CLASSIFYING INVARIANTS FOR E_1: A TAIL OF A GENERIC REAL

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Abstract. Let E be an analytic equivalence relation on a Polish space. We introduce a framework for studying the possible "reasonable" complete classifications and the complexity of possible classifying invariants for E, such that: (1) the standard results and intuitions regarding classifications by countable structures are preserved in this framework; (2) this framework respects Borel reducibility; (3) this framework allows for a precise study of the possible invariants of certain equivalence relations which are not classifiable by countable structures, such as E_1.

In this framework we show that E_1 can be classified, with classifying invariants which are κ-sequences of E_0-classes where $\kappa = \mathfrak{b}$, and it cannot be classified in such a manner if $\kappa < \mathbf{add}(\mathcal{B})$.

These results depend on analyzing the following sub-model of a Cohen real extension, introduced in [KSZ13] and [LZ20]. Let $\langle c_n : n < \omega\rangle$ be a generic sequence of Cohen reals, and define the tail intersection model

$$M = \bigcap_{n<\omega} V[\langle c_m : m \geq n\rangle].$$

An analysis of reals in M will provide lower bounds for the possible invariants for E_1.

We also extend the characterization of turbulence from [LZ20] in terms of intersection models.

1. Introduction

Let E be an equivalence relation on X. A **complete classification** of E is a map $c\colon X \to I$ satisfying

$$x \mathrel{E} y \iff c(x) = c(y).$$

The members of I are said to be **classifying invariants** for E. Given an equivalence relation E, one would like to classify it using the simplest possible invariants.

What constitutes a "reasonable" classification depends on the field and objects of study. One famous example is the classification of compact orientable surfaces, up

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to homeomorphism, by their genus. The reader is referred to [Hjo00, Preface] for more examples and a thorough discussion.

We view the following requirement as a key property of a "reasonable" classification:

($\star$) given $x \in X$, its invariant $c(x)$ is simple to define, and compute, using x.

This should exclude the following two, not useful, "bad", complete classifications:

- Appealing to the axiom of choice, there is a map choosing one element out of each equivalence class. This choice map is a complete classification.
- The map sending x to its equivalence class $[x]_E = \{y \in X : x \mathrel{E} y\}$, is a complete classification.

In the first example, the map is not definable in any reasonable way. In particular, the invariant of a given object cannot be computed from it. In the second, the issue is that the classifying objects are not simple to describe. For example, given a compact orientable surface, one cannot quite simply *describe all* the surfaces which are homeomorphic to it (this is an enormous collection of objects), but its genus, a natural number, is simple to describe and can be directly computed.

Our focus here is on equivalence relations defined on Polish spaces. This generally captures the situation when studying mathematical structures with some inherent separability assumption (see [Kec95, Kec02]). Let E and F be equivalence relations on Polish spaces X and Y respectively. A map $f\colon X \to Y$ is a **reduction** of E to F if

$$x \mathrel{E} x' \iff f(x) \mathrel{F} f(x'), \text{ for any } x, x' \in X,$$

that is, f reduces the problem of determining E-relation to that of F-relation. If $f\colon X \to Y$ is a reduction from E to F and $c\colon Y \to I$ is a complete classification of F, then the composition $c \circ f\colon X \to I$ is a complete classification of E, using the same set of invariants. So if there is a "sufficiently definable" reduction f from E to F, the invariants necessary to classify E are no more complicated than those necessary to classify F.

E is **Borel reducible** to F, denoted $E \leq_B F$, if there is a Borel map $f\colon X \to Y$ reducing E to F. Borel reducibility is the most common notion for comparing the complexity of equivalence relations on Polish spaces, especially when the equivalence relations are Borel or analytic. In particular, Borel reducibility respects the intuitive idea of complete classification.

An equivalence relation E on a Polish space X is said to be **concretely classifiable** (or **smooth**) if there is a Polish space Y and a Borel map $c\colon X \to Y$ which is a complete classification of E; equivalently, if E is Borel reducible to $=_{\mathbb{R}}$, the equality relation on the reals. That is, E admits real numbers as classifying invariants. (See [Gao09, Section 5.4].)

Recall the equivalence relation E_0 on 2^ω, identifying two binary sequences $x, y \in 2^\omega$ if $\exists n \forall m \geq n(x(m) = y(m))$. E_0 is the canonical obstruction for concrete classifications, according to the dichotomy theorem of Harrington, Kechris, and Louveau [HKL90]: for any Borel equivalence relation E, either E is concretely classifiable or $E_0 \leq_B E$.

Hjorth and Kechris [HK95] extended the notion of concrete classifications to Ulm classifications, where the classifying invariants are in $2^{<\omega_1}$, countable subsets of ω_1. They also extended the Harrington-Kechris-Louveau dichotomy: for any analytic equivalence relation E, either E is Ulm classifiable or $E_0 \leq_B E$ (assuming the existence of sharps for reals).

Given a countable first order language $\mathcal{L}$, let $X_{\mathcal{L}}$ be the Polish space of all countable $\mathcal{L}$-structures on the set $\omega = \{0, 1, 2, ...\}$, and let $\cong_{\mathcal{L}}$ be the isomorphism equivalence relation on $X_{\mathcal{L}}$ (see [Gao09, Chapter 11]). An equivalence relation E on a Polish space X is **classifiable by countable structures** if it is Borel reducible to $\cong_{\mathcal{L}}$ for some $\mathcal{L}$ (see [HK96, Kec02, Hjo00]). In this case, the E-classes can be classified using countable structures, such as countable groups or graphs, up to isomorphism. Moreover, for the isomorphism relation, the Scott analysis provides a complete classification with hereditarily countable sets as classifying invariants (see [Gao09, Chapter 12]).

For an equivalence relation E on X, its **Friedman-Stanley jump**, E^+, is defined on $X^{\mathbb{N}}$ by $x \; E^+ \; y \iff \forall n \exists m(x(n) \; E \; y(m))$ and $\forall n \exists m(y(n) \; E \; x(m))$. For example, the map sending $x \in \mathbb{R}^{\mathbb{N}}$ to $\{x(n) : n \in \mathbb{N}\}$ is a complete classification of $=^+_{\mathbb{R}}$. An equivalence relation E is considered "classifiable by countable sets of reals" if it is Borel reducible to $=^+_{\mathbb{R}}$. Similarly, E is considered "classifiable by hereditarily countable sets of sets of reals" if it is Borel reducible to $(=^+_{\mathbb{R}})^+$, and so on. This approach is taken in [HK96, Introduction (E)].

Given a continuous action $a \colon G \curvearrowright X$ of a Polish group G on a Polish space X, let E_a be the induced **orbit equivalence relation**, $x \; E_a \; y \iff \exists g \in G(g \cdot x = y)$. Hjorth's turbulence property provides a condition on the action so that E_a is not classifiable by countable structures (see [Hjo00, Kec02]). Among orbit equivalence relations, Hjorth [Hjo02] showed that turbulence is the canonical obstruction: for a Borel orbit equivalence relation E_a, either E_a is classifiable by countable structures or there is a turbulent action a' such that $E_{a'} \leq_B E_a$.

There are interesting non-orbit Borel equivalence relations. These are referred to as "the dark side" in [Hjo00, Chapter 8]. The equivalence relation E_1 is defined on $\mathbb{R}^{\mathbb{N}}$ by $x \; E_1 \; y \iff \exists n \forall m > n(x(m) = y(m))$. Kechris and Louveau [KL97], extending [Kec92], proved that E_1 is not Borel reducible to any orbit equivalence relation induced by a Polish group action. (In particular, E_1 is not classifiable by countable structures.)

Some natural classification problems in mathematics lie in "the dark side". Solecki [Sol02] showed that for a hereditarily indecomposable continuum, the equivalence relation partitioning it into its composants is Borel bireducible with E_1. Thomas showed that E_1 is Borel bireducible with the quasi-equality relation on finitely generated groups [Tho08, Theorem 5.6], and that the equivalence relation of virtual isomorphism on finitely generated groups, $\approx_{\mathrm{VI}}$, is Borel reducible to E_1^+ [Tho08, Theorem 6.8].

This paper suggests a framework of **generically absolute classifications**, Definition 1.5, in which equivalence relations such as E_1 and E_1^+ admit "reasonable classifying invariants".

1.1. **Abstract complete classification.** Let E be an analytic equivalence relation on a Polish space X. (More generally, we just need that E has an absolute definition.) We consider the following three conditions as the key properties making a complete classification c "reasonable", ensuring the informal requirement ($\star$) above. A good example to keep in mind is $=_{\mathbb{R}}^+$, with the "reasonable" classifying map $\mathbb{R}^{\mathbb{N}} \to \mathcal{P}_{\aleph_0}(\mathbb{R})$ defined by $x \mapsto \{x(i) : i \in \mathbb{N}\}$, and the "bad" classifying map $x \mapsto [x]_{=^+}$.

(1) [Definability] There is a set theoretic formula ψ and a parameter b such that $c(x) = A \iff \psi(x, A, b)$.
(2) [Locality] If $V \subseteq N \subseteq W$ are ZF models, $x, A \in N$ and $\psi^N(x, A, b)$, then $\psi^W(x, A, b)$.
(3) [Absoluteness] If W is a ZF extension of V then c (defined using ψ) is a complete classification of E in W as well.

In this case, say that c is an **absolute complete classification**. We consider Locality to be the most fundamental necessary property. It tells us that given x one can actually calculate/construct the invariant $c(x)$ in a concrete way. For example, it prohibits the classification $x \mapsto [x]_E$, when E is a not-countable Borel equivalence relation.

