

$Q-a.s$, for some function A to be determined.

Similar to Corollary 3.6 the function $A(t,T)$ may be represented as the solution of an ODE by applying the Feynman-Kac formula. However, as this result is not necessary in the sequel we shall proceed directly to our discussion of the forward price.

5 Forward Prices

A forward contract is much the same as a futures contract in that a quantity of the asset is agreed to be delivered at some time in the future. However, unlike a futures contract, the value of a forward contract is only zero to both parties at the initiation of the contract. The forward price is the delivery price that satisfies this constraint. Forward contracts, unlike futures contracts, are not settled daily (marked to market) but only at the delivery time. Basic information on futures and forward contracts and market mechanics can be found in Hull (2002).

By Assumption 2.1 the forward price of the risky asset S is given by

$$F(t,T) = \frac{E_Q[\exp\left(-\int_t^T r_u du\right)S_T|\mathcal{F}_t]}{P(t,T)} \tag{20}$$

at time t for maturity T, where $P(t,T)$ is the zero-coupon bond price (Karatzas and Shreve, 1998, pp. 43-45). In the absence of a stochastic dividend (convenience yield) the numerator of equation (20) reduces to the current spot price S_t by the fact that Q is a martingale measure. In the case of deterministic interest rates the discount factor in the conditional expectation of equation (20) can be brought outside and cancels the denominator. That is, in the case of deterministic interest rates the forward price (20) of the risky asset is equal to the futures price (5) as noted by Cox et al. (1981). Therefore, in both cases the results of the previous two sections may be used to prove that the forward price is an exponential affine function of the factors. As such, we shall only consider models which include stochastic interest rates and a stochastic dividend (convenience) yield given by Assumption 2.2.

The Markov property (Friedman, 1975) gives

Proposition 5.1 *For* $0 \le t \le T$

$$F(t,T) = F(t,T,X_t)$$

Q-almost surely, where for $x \in \mathbf{R}^n$ *we define*

$$F(t,T,x) \triangleq \frac{E_Q[\exp(-\int_t^T r(X_u^{t,x})du)S(X_T^{t,x})]}{P(t,T,x)} \tag{21}$$

and $P(t,T,x)$ *is defined as in equation (14).*

By differentiating $F(t,T,x)$ with respect to x (denoted by $\partial_x F(t,T,x)$), we obtain, subject to regularity conditions that allow the exchange of expectation and differentiation

$$\partial_x F(t,T,x) = \frac{E_Q[\exp(-\int_t^T r(X_u^{t,x})du)S(X_T^{t,x})\left(-B(t,T)+D_{tT}^{'}M\right)] - F(t,T,x)\partial_x P(t,T,x)}{P(t,T,x)}. \tag{22}$$

Therefore, from equations (21) and (22) we find

$$\partial_x F(t,T,x) = \frac{\left(-B(t,T)+D_{tT}^{'}M\right)F(t,T,x)P(t,T,x) - F(t,T,x)\partial_x P(t,T,x)}{P(t,T,x)}. \tag{23}$$

Applying the result of Theorem 4.1, namely $\partial_x P(t,T,x) = -B(t,T)P(t,T,x)$, to equation (23) gives the following result.

Theorem 5.2 *For* $0 \le t \le T$

$$\partial_x F(t,T,x) = D_{tT}^{'}MF(t,T,x) \tag{24}$$

for all $x \in \mathbf{R}^n$.

The solution of the (system of) ODE (24) is an exponential affine function of x and, from equation (21), the terminal condition is $F(T,T,x) = S(x)$.

Corollary 5.3 *For* $0 \le t \le T$

$$F(t,T,x) = \exp(M^{'}D_{tT}\, x + C(t,T)). \tag{25}$$

for all $x \in \mathbf{R}^n$ *for some non-random function* $C(t,T)$ *such that* $C(T,T) = h$ *with* h *from Assumption 2.2 (ii).*

Applying Proposition 5.1 to equation (25) gives

Theorem 5.4 *For* $0 \le t \le T$

$$F(t,T) = \exp(M^{'}D_{tT}\, X_t + C(t,T))$$

Q-almost surely, for some function $C(t,T)$ *to be determined where* $C(T,T) = h$ *and* h *is from Assumption 2.2 (ii).*

We now turn our attention to representing $C(t,T)$ as the solution of an ODE. Write e_i for the unit vector in $\mathbf{R}^n$ with 1 in the i-th position.

