GENERIC DICHOTOMY FOR HOMOMORPHISMS FOR $E_0^{\mathbb{N}}$

ASSAF SHANI

Abstract. We prove the following dichotomy. Given an analytic equivalence relation E, either $E_0^{\mathbb{N}} \leq_B E$ or else any Borel homomorphism from $E_0^{\mathbb{N}}$ to E is "very far from a reduction", specifically, it factors, on a comeager set, through the projection map $(2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^k$ for some $k \in \mathbb{N}$. As a corollary, we prove that $E_0^{\mathbb{N}}$ is a prime equivalence relation, answering a question on Clemens.

1. Introduction

Let E and F be equivalence relations on Polish spaces X and Y respectively. A map $f\colon X \to Y$ is said to be a **reduction** of E to F if for any $x_1, x_2 \in X$,

$$x_1 \; E \; x_2 \iff f(x_1) \; F \; f(x_2).$$

We say that E is **Borel reducible** to F, denoted $E \leq_B F$ if there is a Borel measurable function which is a reduction of E to F. Borel reducibility is the most central concept in the study of equivalence relations on Polish spaces.

A map $f\colon X \to Y$ is a **homomorphism** from E to F, if for any $x_1, x_2 \in X$,

$$x_1 \; E \; x_2 \implies f(x_1) \; F \; f(x_2).$$

We write $f\colon E \to_B F$ to denote that f is a Borel measurable homomorphism from E to F. To prove an irreducibility result, say that some E is not Borel reducible to F, many times the argument takes the following outline: take an arbitrary Borel homomorphism from E to F, and prove that it cannot be a reduction.

This paper concerns the equivalence relation $E_0^{\mathbb{N}}$, often denoted by E_3, which plays a central role in the theory of Borel equivalence relations. For more background the reader is referred to [HK97, HK01, Kan08, Gao09]. For instance, the dichotomy proved in [HK01] implies that $E_0^{\mathbb{N}}$ is an immediate successor of E_0 with respect to $\leq_B$.

Recall that E_0 on $2^{\mathbb{N}}$ is defined as the eventual equality relation between binary sequences. Equivalently, E_0 is the orbit equivalence relation induced by the action $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2 \curvearrowright 2^{\mathbb{N}}$. For a set X, define E_0^X on $(2^{\mathbb{N}})^X$ as the product equivalence relation. Equivalently, E_0^X is the orbit equivalence relation induced by the point-wise action

Date: July 19, 2024.

2010 *Mathematics Subject Classification.* Primary: 03E15, 54H05.

Key words and phrases. Borel reducibility, Borel homomorphisms, Analytic equivalence relations.

Research partially supported by NSF grant DMS-2246746 and NSERC grant RGPIN-2024-05827.

$(\bigoplus_{n\in\mathbb{N}}\mathbb{Z}_2)^X \curvearrowright (2^{\mathbb{N}})^X$, where $(\bigoplus_{n\in\mathbb{N}}\mathbb{Z}_2)^X$ is the (full support) product of X many copies of $\bigoplus_{n\in\mathbb{N}}\mathbb{Z}_2$.

For $k \in \mathbb{N}$, let $\pi_k\colon (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^k$ be the projection map. Note that $\pi_k\colon E_0^{\mathbb{N}} \to_B E_0^k$ is Borel homomorphism. For each $k \in \mathbb{N}$, E_0^k is Borel bireducible with E_0. So $\{\pi_k : k \in \mathbb{N}\}$ is a family of Borel homomorphisms from $E_0^{\mathbb{N}}$ to a Borel equivalence relation which does not reduce $E_0^{\mathbb{N}}$. We prove that, generically, these are essentially all the Borel homomorphisms from $E_0^{\mathbb{N}}$ to analytic equivalence relations which do not reduce $E_0^{\mathbb{N}}$.