These three properties seem to capture the key features of classification by countable structures.

Example 1.1. For an isomorphism relation, the Scott analysis satisfies these properties. This is precisely the point of view taken in [Fri00] and [URL17].

Example 1.2. The Ulm-type classifications of Hjorth and Kechris [HK95] are absolute. Moreover, for the specific case of $I = 2^{<\omega_1}$, and when the parameter b is a member of a Polish space, absolute classifications are essentially the same as Ulm classifications.

Example 1.3. More generally, if $I = \mathrm{HC}$, the space of hereditarily countable sets, and the parameter b is a member of a Polish space, then the notion of absolute classification essentially agrees with the $\mathbf{\Delta}^1_2$-absolute classification of [Hjo00, Chapter 9].

Remark 1.4. Some sort of Absoluteness is necessary for a reasonable notion of classification. For example, in the constructible universe L, any equivalence relation

E admits a definable classification $x \mapsto ([x]_E)^L$ satisfying (1) and (2), but fails (3) whenever new E-classes are added.

To extend this notion beyond classifications by countable structures, we relax the Absoluteness requirement (3), to only some forcing extensions. In the context of equivalence relations defined on Polish spaces, the most natural structure provided is the topology, so it is natural to require absoluteness for Cohen forcing. That is, to require the classification "respects Baire-category arguments".

Definition 1.5. Let E be an analytic equivalence relation on a Polish space X. Say that a complete classification $c\colon X \to I$ is a **generically absolute classification** if

(1) [Definability] There is a set theoretic formula ψ and a parameter b such that $c(x) = A \iff \psi(x, A, b)$.
(2) [Locality] If $V \subseteq W$ are ZF models and $x, A \in V$ with $\psi^V(x, A, b)$, then $\psi^W(x, A, b)$.
(3) (a) [Invariance absoluteness] If W is a ZF extension of V then ψ defines an E-invariant map in W. (That is, the map c defined from ψ is a well defined function and is E-invariant: $x \mathrel{E} y \implies c(x) = c(y)$.)
    (b) [Generic absoluteness] If W is a generic extension of V by a single Cohen real, then ψ defines a complete classification of E in W.

If such a complete classification exists, say that E is **generically classifiable**.

Before going further, let us stress that this notion respects and extends the usual intuitions regarding classification.

Remark 1.6. If $E \leq_B F$ and F admits a generically absolute classification with classifying invariants in I, then so does E (Proposition 7.2).

Example 1.7. The equality relation $=_{\mathbb{R}}$ does not admit a generically absolute classification with ordinal invariants (Proposition 7.5). It then follows from Burgess' theorem [Kec95, 35.21 (ii)] that for an analytic equivalence relation E, the following are equivalent: E is absolutely classifiable with countable ordinal invariants; E is generically classifiable with ordinal invariants; $=_{\mathbb{R}} \not\leq_B E$.

Example 1.8. E_0 does not admit a generically absolute classification with invariants in 2^α for any ordinal α (Proposition 7.7). It then follows from the dichotomy of Hjorth and Kechris [HK95, Theorem 1] that (assuming the existence of sharps for reals) for an analytic equivalence relation E, the following are equivalent: E is absolutely classifiable with invariants in $2^{<\omega_1}$; E is generically classifiable with invariants in 2^α for some ordinal α; $E_0 \not\leq_B E$.

Example 1.9. If E is generically turbulent, then E does not admit a generically absolute classification, with any set of invariants. This follows from the characterization of generic turbulence in [LZ20, Theorem 3.2.2], see Section 4. It then follows from Hjorth's dichotomy [Hjo00, Theorem 9.7] that for an orbit equivalence relation

E_a, the following are equivalent: E_a admits a generically absolute classification; E_a admits an absolute classification with hereditarily countable invariants; there is no turbulent action a' with $E_{a'} \leq_B E_a$.

The following is the main result of this paper. First, E_1 admits "reasonable" invariants. More importantly, we prove lower bounds, suggesting this is a non trivial notion.

Theorem 1.10. (1) There is a generic classification of E_1 with classifying invariants that are sequences of E_0-classes of length $\mathfrak{b}$;
(2) there is no generic classification of E_1 using classifying invariants which are sequences of E_0-classes of length $< \mathbf{add}(\mathcal{B})$;
(3) there is no complete classification of E_1 which is absolute for all forcings.

Here $\mathfrak{b}$ is the bounding number, and $\mathbf{add}(\mathcal{B})$ is the additivity number of the meager ideal. The reader is referred to [Bla10] for their definitions and more background.

Question 1.11. What is the minimal κ such that E_1 admits a generic classification with invariants which are κ-sequences of E_0-classes?

By the above theorem, $\mathbf{add}(\mathcal{B}) \leq \kappa \leq \mathfrak{b}$. We note that the gap between $\mathfrak{b}$ and $\mathbf{add}(\mathcal{B})$ is related to the result of Cichon and Pawlikowsky [CP86], that adding a single Cohen real collapses the bounding number $\mathfrak{b}$ in the generic extension to be $\mathbf{add}(\mathcal{B})^V$, the additivity number as computed in the ground model.

If E can be generically classified with invariants in I, then its Friedman-Stanley jump E^+ can be generically classified, with countable subsets of I as classifying invariants (Proposition 7.4). In particular, E_1^+ is generically classifiable. By the above mentioned result of Thomas, $\approx_{\mathrm{VI}} \leq_B E_1^+$, and therefore $\approx_{\mathrm{VI}}$ is generically classifiable. (It is not classifiable by countable structures, as $E_1 \leq_B \approx_{\mathrm{VI}}$.)

Question 1.12. What are the optimal classifying invariants for a generic classification of $\approx_{\mathrm{VI}}$, the virtual isomorphism relation on finitely generated groups?

We conclude with an ambitious conjecture: when replacing "classification by countable structures" with "generic classification", Hjorth's dichotomy for turbulence [Hjo02] can be extended to non-orbit equivalence relations. We state this conjecture with a generalized definition of turbulence, for non-orbit equivalence relations (Definition 4.5). This definition is motivated by the characterization of turbulence of Larson and Zapletal [LZ20, Theorem 3.2.2], and an extension of this characterization, Theorem 4.3 below.

Conjecture 1.13. Let E be an analytic equivalence relation. Then exactly one of the following hold.

- A turbulent equivalence relation (Definition 4.5) is Borel reducible to E.
- E admits a generically absolute classification.

1.2. **An intersection model.** Let $\mathbb{P}$ be Cohen forcing on $\mathbb{R}^\omega$. Fix $x = \langle x(n) : n < \omega\rangle$ a $\mathbb{P}$-generic over V, and let $x_m = 0^m \frown \langle x(n) : n > m\rangle$, a sequence whose first m values is the number $0 \in \mathbb{R}$, followed by a tail of x. Define the intersection model

$$M = \bigcap_{n<\omega} V[x_n].$$

In a sense, M sees all "generic tail events". Its relationship with the equivalence relation E_1 is immediate: each x_m is E_1-related to x. This model and its relationship with E_1 was introduced in [KSZ13, 4.4] and [LZ20, 4.1]. In both cases, interesting properties of E_1 were deduced from simple properties of this intersection model: essentially that each $x(n)$ is not in M. It follows from [KSZ13, 3.1.1 (i)], or [LZ20, 4.2.9], that M is a model of ZF. However, the precise identity of M remained open, even whether or not it satisfies ZFC.

These questions about M are closely related to questions about possible generic classifications of E_1. For example, we show that the axiom of choice in fact fails in M, and this is closely related to the generic classification of E_1 from Theorem 1.10.

Theorem 1.14. (1) "Choice for $\mathfrak{b}$-sequences of E_0-classes" fails in M;
(2) M satisfies DC;
(3) $M = V(B)$, the minimal ZF-model extending V and containing B, where B is a set of reals.

(Note that $\mathfrak{b}$ as calculated in M is in fact $\mathbf{add}(\mathcal{B})^V$, by the aforemetioned result of Cichon and Pawlikowsky.) We also establish a characterization of reals in M, see Corollary 3.15, which is closely related to the proof of part (2) in Theorem 1.10. Part (3) of Theorem 1.14 can be seen as a weak positive fragment of choice in M, and will be used to conclude part (3) of Theorem 1.10.

Question 1.15. What other forms of choice hold in the intersection model M? Does DC_κ hold for an uncountable $\kappa < \mathbf{add}(\mathcal{B})^V$?

Question 1.16. Suppose $x \in [0,1]^\omega$ is a Random real, with respect to the product measure. Let $N = \bigcap_{n<\omega} V[x_n]$, where x_n is the tail of x past n. What can we say about N? What do reals in N look like? (As in Proposition 3.2, N is a model of DC.)

The rest of the paper is organized as follows. A generically absolute complete classification of E_1 is given in Section 2, establishing part (1) of Theorem 1.10.

In Section 3, the intersection model M is analyzed. Theorem 1.14 is proved there: part (1) in Lemma 3.6; part (2) in Proposition 3.2; part (3) in Theorem 3.19. Using these the rest of Theorem 1.10 is proved: part (2) in Proposition 3.17 and part (3) at the end of the section.