Theorem 5.5 *$F(t,T,x)$ satisfies the PDE*

$$0 = \frac{\partial F(t,T,x)}{\partial t} + \frac{\partial F^{'}(t,T,x)}{\partial x}[A_t x + \gamma_t] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 F(t,T,x)}{\partial x_i \partial x_j} - B^{'}(t,T)\left(e_i \frac{\partial F(t,T,x)}{\partial x_j} + e_j \frac{\partial F(t,T,x)}{\partial x_i}\right)\right][\sigma_t \sigma_t^{'}]_{ij} \tag{26}$$

for all $(t,x) \in [0,T] \times \mathbf{R}^n$ with $F(T,T,x) = S(x)$.

Proof: Similar to the proof of Theorem 3.5. ■

Corollary 5.6 *$C(t,T)$ satisfies the ODE*

$$0 = \frac{\partial}{\partial t}C(t,T) + M^{'}D_{tT}\gamma_t + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{[D_{tT}^{'}MM^{'}D_{tT}]_{ij} - \left(B_i(t,T)[M^{'}D_{tT}]_j + B_j(t,T)[M^{'}D_{tT}]_i\right)\right\}[\sigma_t \sigma_t^{'}]_{ij} \tag{27}$$

for all $t \in [0,T]$, with terminal condition $C(T,T) = h$ and h is from Assumption 2.2 (ii).

Proof: Similar to the proof of Corollary 3.6. ■

6 Hedging a contract for future delivery

As noted by Schwartz (1997) to properly hedge a long term forward commitment in a commodity with a portfolio of shorter term futures contracts and bonds the sensitivity of the present value of the forward commitment with respect to each of the factors must equal the sensitivity of the portfolio with respect to each of the factors. As such, the number of futures contracts and bonds in the portfolio must equal the number of factors. We briefly consider hedging a forward commitment to deliver one unit of the asset at time T.

Let $t_1^G, \ldots, t_{N^G}^G$ denote the maturities, all less than T, of the futures contracts in the portfolio and w_i^G denotes the number of long positions (negative indicating short positions) in the futures contract with maturity t_i^G. Similarly, $t_1^B, \ldots, t_{N^B}^B$ denotes the maturities, all less than T, of the bonds in the portfolio and w_i^B denotes the number of long positions in the bond with maturity t_i^G. Note $n = N^G + N^B$ and to properly hedge the forward commitment we must solve the following system of n equations in n unknowns.

$$\begin{aligned}
\sum_{i=1}^{N^G} w_i^G e_1' D_{tt_i^G}' M G(t,t_i^G,x) - \sum_{i=1}^{N^B} w_i^B e_1' B(t,t_i^B) P(t,t_i^B,x) &= e_1' \left[-B(t,T) + D_{tT}' M \right] P(t,T,x) F(t,T,x) \\
&\vdots \\
\sum_{i=1}^{N^G} w_i^G e_n' D_{tt_i^G}' M G(t,t_i^G,x) - \sum_{i=1}^{N^B} w_i^B e_n' B(t,t_i^B) P(t,t_i^B,x) &= e_n' \left[-B(t,T) + D_{tT}' M \right] P(t,T,x) F(t,T,x)
\end{aligned}$$

The choice of the number of futures contracts, N^G, and the number of bonds, N^B, to include in the portfolio and at which maturities is not unique. However, such considerations are beyond the scope of this paper.

7 Statistical Estimation

One of the key features of Gaussian factor models and the resulting exponential affine form of the associated bond, futures, and forward prices is the ease with which various statistical estimation techniques can be used to calibrate the model parameters to market data. Further, if it is assumed that the factors are unobservable while the term structure of bond, futures, or forward prices is observed over time the Kalman filter can be employed to estimate the model parameters and the time series of the factors given the observed term structure. Under a logarithmic transformation of the bond, futures, or forward price and an Euler discretisation of the factors process a state-space model is obtained to which the discrete-time Kalman filter can be applied. That is, if the term structure of futures prices $\{G(t_n, T_i)\ ;\ i = 1, \ldots, M\}$ for contracts expiring at times $\{T_1, T_2, \ldots, T_M\}$ is observed in the market at discrete times $\{t_0, t_1, \ldots, t_N\}$ then an empirical model to which we can apply the discrete-time (linear) Kalman filter is obtained by an Euler discretisation of the real-world (P-measure) dynamics for the factors X_t and the i^{th} component of the observation vector is $Y_n^i = \log G(t_n, T_i) +$ "*noise*" with $G(t,T)$ given by Theorem 3.4.