Theorem 1.1. Let E be an analytic equivalence relation. Either

- $E_0^{\mathbb{N}}$ is Borel reducible to E, or
- for any Borel homomorphism $f\colon E_0^{\mathbb{N}} \to_B E$ there is $k \in \mathbb{N}$ so that f factors through π_k on a comeager set, that is, there is a Borel homomorphism $h\colon E_0^k \to_B E$, defined on a comeager set, so that for comeager many $x \in (2^{\mathbb{N}})^{\mathbb{N}}$,
$$h \circ \pi_k(x) \; E \; f(x).$$

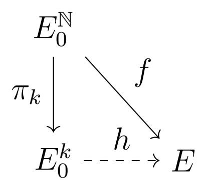

FIGURE 1. $(\forall f\colon E_0^{\mathbb{N}} \to_B E)(\exists k \in \mathbb{N}\, \exists h\colon E_0^k \to_B E)$

1.1. **Primeness for equivalence relations.** From the point of view of the study of Borel reducibility as the study of definable cardinality of quotients of Polish spaces, a Borel homomorphism corresponds to a definable map between two such quotients, and a Borel reduction corresponds to an injective map.

Definition 1.2 (Clemens [Cle22])**.** Let E and F be Borel equivalence relations on Polish spaces X and Y respectively. Say that E **is prime to** F if for any Borel homomorphism $f\colon E \to_B F$, E retains its complexity on a fiber, that is, there is $y \in Y$ so that E is Borel reducible to $E \restriction \{x \in X : f(x) \; F \; y\}$.

Primeness is a strong form of Borel-irreducibility, which holds between many pairs of benchmark equivalence relations (see [Cle22, Theorem 1]).

In the classical context of cardinality, primeness corresponds to a pigeonhole principle: any function $f\colon A \to B$ has a fiber of cardinality $|A|$. This is true if and only if the cardinality $|B|$ is strictly smaller than the cofinality of $|A|$. Recall that the cardinality $|A|$ is regular if it is equal to its cofinality. This is true if and only if for any $|B| < |A|$, any function from A to B has a fiber of size $|A|$.

Following this analogy Clemens defined **regular** equivalence relation as follows. In the context of definable cardinality, when not every two sizes are comparable, the stronger notion of a **prime** equivalence relation is also of interest.

Definition 1.3 (Clemens [Cle22])**.** Let E be a Borel equivalence relation.

- E is **prime** if for any Borel equivalence relation F, either $E \leq_B F$ or E is prime to F.
- E is **regular** if for any Borel equivalence relation F, if $F <_B E$ then E is prime to F.

For example, it follows from the celebrated E_0-dichotomy [HKL90] that E_0 is prime. The $E_0^{\mathbb{N}}$ dichotomy proved by Hjorth and Kechris [HK01] implies that $E_0^{\mathbb{N}}$ is regular. (See [Cle22, Section 4].) Clemens [Cle22, Question 4.2] asked if $E_0^{\mathbb{N}}$ is in fact prime.

Theorem 1.4. For any analytic equivalence relation E, either $E_0^{\mathbb{N}} \leq_B E$ or $E_0^{\mathbb{N}}$ is prime to E. In particular, $E_0^{\mathbb{N}}$ is prime.

Proof. Fix an analytic equivalence relation E which does not Borel reduce $E_0^{\mathbb{N}}$, and fix a Borel homomorphism $f\colon E_0^{\mathbb{N}} \to E$. By Theorem 1.1, there is $k \in \mathbb{N}$ and a comeager set $C \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ so that f factors through π_k on C. We identify $(2^{\mathbb{N}})^{\mathbb{N}}$ with $(2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$. By the Kuratowski-Ulam theorem [Kec95, Theorem 8.41 (iii)] there is $y \in (2^{\mathbb{N}})^k$ so that $C_y = \left\{z \in (2^{\mathbb{N}})^{\mathbb{N}\setminus k} : (y,z) \in C\right\}$ is comeager in $(2^{\mathbb{N}})^{\mathbb{N}\setminus k}$. Note that $\{y\} \times C_y$ is contained in a fiber of f. We conclude the proof by showing that $E_0^{\mathbb{N}} \leq_B E_0^{\mathbb{N}} \restriction \{y\} \times C_y$.