In Section 4, the claim made in Example 1.9, that a turbulent equivalence relation admits no generically absolute classification, is justified (Theorem 4.7). A notion of generic turbulence is defined for arbitrary analytic equivalence relations

(Definition 4.5), which extends the usual notion for orbit equivalence relations (Remark 4.6). In Subsection 4.1 we consider a notion of "intersection numbers" for equivalence relations (Definition 4.9), and use it to give a proof that E_1 is not Borel reducible to an orbit equivalence relation.

In Section 5, another failure of choice in the intersection model M is established: the chromatic number of the shift graph is greater than 2.

Section 7 contains further discussion on why the notion of "generically absolute classification" is a reasonable extension of "classification by countable structures". In particular, some statements made during the introduction are justified.

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2. A complete classification for E_1

Replace $\mathbb{R}$ with the space 2^ω, and consider E_1 as an equivalence relation on $(2^\omega)^\omega$ (see [Gao09, 8.1].) Given $x \in (2^\omega)^\omega$, we consider x as a function $x \colon \omega \times \omega \to \{0,1\}$, a matrix of 0's and 1's, and identify it with a member of the space $2^{\omega\times\omega}$. So for $x, y \in (2^\omega)^\omega$, they are E_1-equivalent if the agree on all but finitely many columns.

We begin by identifying some interesting E_1-invariants. We will reach a complete classification of E_1 by collecting "enough" of these invariants.

Definition 2.1. Given a function $f \colon \omega \to \omega + 1$, define the function $x \restriction f$ whose domain is $\{(n,m) : m < f(n)\}$, the area below the graph of f, by

$$(x \restriction f)(n,m) = x(n,m) \text{ for } m < f(n).$$

Definition 2.2. Given a partial function $y \colon \omega \times \omega \to \{0,1\}$, define $[y]$ to be the set of all functions z with the same domain as y such that z and y differ in at most finitely many places.

Note that $[y]$ can be identified with an E_0-class.

Remark 2.3. Let $f \colon \omega \to \omega$ be some function and suppose that $y, z \colon \omega\times\omega \to \{0,1\}$ are partial functions with domain $\{(n,m) : m < f(n)\}$. Then $[y] = [z]$ if and only if y and z agree on all but finitely many columns.

In conclusion:

Claim 2.4. Given $f \colon \omega \to \omega$, the map $x \mapsto [x \restriction f]$ is E_1-invariant. That is, if x, y are E_1-related, then $[x \restriction f] = [y \restriction f]$.

The reader is refered to either [Bla10, Hal17, Jec03] for the eventual domination order $<^*$ on ω^ω, the space of functions from ω to ω. Fix $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ an $<^*$-increasing and unbounded sequence of functions in ω^ω such that each f_α is increasing. For $x \in (2^\omega)^\omega$, define

$$c(x) = \langle [x \restriction f_\alpha] : \alpha < \mathfrak{b}\rangle .$$

Claim 2.5. c is a complete classification of E_1.

Proof. The map is invariant, as each coordinate is invariant. It remains to show that if $c(x) = c(y)$ then $x \; E_1 \; y$. In fact it suffices to assume that for unboundedly many $\alpha < \mathfrak{b}$, $[x \restriction f_\alpha] = [y \restriction f_\alpha]$. If so, we many find an unbounded subset $X \subseteq \mathfrak{b}$ and some $k < \omega$ such that for any $\alpha \in X$, any $n > k$ and $m < f_\alpha(n)$, $(x \restriction f_\alpha)(n,m) = (y \restriction f_\alpha)(n,m)$. That is,

$$(\star) \text{ for any } \alpha \in X \text{, any } n > k \text{ and } m < f_\alpha(n),\ x(n,m) = y(n,m).$$

As $\langle f_\alpha : \alpha \in X \rangle$ is $<^*$-unbounded, there must be some $k' \geq k$ such that $\{f_\alpha(k') : \alpha \in X\}$ is unbounded in ω. Since each f_α is increasing, it follows that for any $n \geq k'$, $\{f_\alpha(n) : \alpha \in X\}$ is unbounded in ω. Finally, given any $n \geq k'$ and any m, there is some $\alpha \in X$ such that $f_\alpha(n) > m$, and so, by $(\star)$, $x(n,m) = y(n,m)$. That is, x and y agree on all columns past k', and so are E_1-equivalent. □

Note that the map $x \mapsto c(x)$ is definable using the parameter $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$. Furthermore the calculation of $c(x)$ is absolute (local), that is, clause (2) of Definition 1.5 holds. The map c is E_1-invariant in any extension, so clause (3)(a) of Definition 1.5 is satisfied. Finally, in the proof above we only required that $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ is unbounded in the eventual domination order, to conclude that c is a complete classification. Since Cohen forcing does not add dominating reals (see [Hal17, Lemma 22.2]), $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ remains unbounded in a Cohen-real extension, so the map c remains a complete classification, thus (3)(b) of Definition 1.5 is satisfied. This concludes part (1) of Theorem 1.10.

3. The intersection model for E_1

Let $\mathbb{P}$ be Cohen forcing on $2^{\omega\times\omega}$. That is, the conditions of $\mathbb{P}$ are finite partial functions from $\omega \times \omega$ to $\{0,1\}$, ordered by reverse extension. A generic filter for $\mathbb{P}$ can be identified with a generic real $x \in 2^{\omega\times\omega}$. Note that $\mathbb{P}$ is forcing equivalent to Cohen forcing for adding one real in 2^ω.

Fix $x \in 2^{\omega\times\omega}$ a $\mathbb{P}$-generic over V. Define the step functions $\delta_n \colon \omega \to \omega + 1$ by $\delta_n(m) = 0$ if $m < n$ and $\delta_n(m) = \omega$ if $m \geq n$, and let $x_n = x \restriction \delta_n$. That is, x_n is a tail of columns of x. Let $\mathbb{P}^n$ be the poset of all finite partial functions from $(\omega \setminus n) \times \omega$ to $\{0,1\}$. So $\mathbb{P} = \mathbb{P}^0$, and each x_n may be identified as a $\mathbb{P}^n$-generic.

We are interested in the following intersection model, which we fix for the remainder of the paper.

Definition 3.1. $M = \bigcap_{n<\omega} V[x_n]$.

This is the same model as described in Section 1.2 (after identifying $\mathbb{R}^\omega$ with $2^{\omega\times\omega}$). M satisfies ZF by [LZ20, 4.2.9]. Moreover, M satisfies DC (part (2) of Theorem 1.14).

Proposition 3.2. DC holds in M.

Proof. Suppose R is a relation on X, in M, such that $\forall x \exists y (x\ R\ y)$ holds. So X is in $V[x_n]$ for each n. For each n we may choose a well ordering of X in $V[x_n]$, definably in X and x_n, as follows: choose a minimal $\mathbb{P}^n$-name τ (according to some well ordering in V) so that $\tau[x_n]$ is a well ordering of X. This definition can be carried uniformly in any model $V[x_m]$ with $m \leq n$.

Define a sequence $\langle z_n : n < \omega \rangle$ recursively so that z_{n+1} is the minimal, according to the above chosen well ordering of X in $V[x_{n+1}]$, with $z_n\ R\ z_{n+1}$. For each m, the sequence $\langle z_n : n \geq m \rangle$ is definable in $V[x_m]$. Each z_i is in M, and therefore in $V[x_m]$, and so the sequence $\langle z_n : n < \omega \rangle$ is in $V[x_m]$, for each m. It follows that $\langle z_n : n < \omega \rangle \in M$, as required. □

Remark 3.3. More generally, DC holds in the intersection model arising from any countable coherent sequence: see [LZ20], "Supplemental materials", "Updated Theorem 4.2.9".

Next we prove part (1) of Theorem 1.14, that the axiom of choice fails in M.

Remark 3.4. Thinking of x as a matrix of 0's and 1's, one can see that each row of x, as an element of 2^ω, is in M. In fact, given $f \in \omega^\omega$ in V, $x \restriction f$ is in M. More generally, given $f \in \omega^\omega$ in M, $x \restriction f$ is in M. We will see below that any real in M is definable from one of those.

Fix a $<^*$-increasing and unbounded sequence $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ in V. Since Cohen forcing does not add dominating functions, and $V \subseteq M \subseteq V[x]$, the sequence $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ is $<^*$-unbounded in M as well.

By the remark above, each $x \restriction f_\alpha$ is in M, and so is $A_\alpha = [x \restriction f_\alpha]$, the set of all finite changes of $x \restriction f_\alpha$. Define $A = \langle A_\alpha : \alpha < \mathfrak{b} \rangle$.

Claim 3.5. $A \in M$.

Proof. Let y_n be defined as x from the n'th column onward, and be all zeros in the first n columns. Then $y_n \in V[x_n]$ and $A_\alpha = [y_n \restriction f_\alpha]$. We conclude that A is in $V[x_n]$, for each n, and therefore A is in M. □

Lemma 3.6. There is no choice function for A in M.

Proof. Assume towards a contradiction that $\langle a_\alpha : \alpha < \mathfrak{b} \rangle \in M$ where $a_\alpha \in A_\alpha$. Then $[a_\alpha] = [x \restriction f_\alpha]$ for any $\alpha < \mathfrak{b}$. Working in $V[x]$, there is an unbounded $X \subseteq \mathfrak{b}$ and $k < \omega$ such that for all $n > k$, $\alpha \in X$ and $m < f_\alpha(n)$, $x(n, m) = a_\alpha(n, m)$.