This procedure has been employed, for example, by Babbs and Nowman (1999) in the case of interest rates and by Schwartz (1997), Schwartz and Smith (2000), and Manoliu and Tompaidis (2002) in the case of commodity markets and futures prices. In each case parameters were estimated by direct maximization of the likelihood function and the factors were estimated by the Kalman filter from the observable term-structure. An overview of the Kalman filter and maximum likelihood estimation, as well as an alternative to direct maximization of the likelihood using a filter-based expectation-maximization (EM) algorithm, is given by Elliott and Hyndman (2006).

8 Examples

We present a number of examples of commodity models that have appeared in the literature which are special cases of the general Gaussian factor model.

8.1 Models with constant interest rates

The simplest commodity market models are factor models with constant (or deterministic) interest rates and in such cases the futures and forward prices are equal.

8.1.1 Schwartz and Smith (2000): Uncertain Equilibrium

Schwartz and Smith (2000) consider the dynamics

$$\begin{aligned} d\chi_t &= (\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi dB_t^{(1)} \\ d\xi_t &= (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dB_t^{(2)} \\ d\left\langle B^{(1)}, B^{(2)} \right\rangle_t &= \rho_{\chi\xi} dt \end{aligned}$$

and take $S_t = \exp(\chi_t + \xi_t)$. This model does not use convenience yields but instead assumes that χ models short-term deviations in the spot-price and ξ models the equilibrium price level. The Kalman filter and maximum likelihood parameter estimation were employed to study NYMEX oil futures contracts. The model is equivalent to the two-factor model of Gibson and Schwartz (1990) and Schwartz (1997).

Since our discussions have assumed standard Brownian motion ($\rho_{\chi\xi} = 0$) we consider an equivalent formulation of the model. That is, suppose $X_t = (\chi_t, \xi_t)'$ satisfies equation (1) with $B_t = (B_t^{(1)}, B_t^{(2)})'$ a standard Brownian motion taking values in $\mathbf{R}^2$,

$$\gamma_t = \begin{pmatrix} -\lambda_\chi \\ \mu_\xi - \lambda_\xi \end{pmatrix}, \quad A_t = \begin{pmatrix} -\kappa & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_t = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

We consider X_t as our underlying factor in the context of the general Gaussian model. Then with $R = \vec{0} \in \mathbf{R}^2$, $M = \vec{1} \in \mathbf{R}^2$, $N = \vec{0} \in \mathbf{R}^2$, $k = r$, $h = 0$, and $l = 0$ in Assumption 2.2 the market model is equivalent to the formulation given by Schwartz and Smith (2000). The Jacobian of the stochastic flow associated with X_t is

$$D_{tu} = \begin{pmatrix} e^{-\kappa(u-t)} & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that

$$M^{'}D_{tT} = \left(\ e^{-\kappa(T-t)}, 1\ \right).$$

Substituting into equation (11) gives that $L(t,T)$ satisfies the ODE

$$0 = \frac{\partial}{\partial t}L(t,T) + (\mu_\xi - \lambda_\xi + \frac{1}{2}\sigma_2^2) - (\lambda_\chi - \sigma_1\sigma_2\rho)e^{-\kappa(T-t)} + \frac{1}{2}\sigma_1^2 e^{-2\kappa(T-t)}. \tag{28}$$

Equation (28) along with the terminal condition can be solved to obtain

$$L(t,T) = (\mu_\xi - \lambda_\xi + \frac{1}{2}\sigma_2^2)(T-t) - (\lambda_\chi - \sigma_1\sigma_2\rho)\frac{\left(1 - e^{-\kappa(T-t)}\right)}{\kappa} + \frac{1}{4}\sigma_2^2\frac{\left(1 - e^{-2\kappa(T-t)}\right)}{\kappa}. \tag{29}$$

Hence, by Theorem 3.4, the futures price of the risky asset for the Schwartz and Smith (2000) model is

$$G(t,T) = \exp\left(M^{'}D_{tT}X_t + L(t,T)\right) = \exp\left(e^{-\kappa(T-t)}\chi_t + \xi_t + L(t,T)\right) \tag{30}$$

where $L(t,T)$ is given by equation (29). Equations (29) and (30) agree with the futures price given by Schwartz and Smith (2000) which the authors obtained by calculating the conditional expectation (5) using the fact that (χ_t, ξ_t) are jointly normally distributed.

