A NOTE ON E_1 AND ORBIT EQUIVALENCE RELATIONS

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In this note we present a proof of a theorem due to Kechris and Louveau, stating that E_1 is not Borel reducible to an orbit equivalence relation. The proof is a variation of a similar proof in [LZ∞, Theorem 4.1.1]. The main point of the presentation here is isolating a general property of orbit equivalence relations using the **double brackets model** $V[[x]]_E$ defined in Kanovie-Sabok-Zapletal [KSZ13].

Definition 0.1 (Kanovei-Sabok-Zapletal [KSZ13] Definition 3.10)**.** Let E be an analytic equivalence relation on a Polish space X, and let $x \in X$ be generic over V. Then

$$V[[x]]_E = \bigcap \{V[y] : y \text{ is in some further generic extension, } y \in X \text{ and } xEy\}.$$

That is, a set b is in $V[[x]]_E$ if in any generic extension of $V[x]$ and any y in that extension which is E-equivalent to x, b is in $V[y]$.

Kanovei-Sabok-Zapletal [KSZ13] study canonization properties of equivalence relations with respect to various ideals on their domain. In [Sha∞] the double brackets model was further developed and applied to study Borel reducibility, particularly for equivalence relations which are classifiable by countable structures.

Lemma 0.2 ([Sha∞, Lemma 3.5])**.** Suppose E and F are Borel equivalence relations on X and Y respectively, and $f\colon X \longrightarrow Y$ is a partial reduction of E to F. Suppose $x \in \operatorname{dom} f$ is in some generic extension. Then $V[[x]]_E = V[[f(x)]]_F$.

Lemma 0.3 (Folklore)**.** Suppose $N \subseteq M$ are models of ZF, $P \in N$ is a poset. If x is P-generic over M, then $N[x] \cap M = N$.

Theorem 0.4 (Kechris-Louveau [KL97, Theorem 4.2])**.** Suppose $a\colon G \curvearrowright X$ is a continuous action of a Polish group G on a Polish space X, let E_a be the induced orbit equivalence relation on X. Then, on any comeager subset of $\mathbb{R}^\omega$, E_1 is not Borel reducible to E_a.

We give a proof of this theorem based on the following definition.

Given an equivalence relation E on X and $x \in X$ in some generic extension, let **the intersection number of** x **(relative to** E**)** be the minimal size of a finite set B such that

$$V[[x]]_E = \bigcap_{y \in B} V[y],$$

where B is contained in the E-class of x in some further generic extension. If no such set exists say that the intersection number is infinite. For $E = E_1$ and $x \in \mathbb{R}^\omega$ a Cohen-generic, the intersection number can easily be seen to be infinite (Claim 0.8 below). On the other hand, we show that for any orbit equivalence relation E, *for any* x, the intersection number is always 2:

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Lemma 0.5. Suppose $a\colon G \curvearrowright X$ is a continuous action of a Polish group G on X and $E = E_a$ is the induced equivalence relation. Let $x \in X$ be in some generic extension and $g \in G$ be P_I-generic over $V[x]$ where I is the meager ideal over G. Then for $z = g \cdot x$

$$V[[x]]_E = V[x] \cap V[z].$$

Therefore the intersection number of x is always ≤ 2.

This result is a generalization of the following.

Theorem 0.6 ([LZ∞]). Suppose $a\colon G \curvearrowright X$ is a continuous action of a Polish group G on X with dense and meager orbits. The following are equivalent.

- $a\colon G \curvearrowright X$ is generically turbulent;
- If $x \in X$ is Cohen-generic over V and $g \in G$ is Cohen generic over $V[x]$ then $V[x] \cap V[gx] = V$.

Thus turbulent equivalence relations are characterized by having the minimal possible double brackets model. In general, the double brackets model can be quite complex (see [Sha∞]). In order to consider arbitrary Borel reductions we further want to deal with arbitrary generic elements $x \in X$, and not only Cohen-generics.

Proof of Lemma 0.5. By definition, $V[[x]]_E \subseteq V[x] \cap V[z]$. It remains to show that for any y in a generic extension of $V[x]$, if yEx then $V[x] \cap V[z] \subseteq V[y]$. Suppose first that $y \in V[x][H]$ where H is P-generic over $V[x][g]$ for some $P \in V[x]$. In this case, by mutual genericity, g is generic over $V[x][H]$. Let $\gamma \in G$ be such that $y = \gamma \cdot x$, so $\gamma \in V[x][H]$. Since G acts on itself by homeomorphisms and g is P_I-generic over $V[x][H]$, then so is $g\gamma$. Note that $g\gamma \cdot y = g \cdot x = z$ is in $V[y][g\gamma]$. Apply Lemma 0.3 with $N = V[y]$ and $M = V[x][H]$:

$$V[z] \cap V[x] \subseteq V[y][g\gamma] \cap V[x][H] = V[y],$$

as desired.