We can now find in the ground model an unbounded $Y \subseteq \mathfrak{b}$ and $\{p_\alpha : \alpha \in Y\}$ such that $p_\alpha \Vdash \alpha \in \dot{X}$. By thinning out Y, we may assume that there is a fixed $p \in \mathbb{P}$ such that $p_\alpha = p$ for all $\alpha \in Y$.

As in the proof of Claim 2.5, fix $n > k$ such that $\{f_\alpha(n) : \alpha \in Y\}$ is unbounded, and $\{(n, m) : m < \omega\}$ is disjoint from the domain of p. Let $\dot{a}^l$ be a $\mathbb{P}^l$-name such that $\dot{a}^l[x_l] = \langle a_\alpha : \alpha < \mathfrak{b} \rangle$, for $l \in \{0, n+1\}$. Take $q \in \mathbb{P}$ extending p and forcing that $\dot{a}^0[\dot{x}] = \dot{a}^{n+1}[\dot{x}_{n+1}]$. Fix some m such that (n, m) is not in the domain of q, and $\alpha \in Y$ such that $f_\alpha(n) > m$. Let r be a condition in $\mathbb{P}^{n+1}$ deciding $\dot{a}^{n+1}_\alpha(n, m)$

which is compatible with q. Now the condition $r \cup q$ decides $x(n,m) = a_\alpha(n,m)$, yet (n,m) is not in the domain of $r \cup q$, a contradiction. $\square$

We now go on to prove parts (2) and (3) of Theorem 1.10.

Definition 3.7. Given a sequence $\langle p_n : n < \omega \rangle$ such that p_n is a condition in $\mathbb{P}^n$, the **diagonal** of the sequence $\langle p_n : n < \omega \rangle$ is the function $d \in \omega^\omega$ defined by $d(n) =$ the maximal l such that $p_m(n,l)$ is defined, for some $m \leq n$.

Claim 3.8. If $\langle p_n : n < \omega \rangle$ is in M, then the diagonal d is also in M.

We write $y \sqsubset 2^{\omega\times\omega}$ to mean "y is a partial function $\omega \times \omega \to \{0,1\}$". Given $y, z \sqsubset 2^{\omega\times\omega}$, write $z \sqsubseteq y$ if y extends z.

Definition 3.9. For $y \sqsubset 2^{\omega\times\omega}$ say that y forces ψ, $y \Vdash \psi$, if there is some $p \sqsubseteq y$ such that $p \Vdash \psi$.

The following two are the key lemmas towards understanding the reals, and other sets in M.

Lemma 3.10. Suppose $Z \in M$, $Z = \dot{Z}[x]$. Then there is a sequence $\left\langle \tau_n^{\dot{Z}} : n < \omega \right\rangle \in M$, $\tau_0^{\dot{Z}} = \dot{Z}$, where $\tau_n^{\dot{Z}}$ is a $\mathbb{P}^n$-name such that $\tau_n^{\dot{Z}}[x_n] = Z$ and there is a function $d^{\dot{Z}} \in \omega^\omega$ in M such that for each n, $x \upharpoonright d^{\dot{Z}}$ forces that $\dot{Z}[\dot{x}] = \tau_n^{\dot{Z}}[\dot{x}_n]$.

Proof. Working in $V[x]$, define the sequence $\langle \tau_n : n < \omega \rangle$ such that τ_n is the least $\mathbb{P}^n$-name, according to a fixed well order in V, with $\tau_n[x_n] = Z$. Note that $\langle \tau_n : n \geq m \rangle$ is definable in $V[x_m]$ using Z and x_m. It follows that $\langle \tau_n : n < \omega \rangle$ is in $V[x_m]$, for each m, and therefore $\langle \tau_n : n < \omega \rangle$ is in M. Without loss of generality $\tau_0 = \dot{Z}$.

For each n, let q_n be minimal condition in $\mathbb{P}^n$ (according to some fixed well order) such that $q_n \sqsubset x_n$ and $q_n \Vdash \tau_n[\dot{x}_n] = \tau_{n+1}[\dot{x}_{n+1}]$. Note that $\langle q_n : n \geq m \rangle$ is definable in $V[x_m]$ from x_m and Z. It follows that $\langle q_n : n < \omega \rangle$ is in M. Let d be the diagonal of $\langle q_n : n < \omega \rangle$. For any n, large enough initial segments of $x \upharpoonright d$ force that τ_0 and τ_n agree. So $d = d^{\dot{Z}}$ and $\tau_n^{\dot{Z}} = \tau_n$ are as desired. $\square$

Lemma 3.11. Let Z be a set in M, and fix a name $\dot{Z}$ for Z. Given $z = \dot{z}[x] \in M$, there is a function $f^{\dot{z},\dot{Z}}$ in M such that for any g, if $g \geq^* f^{\dot{z},\dot{Z}}$ and $g \geq d^{\dot{Z}}, d^{\dot{z}}$ then

$$z \in Z \implies x \upharpoonright g \Vdash \dot{z} \in \dot{Z},\ z \notin Z \implies x \upharpoonright g \Vdash \dot{z} \notin \dot{Z}.$$

By $g \geq d$ we mean that $g(n) \geq d(n)$ for each n.

Proof. Fix $\left\langle \tau_n^{\dot{Z}} : n < \omega \right\rangle$, $d^{\dot{Z}}$ and $\langle \tau_n^{\dot{z}} : n < \omega \rangle$, $d^{\dot{z}}$ in M as above. Let $\langle p_n : n < \omega \rangle$ be in M such that $p_n \in \mathbb{P}^n$ is minimal deciding $\tau_n^{\dot{z}} \in \tau_n^{\dot{Z}}$, $p_n \sqsubset x_n$. Let $f^{\dot{z},\dot{Z}} \in M$ be the diagonal of $\langle p_n : n < \omega \rangle$. Given g such that $g \geq^* f^{\dot{z},\dot{Z}}$ and $g \geq d^{\dot{Z}}, d^{\dot{z}}$, fix m such that g is above $f^{\dot{z},\dot{Z}}$ beyond m. Then for any $n \geq m$, $x \upharpoonright g$ decides correctly $\tau_n^{\dot{z}} \in \tau_n^{\dot{Z}}$. In addition, $x \upharpoonright g$ extends $x \upharpoonright d^{\dot{Z}}$ and $x \upharpoonright d^{\dot{z}}$, and so forces that $\tau_0^{\dot{Z}} = \tau_n^{\dot{Z}}$ and $\tau_0^{\dot{z}} = \tau_n^{\dot{z}}$. Therefore $x \upharpoonright g$ decides correctly the statement $\dot{z} \in \dot{Z}$. $\square$

The lemma will be applied first to sets Z which are subsets of V. In this case we only consider $z \in V$, and use their canonical names $\check{z}$.

Definition 3.12. Suppose $Z \subseteq U$ is in M, with $U \in V$, and $y \sqsubseteq x$. Say that **y defines Z via $\dot{\mathbf{Z}}$** if for any $v \in U$, if $v \in Z$ then $y \Vdash v \in \dot{Z}$ and if $v \notin Z$ then $y \Vdash v \notin \dot{Z}$. We often suppress $\dot{Z}$ if it is clear from context, or irrelevant.

Remark 3.13. If $y \sqsubseteq x$ defines Z via $\dot{Z}$, and x' is a $\mathbb{P}$-generic extending y, then $\dot{Z}[x'] = Z$.

From Lemma 3.11 we conclude:

Corollary 3.14. Suppose $Z \subseteq \kappa$ is in M. Let $f \in \omega^\omega$ be such that

- f dominates $\left\{ f^{\check{\alpha}, \dot{Z}} : \alpha < \kappa \right\}$;
- f is pointwise above $d^{\dot{Z}}$.

Then $x \restriction f$ defines Z.

Corollary 3.15. Suppose $Z \subseteq \omega$ in M is a real. Then there is $f \in M \cap \omega^\omega$ such that $x \restriction f$ defines Z.

Proof. Using DC, we may find in M a sequence $f^{\check{n}, \dot{Z}}$, each as in Lemma 3.11. Let $f \in M$ diagonalize this sequence, with $f \geq f^{\dot{Z}}$. □

So reals in M are determined by $x \restriction f$ for some $f \in M$. Therefore by changing x above f, the realization of a particular real in M in unchanged. We will want to change x on all columns, to another $\mathbb{P}$-generic.

Lemma 3.16. Suppose $x \in 2^{\omega \times \omega}$ is $\mathbb{P}$-generic over V, $f : \omega \to \omega$ is in $V[x]$. Then there is some $x' \in 2^{\omega \times \omega}$ such that

- x' is $\mathbb{P}$-generic over V;
- x and x' agree below f;
- $x \not{E}_1 x'$.

Moreover, both x, x' live in a Cohen-real extension of V.

Proof. [1] Let $\dot{f}$ be a name such that $\dot{f}[x] = f$. Define a poset $\mathbb{Q}$ as follows. The conditions are pairs (p, q) where $p \in \mathbb{P}$, $q \colon \operatorname{dom} q \to \omega$ is a finite function such that $p \Vdash q(l) \geq \dot{f}(l)$ for any $l \in \operatorname{dom} q$. Say that (p, q) extends (p', q') if $p \leq p'$, q extends q' as a function, and for any $i \in \operatorname{dom} q \setminus \operatorname{dom} q'$, $(i, q(i))$ is above the domain of p'. (That is, if $(i, j) \in \operatorname{dom} p'$ then $j < q(i)$.) Note that $\mathbb{Q}$ is countable and therefore forcing equivalent to Cohen-real forcing.