8.1.2 Manoliu and Tompaidis (2002)

Manoliu and Tompaidis (2002) use a class of multi-factor stochastic models where the spot price is an exponential affine function of Gaussian factors to study energy futures prices. Specifically Manoliu and Tompaidis (2002) assume that $\log S_t$ is of the form

$$\log S_t = \sum_{i=1}^{n} \xi_t^i$$

where

$$d\xi_t^i = (\alpha_t^i - k_t^i \xi_t^i)dt + \sum_{j=1}^{n} \sigma_t^{ij} dB_t^{(j)}$$

and use this formulation to derive the futures price and perform an empirical study of natural gas futures contracts using the Kalman filter and maximum likelihood parameter estimation. This model fits naturally into the general Gaussian framework we have discussed with only minor changes in notation. Set

$$\gamma_t^{'} = (\alpha_t^1, \ldots, \alpha_t^n)^{'}, \quad A_t = diag\left(-k_t^i\right), \quad \text{and } \sigma_t = (\sigma_t^{ij}).$$

Then, with $R = \vec{0} \in \mathbf{R}^n$, $M = \vec{1} \in \mathbf{R}^n$, $N = \vec{0} \in \mathbf{R}^n$, $k = 0$, $h = 0$, and $\ell = 0$ in Assumption 2.2 the market model is equivalent to that of Manoliu and Tompaidis (2002). In particular, for this model since $A_t = diag\left(-k_t^i\right)$ the fundamental matrix is given by

$$\Psi(t) = diag\left(\beta_t^i\right)$$

where β_t^i is the solution of the ODE

$$\frac{d\beta_t^i}{dt} = -k_t^i \beta_t^i, \quad \beta_0^i = 1.$$

Therefore, the Jacobian of the stochastic flow associated with the factors process, X_t, is

$$D_{tu} = diag\big(\beta_u^i (\beta_t^i)^{-1}\big).$$

Note that

$$M^{'} D_{tT} = \left(\beta_T^1 (\beta_t^1)^{-1}, \ldots, \beta_T^n (\beta_t^n)^{-1}\right).$$

Also $D_{tT}^{'} M M^{'} D_{tT}$ is the $n \times n$ matrix whose (i,j)-th entry is

$$[D_{tT}^{'} M M^{'} D_{tT}]_{ij} = \beta_T^i \beta_T^j (\beta_t^i)^{-1} (\beta_t^j)^{-1}.$$

Substituting into equation (11) gives that $L(t,T)$ satisfies the ODE

$$0 = \frac{\partial}{\partial t} L(t,T) + \sum_{i=1}^{n} \beta_T^i (\beta_t^i)^{-1} \alpha_t^i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_T^i \beta_T^j (\beta_t^i)^{-1} (\beta_t^j)^{-1} \sum_{\ell=1}^{n} \sigma_t^{i\ell} \sigma_t^{j\ell} \tag{31}$$

with terminal condition $L(T,T) = 0$. Equation (31) along with the terminal condition can be solved to obtain

$$L(t,T) = \sum_{i=1}^{n} \beta_T^i \int_t^T (\beta_s^i)^{-1} \alpha_s^i ds + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_T^i \beta_T^j \int_t^T (\beta_s^i)^{-1} (\beta_s^j)^{-1} \sum_{\ell=1}^{n} \sigma_s^{i\ell} \sigma_s^{j\ell} ds \tag{32}$$

Hence, by Theorem 3.4, the futures price of the risky asset for the Manoliu and Tompaidis (2002) model is

$$G(t,T) = \exp\left(M^{'} D_{tT} X_t + L(t,T)\right) = \exp\left(\sum_{i=1}^{n} \beta_T^i (\beta_t^i)^{-1} X_t^i + L(t,T)\right) \tag{33}$$

where $L(t,T)$ is given by equation (32). Equations (32) and (33) agree with the futures price given by Manoliu and Tompaidis (2002) which the authors obtained by calculating the conditional expectation (5) using distributional properties of the factors process.

8.2 Models with a stochastic interest rates

Using the general Gaussian factors model presented in Section 2 stochastic interest rates can be easily incorporated into the commodity price models. Doing so results in different futures and forward prices. The most widely studied Gaussian models for stochastic interest rates are the Vašíček (1977) and Hull and White (1994) models. We give one example from the literature. Other models can be studied similarly.