For the general case, let $y \in V[x][H]$ where H is some P-generic over $V[x]$, $P \in V[x]$. H may not be generic over $V[x][g]$. It suffice to show that if $a \in V[x]$ and $a \notin V[y]$ then $a \notin V[z]$. Fix an $a \in V[x]$ and some condition p forcing that $xE\dot{y}$ and $\check{a} \notin V[\dot{y}]$. Let H' be P-generic over $V[x][g]$ extending p. By the argument above $V[z] \cap V[x] \subseteq V[\dot{y}][H']$. Now $a \notin V[\dot{y}[H']]$, hence $a \notin V[z]$. □

Lemma 0.7. Suppose $f\colon E \longrightarrow F$ is a (partial) Borel reduction and $x \in \operatorname{dom} f$ in some generic extension. Then the intersection number of x relative to E is equal to the intersection number of $f(x)$ relative to F.

Proof. By Lemma 0.2, $V[[x]]_E = V[[f(x)]]_F$. Assume first that $V[[f(x)]]_F = \bigcap_{y\in B} V[y]$ where B is contained in the E-class of $f(x)$ in some big generic extension $V[G]$. For each $y \in B$, $f(x)Fy$ in $V[G]$. By absoluteness for the statement $(\exists x)f(x)Fy$ there is $x_y \in V[y]$ such that $f(x_y)Fy$, thus x_yEx for each $y \in B$ (since f is a reduction). Now $\bigcap_{y\in B} V[x_y] \subseteq \bigcap_{y\in B} V[y] = V[[f(x)]]_F = V[[x]]_E$. It follows that $\bigcap_{y\in B} V[x_y] = V[[x]]_E$, so the intersection number of x is $\leq |B|$. Similarly, if $V[[x]]_E = \bigcap_{y\in B} V[y]$ where yEx for each $y \in B$, then $V[[f(x)]]_F = \bigcap_{y\in B} V[f(y)]$ and $f(y)Ff(x)$ for each $y \in B$. We conclude that the intersection numbers of x and $f(x)$ are the same. □

Claim 0.8. Let $x \in \mathbb{R}^\omega$ be Cohen-generic. Suppose $x_1, ..., x_n$, in some further generic extension, are all E_1-equivalent to x. Then

$$V[[x]]_{E_1} \subsetneq V[x_1] \cap ... \cap V[x_n].$$

Proof. Fix k large enough such that $x \downharpoonright k = x_i \downharpoonright k$ for $i = 1, ..., n$, where $x \downharpoonright k = \langle 0, 0, ..., 0, x(k), x(k+1), ... \rangle$. Then $x(k) \in V[x_1] \cap ... \cap V[x_n]$. However, $x \downharpoonright (k+1)$ is also E_1-related to x, and therefore $V[[x]]_{E_1} \subseteq V[x \downharpoonright (k+1)]$. Since $x(k)$ is generic over $\langle x_j : j \neq k \rangle$, $x(k) \notin V[x \downharpoonright (k+1)]$ and therefore $x(k) \notin V[[x]]_{E_1}$. □

Proof of Theorem 0.4. Assume for contradiction that there is a reduction f, defined on a comeager subset of $\mathbb{R}^\omega$, reducing E_1 to some equivalence relation E induced by a Polish group action. Let $x \in \mathbb{R}^\omega$ be Cohen generic over V, so x is in the domain of f. By lemmas 0.5 and 0.7 it follows that the intersection number of x is 2, contradicting the claim above. □

Question 0.9 (see [KL97])**.** If E is an analytic equivalence relation, is it true that either E is reducible to an orbit equivalence relation or $E_1 \leq E$?

The proof above suggests the following strategy for a counterexample: suppose we can find an analytic equivalence relation E such that:

(1) For any $x \in \operatorname{dom} E$ in a generic extension the intersection number of x is finite;
(2) There is $x \in \operatorname{dom} E$ in a generic extension whose intersection number is strictly greater than 2.

(1) would imply that $E_1 \not\leq_B E$ and part (2) implies that E is not reducible to an orbit equivalence relation. On the other hand, the following would support a positive answer to Question 0.9.

Question 0.10. If E is an analytic equivalence relation, x in the domain of E in some generic extension, is it true that the intersection number of x must be either 2 or infinite?

References

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