[1] Another way of finding x' as in the lemma, due to Paul Larson, is as follows. Let $y \in 2^{\omega \times \omega}$ be $\mathbb{P}$-generic over $V[x]$, and define x' to agree with x below the graph of f, and to agree with y otherwise. It can be shown that x' is $\mathbb{P}$-generic over V.

Define $\pi\colon \mathbb{Q} \to \mathbb{P}$ as follows. $\pi(p,q) = r$ where $\operatorname{dom} r = \operatorname{dom} p$ and

$$r(i,j) = \begin{cases} 1 - p(i,j) & i \in \operatorname{dom} q \wedge j = q(i); \\ p(i,j) & \text{otherwise.} \end{cases}$$

That is, r flips the values of p along the graph of q, and is equal to p elsewhere. It follows from the definitions that π is a forcing projection (see [Cum10, 5.2]).

A $\mathbb{Q}$-generic is naturally associated with a pair (y,g) such that $y \in 2^{\omega\times\omega}$ is $\mathbb{P}$-generic and $g\colon \omega \to \omega$ is a function which is pointwise above $\dot{f}[y]$. Note also that the map $\pi_0\colon \mathbb{Q} \to \mathbb{P}$ defined by $(p,q) \mapsto p$ is a projection, so we may find g so that (x,g) is $\mathbb{Q}$-generic over V. Let $x' = \pi(x,g)$, a $\mathbb{P}$-generic over V. Then x and x' agree everywhere but on the graph of g. Since g is pointwise above $f = \dot{f}[x]$, x and x' satisfy the desired properties. □

We now prove part (2) of Theorem 1.10.

Proposition 3.17. There is no generic classification of E_1 using sequences of E_0-classes of length $< \mathbf{add}(\mathcal{B})$.

Let $E = E_0$ for this proof. Note that the proof works without change for any countable Borel equivalence relation E.

Proof. Assume for a contradiction that there is such a classification $c\colon (2^\omega)^\omega \to I$, defined by some $\psi(x,B,e)$, where e is a parameter and I is the set of sequences of E-classes of length $< \mathbf{add}(\mathcal{B})$. Let $x \in (2^\omega)^\omega$ be Cohen generic and $c(x) = \langle B_\alpha : \alpha < \kappa\rangle = B$ its invariant, where $\kappa < \mathbf{add}(\mathcal{B})^V$ and each B_α is an E-class. Then B is in M. Indeed, let y_n be defined as x on the n'th column forward, and be all 0's on the first n columns. Then $y_n \; E_1 \; x$ and $y_n \in V[x_n]$. Now $c(x) = c(y_n) \in V[x_n]$ for each n, so B is in the intersection model M.

Working in $V[x]$, fix $\langle b_\alpha : \alpha < \kappa\rangle$ such that $b_\alpha \in B_\alpha$. Each b_α is a member of a Polish space, and therefore can be identified as a subset of ω. Since $b_\alpha \in M$, by Corollary 3.15, there is a function f_α such that for any h, if h is above f_α then $x \restriction h$ defines b_α.

By a result of Cichon and Pawlikowsky [CP86], the bounding number $\mathfrak{b}$ in a Cohen-real extension $V[x]$ is $\mathbf{add}(\mathcal{B})^V$, the additivity of the meager ideal as calculated in V. Since $\kappa < \mathbf{add}(\mathcal{B})^V$, $\kappa < \mathfrak{b}^{V[x]}$, so in $V[x]$ we may find $g \in \omega^\omega$ which dominates $\{f_\alpha : \alpha < \kappa\}$. Fix a condition $p \in \mathbb{P}$ forcing that

$$\psi(\dot{x}, \dot{B}, \check{e}) \wedge \forall \alpha < \kappa (B_\alpha = [\dot{b}_\alpha]_E).$$

By Lemma 3.16 we may find x', in a further generic extension, such that x' is $\mathbb{P}$-generic over V, x' and x agree below g, yet x and x' are not E_1-equivalent. By a finite change of x', we may assume that x' extends p.

Claim 3.18. For each $\alpha < \kappa$, $\dot{B}_\alpha[x'] = \dot{B}_\alpha[x] = B_\alpha$.

Proof. Fix $\alpha < \kappa$. Define $y \in 2^{\omega\times\omega}$ by changing finitely many coordinates of x', so that y still extends p and agrees with x below g, and moreover y and x agree below f_α. (This is possible since g eventually dominates f_α.) Then y is $\mathbb{P}$-generic over V and $\dot{b}_\alpha[y] = \dot{b}_\alpha[x] = b_\alpha$, by the choice of f_α. Furthermore, y and x' are E_1-related. Since the map c is a generic classification, $\dot{B}[y] = \dot{B}[x']$. In particular, $\dot{B}_\alpha[x'] = \dot{B}_\alpha[y] = [b_\alpha]_E = B_\alpha$. □

It follows that $c(x) = B = c(x')$. In conclusion, we found x and x' in a Cohen-real generic extension which are not E_1-equivalent, yet they are assigned the same invariant by c. Thus c is not a complete classification in this extension, so c is not a generically absolute classification. □

The following establishes part (3) of Theorem 1.14. Recall that, working in some generic extension of V, given a set A, $V(A)$ is the minimal transitive extension of V which satisfies ZF. Showing that a model M is of the form $V(A)$ for a set A, where V is a model of ZFC, can be seen as some weak fragment of choice for M. If A is a set of ordinals, then $V(A)$ satisfies the axiom of choice. The next level of complexity, in which choice can fail, is when A is a set of reals (a set of subsets of ω).

Theorem 3.19. Let $\mathcal{F} = (\omega^\omega)^M$ and $D = \{[x \restriction f] : f \in \mathcal{F}\}$. Then

$$M = V(D).$$

Note that $V(D) = V(\bigcup D)$ where $\bigcup D$ is a set of reals.

Proof. As before, since the map $x \mapsto [x \restriction f]$ is E_1-invariant for each f, we see that $D \in M$, and therefore $V(D) \subseteq M$. It remains to show that $M \subseteq V(D)$. Note that each $f \in \mathcal{F}$ is encoded by $[x \restriction f]$, as the domain of all members in $[x \restriction f]$, and so $\mathcal{F} \in V(D)$ as well.

Claim 3.20. There is a poset $\mathbb{Q} \in V(D)$ and a filter $G \subseteq \mathbb{Q}$ which is $\mathbb{Q}$-generic over $V[x]$, such that in $V(D)[G]$ there a $y \in (2^\omega)^\omega$, Cohen generic over V, and there is a function $g \in V[x][G]$ dominating all functions in $\mathcal{F}$ so that $y \restriction g = x \restriction g$.

Proof. Consider the poset $\mathbb{Q} \in V(D)$ to enumerate $\mathcal{F}$, D, and $(\mathcal{P}(\mathbb{P}))^V$. Fix $G \subseteq \mathbb{Q}$ generic over $V[x]$. In the extension $V(D)[G]$ let $\langle E_n : n < \omega \rangle$ be an enumeration of the dense open subsets of $\mathbb{P}$ in V. Next, find in $V(D)[G]$ sequences $\langle y_n : n < \omega \rangle$ and $\langle h_n : n < \omega \rangle$ such that

- $h_n \in \mathcal{F}$, $y_n \in [x \restriction h_n]$;
- $\langle h_n : n < \omega \rangle$ is $<$-increasing and $<^\star$-dominating $\mathcal{F}$;
- y_{n+1} extends y_n;
- y_n extends a condition in E_n.

This can be done as follows. Take $h_{n+1} \in \mathcal{F}$ which is strictly above h_n, such that h_{n+1} dominates the $n + 1$'th member of $\mathcal{F}$ (under some fixed enumeration in $V(D)[G]$), and such that there exists some member of $[x \restriction h_{n+1}]$ extending a condition in E_{n+1}. To find y_{n+1}, take any member of $[x \restriction h_{n+1}]$ (the minimal according to the enumeration given by the collapse), and change it in finitely many positions so that it extends y_n.

Define $y = \bigcup_n y_n$, so $y \in 2^{\omega\times\omega}$ is $\mathbb{P}$-generic over V. Work now in $V[x][G]$. For each n, there is some m_n such that y_n and $x \restriction h_n$ agree past m_n. We may assume that $m_n < m_{n+1}$. Define $g \in \omega^\omega$ by $g(k) = h_n(k)$ if n is such that $m_n < k \leq m_{n+1}$, and $g(k) = 0$ if $k \leq m_0$. Then $x \restriction g = y \restriction g$. Given $f \in \mathcal{F}$, there are n and m such that for $k > m$, $f(k) < h_n(k) < g(k)$. We therefore established that g dominates each $f \in \mathcal{F}$, which concludes the proof of the claim. □

Lemma 3.21. Suppose $y \in 2^{\omega\times\omega}$ is $\mathbb{P}$-generic over V, in some further generic extension, and y extends $x \restriction g$ for some g which dominates $\omega^\omega \cap M$. Them $M \subseteq V[y]$.

Proof. Note that for any $Z \in M$ we can find some y', a finite alteration of y, so that y' and x agree above $d^{\dot{Z}}$ and above g. This is because g dominates $d^{\dot{Z}}$, so y and x agree eventually above $d^{\dot{Z}}$.

Claim 3.22. For any Z in M, if y' is a finite alteration of y such that y' agrees with x above $d^{\dot{Z}}$ and g, then $\dot{Z}[y'] = Z$.