8.2.1 Schwartz (1997): Model 3 (Three-factor model)

We shall derive the futures price and forward price for a three-factor commodity market model considered by Schwartz (1997) which includes stochastic interest rates and stochastic dividends. Schwartz (1997) considers the dynamics

$$dS_t = (r_t - \delta_t)S_t dt + \sigma_1 S_t dB_t^{(1)} \tag{34}$$

$$d\delta_t = \kappa(\alpha - \delta_t)]dt + \sigma_2 dB_t^{(2)} \tag{35}$$

$$dr_t = a(m - r_t)dt + \sigma_3 dB_t^{(3)} \tag{36}$$

with $d\left\langle B^{(1)}, B^{(2)}\right\rangle_t = \rho_{12}dt$, $d\left\langle B^{(1)}, B^{(3)}\right\rangle_t = \rho_{13}dt$, and $d\left\langle B^{(2)}, B^{(3)}\right\rangle_t = \rho_{23}dt$. Schwartz (1997) used the Kalman filter and maximum likelihood parameter estimation to study futures contracts on oil, copper, and gold.

Since our discussions have assumed standard Brownian motion ($\rho_{12} = \rho_{23} = \rho_{13} = 0$) we consider an equivalent formulation of the model. That is, suppose $X_t = (\log S_t, \delta_t, r_t)'$ satisfies equation (1) where $B_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})'$ is a standard Brownian motion taking values in $\mathbf{R}^3$,

$$\gamma_t = \begin{pmatrix} -\frac{1}{2}\sigma_1^2 \\ \kappa\alpha \\ am \end{pmatrix}, \quad A_t = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -\kappa & 0 \\ 0 & 0 & -a \end{pmatrix},$$

and

$$\sigma_t = \begin{pmatrix} \sigma_1 & 0 & 0 \\ \sigma_2\rho_{12} & \sigma_2\sqrt{1-\rho_{12}^2} & 0 \\ \sigma_3\rho_{13} & \frac{\sigma_3(\rho_{23}-\rho_{12}\rho_{13})}{\sqrt{1-\rho_{12}^2}} & \sigma_3\sqrt{1-\frac{\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23}+\rho_{23}^2}{1-\rho_{12}^2}} \end{pmatrix}.$$

We consider X_t as our underlying factor in the context of the general Gaussian model. Then with $R = e_3 \in \mathbf{R}^3$, $M = e_1 \in \mathbf{R}^3$, $N = e_2 \in \mathbf{R}^3$, $k = 0$, $h = 0$, and $l = 0$ in Assumption 2.2 the market model is equivalent to the formulation given by Schwartz (1997), i.e., equations (34)-(36).

The Jacobian of the stochastic flow associated with X_t is

$$D_{tu} = \begin{pmatrix} 1 & -\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right) & \frac{1}{a}\left(1-e^{-a(T-t)}\right) \\ 0 & e^{-\kappa(T-t)} & 0 \\ 0 & 0 & e^{-a(T-t)} \end{pmatrix}.$$

Note that

$$M' D_{tT} = \begin{pmatrix} 1 & -\frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right) & \frac{1}{a}\left(1-e^{-a(T-t)}\right) \end{pmatrix}.$$

Substituting into equation (11) gives that $L(t,T)$ satisfies the ODE

$$0 = \frac{\partial}{\partial t}L(t,T) - [\kappa\alpha + \sigma_1\sigma_2\rho_{12}]\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa} + [am + \sigma_1\sigma_3\rho_{13}]\frac{\left(1-e^{-a(T-t)}\right)}{a} \\ + \frac{1}{2}\frac{\sigma_2^2}{\kappa^2}\left(1-e^{-\kappa(T-t)}\right)^2 + \frac{1}{2}\frac{\sigma_3^2}{a^2}\left(1-e^{-a(T-t)}\right)^2 - \sigma_2\sigma_3\rho_{23}\frac{\left(1-e^{-\kappa(T-t)}\right)\left(1-e^{-a(T-t)}\right)}{\kappa a} \tag{37}$$

with terminal condition $L(T,T) = 0$. Equation (37) along with the terminal condition can be solved to obtain