Since $E_0^{\mathbb{N}} \restriction \{y\} \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$ is isomorphic, via a homeomorphism, to $E_0^{\mathbb{N}}$, it suffices to show that $E_0^{\mathbb{N}} \leq_B E_0^{\mathbb{N}} \restriction C$ for any comeager set C in the domain of $E_0^{\mathbb{N}}$. We will give a proof of this fact in Section 3.5 below. Here we mention that it follows from the $E_0^{\mathbb{N}}$ dichotomy of Hjorth and Kechris [HK01], as $E_0^{\mathbb{N}} \restriction C$ is not Borel reducible to E_0, for any comeager set C. □

2. A generic dichotomy for homomorphisms for E_0

In this section we note that the primeness dichotomy of E_0 is also true for all analytic equivalence relations. This follows from the following generic dichotomy for Borel homomorphisms.

Theorem 2.1. Let E be an analytic equivalence relation. Then either

- $E_0 \leq_B E$ or
- any Borel homomorphism $f\colon E_0 \to_B E$ sends a comeager subset of $2^{\mathbb{N}}$ into a single E-class.

Remark 2.2. The theorem follows from the Ulm-invariants dichotomy of Hjorth and Kechrish [HK95], since if E is Ulm classifiable then the second bullet holds. The dichotomy in [HK95] is proved assuming the existence of sharps for reals.

We include a direct proof below (not using any set theoretic assumptions). The dichotomy for homomorphisms is much easier to prove than the E_0-dichotomies. In fact, these ideas can be found as part of the proof of any E_0-dichotomy.

This is one of the motivations to study such generic dichotomies for homomorphisms. While, beyond E_0, there are no more dichotomies quite like the E_0-dichotomy (see [KL97, Theorem 5.1]), a generic analysis of all Borel homomorphisms is one aspect of the E_0-dichotomy which we can be generalized beyond E_0.

Given two equivalence relations F and E on the same domain, say that E **extends** F if $F \subseteq E$. Let F and E be equivalence relations on domains X and Y respectively. Given a function $f\colon X \to Y$, define the pullback of E as the equivalence relation E^* on X defined by $x \mathrel{E^*} y \iff f(x) \mathrel{E} f(y)$. Note that f is a Borel homomorphism from E to F if and only if E^* extends E. Furthermore, f sends a comeager subset of X to a single E class if and only if E^* has a comeager class. Theorem 2.1 is equivalent to the following.

Theorem 2.3. Let E be an analytic equivalence relation on $2^{\mathbb{N}}$ which extends E_0. Then either

- $E_0 \leq_B E$ or
- E has a comeager class.

Fact 2.4. Let $C \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ be comeager. There is a Borel homomorphism $f\colon E_0 \to_B E_0$ so that if $x \not\mathrel{E_0} y$ then $(f(x), f(y)) \in C$.

This fact is commonly used when building reductions from E_0. For example, the map α_1 defined in Section 3.3, using the set $D_{0,1} = C \subseteq (2^{\mathbb{N}})^2$ in that construction, satisfies the conclusion in Fact 2.4.

To prove Theorem 2.3, let E be an analytic equivalence relation which extends E_0 and does not have a comeager class. It follows that $C = 2^{\mathbb{N}} \times 2^{\mathbb{N}} \setminus E$ is comeager. Then f as above is a reduction of E_0 to E.

3. A generic dichotomy for homomorphisms for $E_0^{\mathbb{N}}$

Towards the proof of Theorem 1.1, we begin with some technical lemmas.

3.1. Symmetries of $E_0^{\mathbb{N}}$.

In this section we prove a lemma regarding Vaught transforms for the action $(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^{\mathbb{N}} \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}}$.

Let X and Y be infinite sets and consider the space $(2^X)^Y$ with the product topology. Let $G = (\bigoplus_{x\in X} \mathbb{Z}_2)^Y$, acting on $(2^X)^Y$ in the natural way. We consider G as a topological group with the product topology, where $\bigoplus_{x\in X} \mathbb{Z}_2$ is taken with the discrete topology. Let Γ be the subgroup of G of all finite support sequences. That is, $g \in \Gamma$ if $g(y)$ is the identity for all but finitely many $y \in Y$.

Given a subset $Y_0 \subseteq Y$, identify $(2^X)^Y$ with $(2^X)^{Y_0} \times (2^X)^{Y\setminus Y_0}$. For $a \in (2^X)^{Y_0}$ and $D \subseteq (2^X)^Y$, define $D_a = \left\{b \in (2^X)^{Y\setminus Y_0} : (a,b) \in D\right\} \subseteq (2^X)^{Y\setminus Y_0}$.