In this case, $V[y'] = V[y]$, so we conclude that $Z \in V[y]$ for any $Z \in M$, as the lemma requires. We now prove the claim, by induction on the rank of the set Z.

Fix Z in M, and let y' be as above. Fix some $z \in M$ of lower rank, and assume that $z \in Z$. We need to show that $z \in \dot{Z}[y']$. (The argument for when $z \notin Z$ is similar.) Let $\dot{z}$ be a name such that $\dot{z}[x] = z$, and y'' a finite alteration of y' which agrees with x also above $d^{\dot{z}}$ as well as $d^{\dot{Z}}$.

Since y' and y'' differ by only finitely many coordinates, and both force that $\dot{Z} = \tau_n^{\dot{Z}}$ for all n, then $\dot{Z}[y'] = \dot{Z}[y'']$. y'' agrees with x above g and $d^{\dot{z}}$, so $\dot{z}[y''] = z$ by the inductive hypothesis. Furthermore, since g dominates $d^{\dot{z},\dot{Z}}$, then $\dot{z}[y''] \in \dot{Z}[y'']$ by Lemma 3.11. It follows that $z \in \dot{Z}[y']$, as desired. □

To conclude the theorem we will use the following well known lemma.

Lemma 3.23 (Folklore)**.** Suppose $N \subseteq M$ are transitive models of ZF, $\mathbb{P} \in N$ is a poset. If G is $\mathbb{P}$-generic over M, then $N[G] \cap M = N$.

Proof sketch of the lemma. Suppose $\tau \in N$ is a $\mathbb{P}$-name and $\tau[G] = X \in M$. Using an inductive argument on the rank, it suffices to assume that $X \subseteq N$, and conclude that $X \in N$. Since G is generic over M, we have $\tau[G] = X$ in $M[G]$ as well. Working in M, we may find $p \in \mathbb{P}$ forcing that $\tau = \check{X}$. Now the set X can be defined in N as the set of all x for which $p \Vdash \check{x} \in \tau$. (The point is that for any $q \in \mathbb{P}$ and $x \in N$, the statement $q \Vdash \check{x} \in \tau$ is absolute between M and N.) □

Finally, take G and $y \in V(D)[G]$ as above. Since G is generic over $V[x]$, it is generic over M, and therefore $V(D)[G] \cap M \subseteq V(D)$ by Lemma 3.23. Since $M \subseteq V[y] \subseteq V(D)[G]$, it follows that $M \subseteq V(D)$, concluding the proof of the theorem. □

We now prove part (3) of our main Theorem 1.10, that there is no complete classification for E_1 which is absolute for all forcing.

Proof of part Theorem 1.10 part (3). Assume towards a contradiction that $x \mapsto A_x$ is an absolute classification of E_1. Let $A = A_x$, the classifying invariant of the Cohen generic $x \in (2^\omega)^\omega$. Then $A \in M = V(D)$. By [Gri75, Theorem B] there is some poset $\mathbb{Q}$ in $V(D)$ and a $\mathbb{Q}$-generic filter over $V(D)$, H, such that $V(D)[H] = V[x]$. Fix a $\mathbb{Q}$-name σ such that $\sigma[H] = x$ and a condition $q \in \mathbb{Q}$ forcing, in $V(D)$, that $A_\sigma = \check{A}$.

Let $q \in H'$ be $\mathbb{Q}$-generic over $V(D)[H]$, and let $x' = \sigma[H']$, so $A_x = A = A_{x'}$. We claim that x and x' are not E_1-related, showing that $x \mapsto A_x$ is not a complete classification in this extension. Indeed, by Lemma 3.23, $V(D)[H] \cap V(D)[H'] = V(D) = M$. If x were E_1-related to x', then for large enough n, they would agree on their n'th columns. This would imply that for large n, the n'th column $x(n)$ is in $V[x]$ and $V[x']$, and therefore is in M. This is a contradiction, since $x(n) \notin V[x_{n+1}]$ and $M \subseteq V[x_{n+1}]$.

□

4. Turbulence

We now show that turbulent equivalence relations remain completely unclassifiable in the context of generically absolute classifications. When $E = E_a$ is the orbit equivalence relation of a turbulent action a, this follows directly from the characterization of turbulence in [LZ20] (Theorem 4.1 below). In this section we extend the characterization of Larson and Zapletal (Theorem 4.3), and use this to extend the definition of turbulence to arbitrary analytic equivalence relations (Definition 4.5). Finally, we prove in this generality that turbulent equivalence relations are not generically classifiable (Proposition 4.7).

Theorem 4.1 (Larson-Zapletal [LZ20, Theorem 3.2.2])**.** Suppose $a \colon G \curvearrowright X$ is a continuous action of a Polish group G on X with dense and meager orbits. The following are equivalent.

- $a \colon G \curvearrowright X$ is generically turbulent;
- If $x \in X$ is Cohen-generic over V and $g \in G$ is Cohen generic over $V[x]$ then $V[x] \cap V[g \cdot x] = V$.

Definition 4.2 ([KSZ13, Definition 3.10])**.** Let E be an analytic equivalence relation on a Polish space X, and let $x \in X$ be in some generic extension of V. Define

$$V[[x]]_E = \bigcap \{V[y] : y \text{ is in some further generic extension, } y \in X \text{ and } x \mathrel{E} y\}.$$

That is, a set b is in $V[[x]]_E$ if in any generic extension of $V[x]$ and any y in that extension which is E-equivalent to x, b is in $V[y]$.

We will call this model $V[[x]]_E$ the **intersection model** of x, with respect to E. For example, our tail intersection model M is precisely $V[[x]]_{E_1}$. Indeed, it follows from the definition that $V[[x]]_{E_1} \subseteq V[x_n]$ for each n, and therefore $V[[x]]_{E_1} \subseteq M$.

On the other hand, for any $y \; E_1 \; x$, in some generic extension, there is some n such that $V[x_n] \subseteq V[y]$. Thus $M \subseteq V[y]$ for any such y, and so $M \subseteq V[[x]]_{E_1}$.

In the setting of Theorem 4.1, note that $V[[x]]_{E_a} \subseteq V[x] \cap V[g \cdot x]$, for any g, where E_a is the orbit equivalence relation induced by the action a. It follows from Theorem 4.1 that if $a \colon G \curvearrowright X$ is a generically turbulent action, then for a Cohen generic $x \in X$, $V[[x]]_{E_a} = V$ is as small as can be. We show that the latter already characterizes turbulence.

Theorem 4.3. Suppose $a \colon G \curvearrowright X$ is a continuous action of a Polish group G on X with dense and meager orbits. The following are equivalent.

- $a \colon G \curvearrowright X$ is generically turbulent.
- If $x \in X$ is Cohen-generic over V then $V[[x]]_{E_a} = V$.

The key lemma in the proof of Theorem 4.3 is the following.

Lemma 4.4. Suppose $a \colon G \curvearrowright X$ is a continuous action of a Polish group G on X and E_a is the induced equivalence relation. Let $x \in X$ be in *some* generic extension and $g \in G$ be Cohen generic over $V[x]$. Then

$$V[[x]]_{E_a} = V[x] \cap V[g \cdot x].$$

Note that here x is not required to be Cohen generic.

Proof. Let $z = g \cdot x$. By definition, $V[[x]]_{E_a} \subseteq V[x] \cap V[z]$. It remains to show that for any y in a generic extension of $V[x]$, if $y \; E \; x$ then $V[x] \cap V[z] \subseteq V[y]$. Suppose first that $y \in V[x][H]$ where H is $\mathbb{P}$-generic over $V[x][g]$ (not just over $V[x]$) for some $\mathbb{P} \in V[x]$. In this case, by mutual genericity, g is generic over $V[x][H]$. In $V[x][H]$, fix $\gamma \in G$ such that $y = \gamma \cdot x$. Since G acts on itself by homeomorphisms and $g \in G$ is Cohen over $V[x][H]$, then so is $g\gamma$, and therefore $V[y][g\gamma] \cap V[x][H] = V[y]$, by Lemma 3.23. Finally, $g\gamma \cdot y = g \cdot x = z$ is in $V[y][g\gamma]$, so $V[z] \cap V[x] \subseteq V[y][g\gamma] \cap V[x][H] = V[y]$, as desired.

For the general case, let $y \in V[x][H]$ where H is some $\mathbb{P}$-generic over $V[x]$, $\mathbb{P} \in V[x]$. H may not be generic over $V[x][g]$. It suffice to show that if $a \in V[x]$ and $a \notin V[y]$ then $a \notin V[z]$. Fix an $a \in V[x]$ and some condition p forcing that $x \; E \; \dot{y}$ and $\check{a} \notin V[\dot{y}]$. Let H' be $\mathbb{P}$-generic over $V[x][g]$ extending p. By the argument above $V[z] \cap V[x] \subseteq V[\dot{y}[H']]$. Since $a \notin V[\dot{y}[H']]$ then $a \notin V[z]$. □

Proof of Theorem 4.3. For the forward direction: take a Cohen generic $g \in G$ over $V[x]$, then $V[[x]]_{E_a} \subseteq V[x] \cap V[g \cdot x] = V$ (by Theorem 4.1). For the other direction, assume that for any Cohen generic $x \in X$ the intersection model is trivial: $V[[x]]_{E_a} = V$. Then for any $g \in G$, Cohen generic over $V[x]$, it follows from Lemma 4.4 that $V[x] \cap V[g \cdot x] = V[[x]]_E = V$. Now Theorem 4.1 implies that $a \colon G \curvearrowright X$ is generically turbulent. □

These results naturally motivate the following definition of turbulence.