$$\begin{aligned} L(t,T) &= (\kappa\alpha + \sigma_1\sigma_2\rho_{12})\left[\frac{(1-e^{-\kappa(T-t)}) - \kappa(T-t)}{\kappa^2}\right] \\ &- (ma + \sigma_1\sigma_3\rho_{13})\left[\frac{(1-e^{-a(T-t)}) - a(T-t)}{a^2}\right] \\ &- \sigma_2^2\left[\frac{4(1-e^{-\kappa(T-t)}) - (1-e^{-2\kappa(T-t)}) - 2\kappa(T-t)}{4\kappa^3}\right] \\ &- \sigma_3^2\left[\frac{4(1-e^{-a(T-t)}) - (1-e^{-2a(T-t)}) - 2a(T-t)}{4a^3}\right] \\ &+ \frac{\sigma_2\sigma_3\rho_{23}}{\kappa a}\left[\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa} + \frac{\left(1-e^{-a(T-t)}\right)}{a} - \frac{\left(1-e^{-(\kappa+a)(T-t)}\right)}{(\kappa+a)} - (T-t)\right]. \end{aligned} \tag{38}$$

Hence, by Theorem 3.4, the futures price of the risky asset for the three-factor model of Schwartz (1997) is

$$\begin{aligned} G(t,T) &= \exp\left(M^{'}D_{tT}X_t + L(t,T)\right) \\ &= S_t \exp\left(-\frac{(1-e^{-\kappa(T-t)})}{\kappa}\delta_t + \frac{(1-e^{-a(T-t)})}{a}r_t + L(t,T)\right) \end{aligned} \tag{39}$$

where $L(t,T)$ is given by equation (38). Equations (38) and (39) agree with the futures price which can be obtained from the results given by Schwartz (1997). The results presented by Schwartz (1997) are stated to be verifiable by substitution into the PDE for the futures price.

Substituting into equation (27) gives that $C(t,T)$ satisfies the ODE

$$0 = \frac{\partial}{\partial t}C(t,T) - \alpha\left(1-e^{-\kappa(T-t)}\right) + m\left(1-e^{-a(T-t)}\right) \\ + \frac{1}{\kappa^2}\sigma_2^2\left(1-e^{-\kappa(T-t)}\right)^2 - \frac{1}{a^2}\sigma_3^2\left(1-e^{-a(T-t)}\right)^2 - \frac{\sigma_1\sigma_2\rho_{12}}{\kappa}\left(1-e^{-\kappa(T-t)}\right) \tag{40}$$

with terminal condition $C(T,T) = 0$. Equation (40) along with the terminal condition can be solved to obtain

$$\begin{aligned} C(t,T) &= (\kappa\alpha+\sigma_1\sigma_2\rho)\left[\frac{(1-e^{-\kappa(T-t)})-\kappa(T-t)}{\kappa^2}\right]-m\left[\frac{(1-e^{-a(T-t)})-a(T-t)}{a}\right] \\ &- \sigma_2^2\left[\frac{4(1-e^{-\kappa(T-t)})-(1-e^{-2\kappa(T-t)})-2\kappa(T-t)}{4\kappa^3}\right] \\ &+ \sigma_3^2\left[\frac{4(1-e^{-a(T-t)})-(1-e^{-2a(T-t)})-2a(T-t)}{4a^3}\right]. \end{aligned} \tag{41}$$

Hence, by Proposition 5.1, the forward price of the risky asset for the three-factor model of Schwartz (1997) is

$$F(t,T) = S_t \exp\left(-\frac{(1-e^{-\kappa(T-t)})}{\kappa}\delta_t+\frac{(1-e^{-a(T-t)})}{a}r_t+C(t,T)\right) \tag{42}$$

where $C(t,T)$ is given by equation (41). Equations (41) and (42) agree with the forward price which which can be obtained from the results given by Schwartz (1997) after some algebraic simplification. Specifically, Schwartz (1997) did not present the forward price but considered the present value of a forward commitment to deliver a unit of the asset as the solution to a PDE. The present value when normalized by the bond price corresponding to the Vašíček (1977) interest rate model gives the forward price.

Specialising the results of Section 6 to this model provides the same system of equations that must be solved in order to hedge a long term forward commitment on the commodity using short term futures contracts that are presented by Schwartz (1997).

9 Summary

We have provided a complete characterisation of futures and forward prices based on the method of stochastic flows when the factors of the market are Gaussian. This characterisation generalises the approach of Elliott and van der Hoek (2001) for the bond price in the case of Gaussian factors. Our approach shows why the futures and forward prices are exponential affine functions of the factors. The approach is based on the assumption that the interest rate and dividend yield are affine functions of the factors while the asset price is an exponential affine function of the factors. These assumptions include many of the popular commodity price models that have appeared in the literature. Three such examples illustrated the approach.

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