Lemma 3.1. Fix a dense open $D \subseteq (2^X)^Y$ and $\zeta \in (2^X)^Y$. Assume that for any finite $Y_0 \subseteq Y$ and for any $\gamma \in \Gamma$,

$$D_{\gamma\cdot\zeta\restriction Y_0} \subseteq (2^X)^{Y\setminus Y_0}$$

is not empty. Then

$$\{g \in G :\ g \cdot \zeta \in D\}$$

is dense open in G.

In particular, if D is assumed to be comeager, then we conclude that $\{g \in G :\ g \cdot \zeta \in D\}$ is comeager.

Proof. First, as the map $G \to (2^X)^Y$, $g \mapsto g{\cdot}\zeta$, is continuous, then $\{g \in G :\ g \cdot \zeta \in D\}$ is open as the pre-image of D. To show that $\{g \in G :\ g \cdot \zeta \in D\}$ is dense, fix a finite set $Y_0 \subseteq Y$ and some $\pi \in (\bigoplus_{x\in X} \mathbb{Z}_2)^{Y_0}$. We need to find some $g \in G$ extending π so that $g \cdot \zeta \in D$.

By assumption, $D_{\pi\cdot\zeta\restriction Y_0} \subseteq (2^X)^{Y\setminus Y_0}$ is a non-empty open set. As all orbits of the action $(\bigoplus_{x\in X} \mathbb{Z}_2)^{Y\setminus Y_0} \curvearrowright (2^X)^{Y\setminus Y_0}$ are dense, we may find some $g \in G$ extending π so that $(g \restriction Y \setminus Y_0) \cdot (\zeta \restriction Y \setminus Y_0) \in D_{\pi\cdot\zeta\restriction Y_0}$, and therefore $g \cdot \zeta \in D$. ☐

3.2. **A reformulation.** For $k \in \mathbb{N}$, we can view E_0^k as an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$, defined by $x\ E_0^k\ y \iff x \restriction k\ E_0^k\ y \restriction k$. That is, by identifying E_0^k with its pullback via the homomorphism $\pi_k \colon E_0^{\mathbb{N}} \to_B E_0^k$. We therefore view

$$E_0 \supseteq E_0^2 \supseteq E_0^3 \supseteq \cdots \supseteq E_0^{\mathbb{N}}$$

as a descending sequence of equivalence relations on $(2^{\mathbb{N}})^{\mathbb{N}}$.

We will prove Theorem 1.1 in the following equivalent form.

Theorem 3.2. Let E be an analytic equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ which extends $E_0^{\mathbb{N}}$. Then either

- $E_0^{\mathbb{N}} \leq_B E$ or
- there is $k \in \mathbb{N}$ so that E extends E_0^k on a comeager set.

Proof of Theorem 1.1 from Theorem 3.2. Let E be an analytic equivalence relation and $f \colon E_0^{\mathbb{N}} \to_B E$ a Borel homomorphism, and assume that $E_0^{\mathbb{N}}$ is not Borel reducible to E. Let E^* on $(2^{\mathbb{N}})^{\mathbb{N}}$ be the pullback of E. Then E^* extends $E_0^{\mathbb{N}}$, and $E_0^{\mathbb{N}}$ is not Borel reducible to E^*. By Theorem 3.2 there is some $k \in \mathbb{N}$ and a comeager $C \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ so that for $x, y \in C$,

$$x \restriction k\ E_0^k\ y \restriction k \implies x\ E^*\ y.$$

Let C_k be the set of all $\xi \in (2^{\mathbb{N}})^k$ so that the fiber $C_\xi \subseteq (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$ is comeager. Fix a Borel map $g \colon C_k \to C$ so that $g(\xi) \restriction k = \xi$. Then g is a homomorphism from E_0^k to E^*, and therefore $h = f \circ g$ is a homomorphism from E_0^k to E, defined on a comeager set. Now for any $x \in C$, $h \circ \pi_k(x)\ E\ f(x)$, as required. ☐

Towards the proof of Theorem 3.2, fix an analytic equivalence relation E on $(2^{\mathbb{N}})^{\mathbb{N}}$ which extends $E_0^{\mathbb{N}}$. Assume that for every $k \in \mathbb{N}$, E does not extend E_0^k on a comeager set. We need to prove that $E_0^{\mathbb{N}}$ is Borel reducible to E.