Definition 4.5. Let E be an analytic equivalence relation on a Polish space X. Say that E is **generically turbulent** if for any Cohen generic $x \in X$,

(1) the E-class of x is dense and meager in X and
(2) $V[[x]]_E = V$.

Remark 4.6. If $a\colon G \curvearrowright X$ is a continuous action and E is the orbit equivalence relation, it follows from Theorem 4.3 that the action is generically turbulent if and only if E is generically turbulent with respect to the above definition.

Theorem 4.7. If E is generically turbulent (as in Definition 4.5) then E does not admit a generically absolute classification.

Proof. Assume for a contradiction that $x \mapsto A_x$ is a generically absolute classification of E. Let $x \in X$ be a Cohen generic over V. Then $A = A_x$ is in $V[[x]]_E$. By assumption, $A \in V$. Fix a condition p in the Cohen forcing for X such that $p \Vdash A_{\dot{x}} = \check{A}$. Now any two Cohen generics extending p are E-related, contradicting the fact that the E-classes are meager. $\square$

Since E_1 admits a generically absolute classification (Claim 2.5), then so does E_1^+ (see Proposition 7.4). This implies the result of Kanovei and Reeken [KR03], that no turbulent orbit equivalence relation is Borel reducible to E_1^+. This result is vastly generalized in [LZ20, Theorem 3.3.5].

4.1. Intersection numbers and a theorem of Kechris and Louveau.

Theorem 4.8 (Kechris-Louveau [KL97, Theorem 4.2])**.** Suppose $a\colon G \curvearrowright X$ is a continuous action of a Polish group G on a Polish space X, let E_a be the induced orbit equivalence relation on X. Then, on any comeager subset of $(2^\omega)^\omega$, E_1 is not Borel reducible to E_a.

Larson and Zapletal [LZ20, Theorem 4.1.1] gave a proof of this theorem using the intersection model M. We give a different proof here, using the following definition, motivated by Lemma 4.4.

Definition 4.9. Given an equivalence relation E on X and $x \in X$ in some generic extension, let **the intersection number of** x **(relative to** E**)** be the minimal size of a finite set B such that

$$V[[x]]_E = \bigcap_{y \in B} V[y],$$

where B is contained in the E-class of x in some generic extension of $V[x]$. If no such set exists say that the intersection number is infinite.

It follows from Lemma 4.4 that if E is an orbit equivalence relation then the intersection number of x is always ≤ 2, for any $x \in X$ in some generic extension. On the other hand, for a Cohen generic $x \in (2^\omega)^\omega$, the intersection number of x relative to E_1 is infinite:

Claim 4.10. Let $x \in (2^\omega)^\omega$ be Cohen-generic. Suppose $x_1, ..., x_n$, in some further generic extension, are all E_1-equivalent to x. Then

$$V[[x]]_{E_1} \subsetneq V[x_1] \cap ... \cap V[x_n].$$

Proof. Fix k large enough such that x and $x_1, ..., x_n$ all agree starting the k'th column onwards. Then $x(k) \in V[x_1] \cap ... \cap V[x_n]$. However, $x(k) \notin V[[x]]_{E_1} = M$, so M is strictly smaller than $V[x_1] \cap ... \cap V[x_n]$. □

Borel reductions respect the intersection number:

Lemma 4.11. Let E and F be analytic equivalence relations. Suppose $f : E \to_B F$ is a (partial) Borel reduction and $x \in \operatorname{dom} f$ in some generic extension. Then the intersection number of x relative to E is equal to the intersection number of $f(x)$ relative to F.

By a partial Borel reduction we mean that the domain of f is a Borel subset of the domain of E, and f is a reduction of $E \restriction \operatorname{dom} f$ to F. The lemma is a generalization of [Sha21, Lemma 3.5], which states that if $f : X \to Y$ is a (partial) Borel reduction of E to F, and $x \in \operatorname{dom} f$, in some generic extension, then $V[[x]]_E = V[[f(x)]]_F$.

Proof. Assume first that $V[[f(x)]]_F = \bigcap_{y \in B} V[y]$ where B is contained in the F-class of $f(x)$ in some big generic extension $V[G]$. For each $y \in B$, $f(x)\ F\ y$ in $V[G]$. By absoluteness for the statement $(\exists x) f(x)\ F\ y$ there is $x_y \in V[y]$ such that $f(x_y)\ F\ y$, thus $x_y\ E\ x$ for each $y \in B$ (since f is a reduction). Now $\bigcap_{y \in B} V[x_y] \subseteq \bigcap_{y \in B} V[y] = V[[f(x)]]_F = V[[x]]_E$. It follows that $\bigcap_{y \in B} V[x_y] = V[[x]]_E$, so the intersection number of x is $\leq |B|$.

Similarly, suppose $V[[x]]_E = \bigcap_{y \in D} V[y]$ where D is contained in the E-class of x, in some big generic extension. Then $\{f(y) : y \in D\}$ is contained in the F-class of $f(x)$, and $V[[f(x)]]_F = \bigcap_{y \in B} V[f(y)]$, so the intersection number of $f(x)$ is $\leq |D|$. We conclude that the intersection numbers of x and $f(x)$ are the same. □

Proof of Theorem 4.8. Assume for contradiction that there is a reduction f, defined on a comeager subset of $\mathbb{R}^\omega$, reducing E_1 to some equivalence relation E induced by a Polish group action. Let $x \in \mathbb{R}^\omega$ be Cohen generic over V, so x is in the domain of f. By lemmas 4.4 and 4.11 it follows that the intersection number of x is 2, contradicting claim 4.10. □

Remark 4.12. In the proof above, x is taken to be a Cohen generic real in the domain of E_1. However, we have no control over what $f(x)$ looks like. This is why we need to consider arbitrary reals $x \in X$ in Lemma 4.4.

Question 4.13 (see [KL97])**.** If E is an analytic equivalence relation, is it true that either E is Borel reducible to an orbit equivalence relation or $E_1 \leq E$?

The proof above suggests the following strategy for a counterexample: suppose we can find an analytic equivalence relation E such that:

(1) For any $x \in \operatorname{dom} E$ in a generic extension the intersection number of x is finite;
(2) there is $x \in \operatorname{dom} E$ in a generic extension whose intersection number is strictly greater than 2.

Part (1) would imply that $E_1 \not\leq_B E$ and part (2) that E is not Borel reducible to an orbit equivalence relation. On the other hand, the following would support a positive answer to Question 4.13.

Question 4.14. If E is an analytic equivalence relation, x in the domain of E in some generic extension, must the intersection number of x be either ≤ 2 or infinite?

5. Chromatic numbers in the intersection model

Consider the shift graph $(\mathcal{P}(\omega), S)$, where for $X \subseteq \omega$, $S(X) = X \setminus \min X$. (We identify $\mathcal{P}(\omega)$ with 2^ω in the usual way.) This is an acyclic graph, so its chromatic number is 2 in a ZFC model.

Proposition 5.1. In M, the chromatic number of the shift graph is not 2.

Remark 5.2. The following alternative intersection model was studied in [LZ20]. Let $c\colon \omega \times \omega_1 \to \{0,1\}$ be Cohen-generic over V. Define $M' = \bigcap_{n<\omega} V[c \restriction (\omega \setminus n) \times \omega_1]$. This model was considered precisely to provide an example of an intersection model in which the axiom of choice fails. Indeed, they show that in M' the shift graph does not have chromatic number 2. The argument below is similar, using functions in ω^ω to get uncountably many reals out of ω many Cohen reals, as before. Note that both M' and M are models of DC.

Proof of Proposition 5.1. In V, let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be an $<^*$-increasing and unbounded sequence of functions in ω^ω such that each f_α is increasing. Let $x \in 2^{\omega\times\omega}$ be the Cohen generic over V, where $M = \bigcap_{n<\omega} V[x_n]$, $x_n = x \restriction (\omega \setminus n) \times \omega$. Define $z_\alpha \in 2^\omega$ by $z_\alpha(n) = x(n, f_\alpha(n))$, the restriction of x to the graph of f_α.

As before, each z_α is in M, but $\langle z_\alpha : \alpha < \mathfrak{b}\rangle$ is not in M (similar to Lemma 3.6). For $m < \omega$, define $z_{\alpha,m} \in 2^\omega$ by $z_{\alpha,m}(k) = \begin{cases} 0 & k < m; \\ z_\alpha(k) & k \geq m. \end{cases}$ So $z_{\alpha,0} = z_\alpha$ and $z_{\alpha,m}$ is the result of finitely many application of S to z_α. Let $A = [\{z_\alpha : \alpha < \mathfrak{b}\}]_{E_0}$, all reals for which there is some z_α with which they agree up to finitely many values. Note that $A \in M$ and $\{z_{\alpha,m} : \alpha < \mathfrak{b},\, m < \omega\} \subseteq A$. We will show that in M there is no 2-coloring of (A, S).

Assume towards a contradiction that $\sigma\colon A \to \{0,1\}$ is a coloring, $\sigma \in M$. Let $\dot\sigma$ be a name forced to be a 2-coloring of A in M. Working in V, find conditions $p_\alpha \in \mathbb{P}$ and $\epsilon_\alpha \in \{0,1\}$ such that $p_\alpha \Vdash \dot\sigma(\dot z_\alpha) = \epsilon_\alpha$ for any $\alpha \in X$. Since $\mathbb{P}$ is countable, we may find an unbounded $X \subseteq \mathfrak{b}$, a single condition $p \in \mathbb{P}$ and $\epsilon \in \{0,1\}$ such that for $\alpha \in X$, $p_\alpha = p$ and $\epsilon_\alpha = \epsilon$. Since $\langle f_\alpha : \alpha \in X\rangle$ is $<^*$-unbounded there is $m < \omega$ for which $\{f_\alpha(m) : \alpha \in X\}$ is unbounded in ω.