Lemma 3.3. For every k, there is a comeager set $C_k \subseteq (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k} \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$ so that $(x,y) \not\mathrel{E} (x,z)$ for any $(x,y,z) \in C_k$.

Proof. Otherwise, since the actions $(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^k \curvearrowright (2^{\mathbb{N}})^k$ and $(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^{\mathbb{N}\setminus k} \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$ have dense orbits, we would get a comeager set $C \subseteq (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k} \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$, which we may assume is invariant, so that $(x,y) \mathrel{E} (x,z)$ for all $(x,y,z) \in C$. Now E extends E_0^k on the comeager set of all $(x,y) \in (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$ for which $\left\{z \in (2^{\mathbb{N}})^{\mathbb{N}\setminus k} : (x,y,z) \in C\right\}$ is comeager, contradicting our assumption. □

We identify each C_k as a subset of

$$C_k \subseteq (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{2\times(\mathbb{N}\setminus k)}.$$

For each $k \in \mathbb{N}$, $m \geq k$, and $\gamma \in (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^k \times (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^{2\times(m\setminus k)}$, consider the set

$$\{(\eta_0, \dots, \eta_{k-1}, \xi_k, \zeta_k, \dots, \xi_{m-1}, \zeta_{m-1}) \in (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{2\times(m\setminus k)} :$$
$$(C_k)_{\gamma\cdot(\eta_0,\dots,\eta_{k-1},\xi_k,\zeta_k,\dots,\xi_{m-1},\zeta_{m-1})} \subseteq (2^{\mathbb{N}})^{2\times\mathbb{N}\setminus m} \text{ is comeager}\},$$

a comeager subset of $(2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{2\times(m\setminus k)}$. Let

$$D_{k,m} \subseteq (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{2\times(m\setminus k)}$$

be the intersection of all (countably many) such sets. Write

$$D_{k,m} = \bigcap_{l\in\mathbb{N}} D^l_{k,m},$$

an intersection of dense open sets. We may assume that, for $k < m < l < h$,

$$D^l_{k,m} \times (2^{\mathbb{N}})^{2\times(l\setminus m)} \supseteq D^l_{k,l} \supseteq D^h_{k,l}.$$

We will identify members of $(2^{<\mathbb{N}})^k \times (2^{<\mathbb{N}})^{2\times(m\setminus k)}$ with the basic open subsets of $(2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{2\times(m\setminus k)}$ which they define.

3.3. **A construction.** We want to find a Borel homomorphism $f\colon E_0^{\mathbb{N}} \to_B E_0^{\mathbb{N}}$ which is a reduction from $E_0^{\mathbb{N}}$ to E. Roughly speaking, we hope to construct such f so that, given $x \not\mathrel{E_0^{\mathbb{N}}} y$, then $f(x)$ and $f(y)$ can be written as $(a,b),(a,c) \in (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$ respectively, so that $(a,b,c) \in C_k$. Instead we will ensure that (a,b,c) satisfies the assumptions in Lemma 3.1, with respect to the comeager set C_k.

We will construct f so that $f(x)(n)$ will depend on $x \restriction n{+}1$. We view $x \restriction n \in (2^{\mathbb{N}})^n$ which we identify as $(2^n)^{\mathbb{N}}$. The equivalence relation E_0^n, identified on $(2^n)^{\mathbb{N}}$, is still "eventual equality", between sequences of members of 2^n. The point here is that given $x,y \in (2^{\mathbb{N}})^{\mathbb{N}}$, $k,l \in \mathbb{N}$, for which $x(k)(l) \neq y(k)(l)$, then $(x \restriction n)(k)(l) \neq (y \restriction n)(k)(l)$ for all $n \geq k+1$, where $(x \restriction n)(k)(l) \in 2^n$.

First, we define maps $\alpha_n\colon (2^n)^{\mathbb{N}} \to 2^{\mathbb{N}}$ as follows. We define recursively maps $\alpha_n\colon (2^n)^r \to 2^{<\mathbb{N}}$ which cohere, that is, for $r_1 < r_2$, $t_1 \in (2^n)^{r_1}$, and $t_2 \in (2^n)^{r_2}$ extending t_1, $\alpha_n(t_2)$ extends $\alpha_n(t_1)$. Then $\alpha_n\colon (2^n)^{\mathbb{N}} \to 2^{\mathbb{N}}$ will be defined as the limit.

At stage r, assume that we have defined

$$\alpha_n\colon (2^n)^r \to 2^{<\mathbb{N}},\ n \le r.$$

For each $k < m \le r$, since $D^{r+1}_{k,r+1} \subseteq (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{2\times(r+1\setminus k)}$ is dense open, any member of $(2^{<\mathbb{N}})^k \times (2^{<\mathbb{N}})^{2\times(r+1\setminus k)}$ can be extended to define a subset of $D^{r+1}_{k,r+1}$. By extending finitely many times, we may find

$$a^n_\xi \in 2^{<\mathbb{N}},\ \text{for } \xi \in 2^n, n \le r+1,$$

so that for any $k < r+1$, for any

$$(t_n \in (2^n)^r : n < k), (t_n, s_n \in (2^n)^r : k \le n \le r),\ \text{and any}$$

$$(\xi_n \in 2^n : n < k), (\xi_n \neq \zeta_n \in 2^n : k \le n \le r+1),$$

$$\Big(\big(\alpha_n(t_n) \frown a^n_{\xi_n} : n < k\big), \big(\alpha_n(t_n) \frown a^n_{\xi_n}, \alpha_n(s_n) \frown a^n_{\zeta_n} : k \le n \le r\big), (a^{r+1}_{\xi_{r+1}}, a^{r+1}_{\zeta_{r+1}})\Big) \in D^{r+1}_{k,r+1}.$$

This concludes the definition of the maps α_n, $n \in \mathbb{N}$. Note that for each $n \in \mathbb{N}$, $\alpha_n\colon E_0^n \to_B E_0$ is a Borel homomorphism.

Remark 3.4. We may assume that $D_{0,n} \subseteq (2^{\mathbb{N}})^{2\times n} \setminus E_0^n$, and so α_n is a reduction of E_0^n to E_0.

Claim 3.5. Suppose $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$ are such that $x \restriction k = y \restriction k$ and $x(k) \not E_0\ y(k)$. Then for any $k < m$,

$$((\alpha_n(x \restriction n+1) : n < k), (\alpha_n(x \restriction n+1), \alpha_n(y \restriction n+1) : k \le n < m)) \in D_{k,m}$$

Proof. It suffices to prove membership in $D^r_{k,m}$ for infinitely many $r \in \mathbb{N}$, since $D^r_{k,m}$ is decreasing in r. For each r so that $x(k)(r) \neq y(k)(r)$, we have that $(x \restriction n+1)(r) \neq (y \restriction n+1)(r)$, as members of 2^{n+1}, for all $k \le n$. Therefore at stage r of the construction we ensure that

$$((\alpha_n(x \restriction n+1) : n < k), (\alpha_n(x \restriction n+1), \alpha_n(y \restriction n+1) : k \le n < m)) \in D^r_{k,m},$$

since $D^r_{k,m} \times (2^{\mathbb{N}})^{2\times(r\setminus m)} \supseteq D^r_{k,r}$. □

Finally, define $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ by

$$f(x)(n) = \alpha_n(x \restriction n+1).$$

Then $f\colon E_0^{\mathbb{N}} \to_B E_0^{\mathbb{N}}$ is Borel homomorphism. To conclude the proof of the main theorem, we prove that f is a reduction of $E_0^{\mathbb{N}}$ to E.

3.4. Concluding the proof.

Since E extends $E_0^{\mathbb{N}}$, it remains to prove that if $x \not E_0^{\mathbb{N}}\ y$ then $f(x) \not E\ f(y)$. Since $f\colon E_0^{\mathbb{N}} \to_B E$ is a homomorphism, it suffices to prove the following.

Claim 3.6. Suppose $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, $x \restriction k = y \restriction k$ and $x(k) \not E_0\ y(k)$. Then $f(x) \not E$ $f(y)$.

Proof. By the definition of f, Claim 3.5, and the choice of the sets $D_{k,m}$, we may write $f(x) = (a, b)$ and $f(y) = (a, c)$ where $a \in (2^{\mathbb{N}})^k$, $b, c \in (2^{\mathbb{N}})^{\mathbb{N}\setminus k}$, so that the triplet (a, b, c) satisfies the conditions in Lemma 3.1 with respect to the comeager set $C_k \subseteq (2^{\mathbb{N}})^Y$, where $Y = k \sqcup (\mathbb{N} \setminus k) \sqcup (\mathbb{N} \setminus k)$. It follows from Lemma 3.1 that there is some

$$(g, h_1, h_2) \in (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^k \times (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^{\mathbb{N}\setminus k} \times (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^{\mathbb{N}\setminus k}$$

so that

$$(g \cdot a, h_1 \cdot b, h_2 \cdot c) \in C_k,$$

and so

$$(g \cdot a, h_1 \cdot b) \not\mathrel{E} (g \cdot a, h_2 \cdot b).$$

Since E extends $E_0^{\mathbb{N}}$, it is invariant under the action, and so

$$f(x) = (a, b) \not\mathrel{E} (a, c) = f(y),$$

as required. □

3.5. **Complexity on comeager sets.** In the proof of Theorem 1.4 we used the fact that for any comeager $C \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $E_0^{\mathbb{N}} \leq_B E_0^{\mathbb{N}} \restriction C$. We sketch a proof of this using the construction above.

Let $C \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ be comeager. Similar to the above, define $D_m \subseteq (2^{\mathbb{N}})^m$ as the intersection of all sets of the form

$$\left\{(\eta_0, \dots, \eta_{m-1}) \in (2^{\mathbb{N}})^m : C_{\gamma\cdot(\eta_0,\dots,\eta_{m-1})} \subseteq (2^{\mathbb{N}})^{\mathbb{N}\setminus m} \text{ is comeager}\right\},$$

for $\gamma \in (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^m$. Define $f\colon E_0^{\mathbb{N}} \to_B E_0^{\mathbb{N}}$ as above, so that for any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ and any $m \in \mathbb{N}$, $f(x) \restriction m \in D_m$. It follows from Lemma 3.1 that $\left\{g \in (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^{\mathbb{N}} : g \cdot f(x) \in C\right\}$ is comeager. It follows from [Kec95, Theorem 18.6] that there is a Borel map $h\colon (2^{\mathbb{N}})^{\mathbb{N}} \to (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2)^{\mathbb{N}}$ so that $h(x) \cdot f(x) \in C$ for all $x \in (2^{\mathbb{N}})^{\mathbb{N}}$. We conclude that $x \mapsto h(x) \cdot f(x)$ is a Borel reduction of $E_0^{\mathbb{N}}$ to $E_0^{\mathbb{N}} \restriction C$.

References

[Cle22] John D. Clemens, *Relative primeness and borel partition properties for equivalence relations*, Trans. Amer. Math. Soc. **375** (2022), no. 1, 111–149.

[Gao09] S. Gao, *Invariant descriptive set theory*, CRC Press, 2009.

[HK01] Greg Hjorth and Alexander S. Kechris, *Recent developments in the theory of Borel reducibility*, Fund. Math. **170** (2001), no. 1-2, 21–52. MR1881047

[HK95] Greg Hjorth and Alexander S. Kechris, *Analytic equivalence relations and Ulm-type classifications*, J. Symbolic Logic **60** (1995), no. 4, 1273–1300. MR1367210

[HK97] Greg Hjorth and Alexander S. Kechris, *New dichotomies for borel equivalence relations*, The Bulletin of Symbolic Logic **3** (1997), no. 3, 329–346.

[HKL90] L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928. MR1057041

[Kan08] Vladimir Kanovei, *Borel equivalence relations*, University Lecture Series, vol. 44, American Mathematical Society, Providence, RI, 2008. Structure and classification. MR2441635

[Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597

[KL97] Alexander S. Kechris and Alain Louveau, *The classification of hypersmooth Borel equivalence relations*, J. Amer. Math. Soc. **10** (1997), no. 1, 215–242. MR1396895

Department of Mathematics and Statistics, Concordia University University, Montreal, QC H3G 1M8, Canada

Email address: `assaf.shani@concordia.ca`

URL: `https://sites.google.com/view/assaf-shani/`