Let $\dot\sigma_{m+1}$ be a $\mathbb{P}_{m+1}$-name and $q \in \mathbb{P}$ extending p such that $q \Vdash \dot\sigma[\dot x] = \dot\sigma_{m+1}[\dot x_{m+1}]$. (This is possible since σ is forced to be in $V[\dot x_{m+1}]$.) Fix $\alpha \in X$ such that $(m, f_\alpha(m))$ is not in the domain of q. Working with the poset $\mathbb{P}_{m+1}$, consider the condition $q \restriction (\omega \setminus (m+1) \times \omega)$ and extend it to $t \in \mathbb{P}_{m+1}$ such that $t \Vdash \dot\sigma_{m+1}(\dot z_{\alpha,m+1}) = \delta$

for some $\delta \in \{0,1\}$. Now $q \cup t$ is a condition in $\mathbb{P}$ forcing that $\dot{\sigma}(\dot{z}_\alpha) = \epsilon$ and $\dot{\sigma}(\dot{z}_{\alpha,m+1}) = \delta$.

The only coordinates on which z_α and $z_{\alpha,m+1}$ differ are $0, ..., m$, and at least one of these, the m'th coordinate, is undecided by $q \cup t$. So we can find two extensions of $q \cup t$, r_0, r_1 such that for each $i = 0, 1$, r_i decides $z_\alpha(0), ..., z_\alpha(m)$ and that the parity of $|\{k \in \{0, ..., m\} : z_\alpha(k) = 1\}|$ is i, respectively. Since σ is a 2-coloring, r_0 forces that z_α and $z_{\alpha,m+1}$ have the same color, that is, $\epsilon = \delta$, and r_i forces that $\epsilon \neq \delta$, a contradiction. □

Question 5.3. What is the chromatic number of $(\mathcal{P}(\omega), S)$ in M?

Remark 5.4. When restricting to just the set A defined above, the graph (A, S) has chromatic number precisely 3 in M. Conley and Miller [CM16] proved in a very general setting that acyclic graphs, such as the shift graph, admit Borel 3-colorings, when restricted to a comeager set. The set A is "sufficiently generic" for their argument, as follows. It suffices to find a forward recurring independent subset of A. Let $B =$ all $z \in A$ of the form $z = 0...0110 * * * ....$ That is, all z's in which the first appearance of 1 is followed by 10. B is forwards recurring, by genericity of the members of A, and $B \cap S(B) = \emptyset$, as any $y \in S(B)$ is of the form $y = 0...010 * **$.

6. Further questions

Let $\mathcal{P}$ be a collection of forcing notions. Say that a map c is a **$\mathcal{P}$-absolute classification** of E if, as in Definition 1.5, it satisfies (1), (2), and (3)(a), and (3)(b) for all generic extensions by posets in $\mathcal{P}$. This notion still respects Borel reducibility, as in Remark 1.6. It may not respect the usual non-classifiability results. For example, a turbulent equivalence relation may become classifiable.

Following [Zap08] and [KSZ13], it seems reasonable to study $\{P_I\}$-absolute classifications, where I is a proper ideal on a Polish space, and P_I is the poset of Borel sets modulo I. A particularly natural notion is a **random absolute classification**, where I is the ideal of measure zero sets. In this measure theoretic context, Larson and Zapletal developed an analogue for turbulence [LZ20, Definition 3.6.1]. By [LZ20, Theorem 3.6.2], this measure-theoretic turbulence implies that $V[[x]]_E = V$ when x is random-generic over V. It follows that for such equivalence relations no random absolute classification exists.

Similarly, given a proper ideal I on X^ω, it would be interesting to understand the tail intersection model corresponding to a P_I-generic $x \in X^\omega$. This also relates to the study of intersection models of coherent sequences as in [LZ20, Chapter 4].

7. Appendix

In this section we prove some statements about generic classifications which were mentioned in the introduction, Remark 1.6, Example 1.7, and Example 1.8. These follow from well known facts about forcing. We also provide some more context and justification to the fact that "absolute generic classification" extends the notion of "classification by countable structures". Some basic facts are presented in the more general context of $\mathcal{P}$-absolute classifications. Fix a collections of posets $\mathcal{P}$.

Remark 7.1. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are posets such that $\mathbb{Q}$ embeds into $\mathbb{P}$, as a forcing poset. Then any $\mathbb{Q}$-generic extension of V can be further extended to a $\mathbb{P}$-generic extension of V. If c is a complete classification which remains absolute in all $\mathbb{P}$-generic extensions (as in (3)(b) of Definition 1.5), then it is also true in all $\mathbb{Q}$-generic extensions. In conclusion, we may assume that $\mathcal{P}$ is closed under subforcings and forcing equivalence.

Proposition 7.2. Suppose E and F are analytic equivalence relations on Polish spaces X and Y, $f\colon X \to Y$ is a Borel reduction of E to F, and $c\colon Y \to I$ is a $\mathcal{P}$-absolute classification of F. Then $c \circ f$ is a $\mathcal{P}$-absolute classification of E.

Proof. Since f is a reduction, $c \circ f$ is a complete classification of E whenever c is a complete classification of F. The proposition follows as the statement "f is a Borel reduction from E to F" is absolute in all forcing extensions. □

Recall the definition of the Friedman-Stanley jump operator $E \mapsto E^+$ from the introduction. Consider also the following product operation on equivalence relations. Given an equivalence relation E on X, E^ω is defined on X^ω by $\langle x_n : n < \omega\rangle \; E^\omega \; \langle y_n : n < \omega\rangle$ if $x_n \; E \; y_n$ for all $n < \omega$. Both these operation preserve "classifiability by countable structures". Similarly, they preserve generic classifications.

Proposition 7.3. Suppose E is an analytic equivalence relation on a Polish space X and $c\colon X \to I$ is a $\mathcal{P}$-absolute classification of E. Then the map $c^\omega\colon X^\omega \to I^\omega$ defined by $c^\omega(\langle x_n : n < \omega\rangle) = \langle c(x_n) : n < \omega\rangle$ is a $\mathcal{P}$-absolute classification of E^ω.

Proof. The map c^ω is definable, using a definition of c. Parts (1) and (2) of Definition 1.5 follow. Since c is an E-invariant map in any extension, so is c^ω. Finally, in any generic extension by a poset in $\mathcal{P}$, if $c^\omega(\langle x_n : n < \omega\rangle) = c^\omega(\langle y_n : n < \omega\rangle)$, then $c(x_n) = c(y_n)$ for each n, and therefore $x_n \; E \; y_n$ for each n, since c is a $\mathcal{P}$-absolute classification. □

Proposition 7.4. Suppose E is an analytic equivalence relation on a Polish space X and $c\colon X \to I$ is a $\mathcal{P}$-absolute classification of E. Then the map $c^+\colon X^\omega \to \mathcal{P}(I)$ defined by $c^+(\langle x_n : n < \omega\rangle) = \{c(x_n) : n \in \omega\}$ is a $\mathcal{P}$-absolute classification of E^+.

Proof. Note that c^+ can be written as the composition $c^+ = s \circ c^\omega$, where $s\colon I^\omega \to \mathcal{P}(I)$ is defined by $s(\langle A_n : n < \omega\rangle) = \{A_n : n \in \omega\}$. Note that s is definable in an absolute way in all extensions. It follows that c^+ is E^+-invariant whenever c^ω is E^ω-invariant, and c^+ is a classification of E^+ whenever c^ω is a classification of E^ω. In particular, c^+ is E^+-invariant in all extensions, and is a classification in any generic extension by a poset in $\mathcal{P}$. □

Proposition 7.5. Assume that there is some poset $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}$ adds a new real and $\mathbb{P} \times \mathbb{P}$ is in $\mathcal{P}$. Then there is no $\mathcal{P}$-absolute classification of $=_{\mathbb{R}}$ with ordinal classifying invariants.

Proof. Assume for contradiction that c is a $\mathcal{P}$-absolute classification taking ordinal values. Fix $\mathbb{P}$ as above and let σ be a $\mathbb{P}$-name for a new real. There is some condition $p \in \mathbb{P}$ and an ordinal α such that $p \Vdash_{\mathbb{P}} c(\sigma) = \check{\alpha}$. We claim that p forces σ to be in the ground model, contradicting our assumption. Indeed, given any generic filter G extending p, let H be $\mathbb{P}$-generic over $V[G]$ with $p \in H$. Then $c(\sigma[G]) = \alpha = c(\sigma[H])$, and so $\sigma[G] = \sigma[H]$, as c is a complete classification in the $\mathbb{P} \times \mathbb{P}$ extension $V[G][H]$. Finally, by Lemma 3.23, $\sigma[G] \in V[G] \cap V[H] = V$. □

The hypothesis in the above proposition holds if $\mathcal{P}$ contains the poset $\mathbb{P}$ for adding a single Cohen real, since $\mathbb{P} \times \mathbb{P}$ and $\mathbb{P}$ are forcing equivalent. In particular, there is no generically absolute classification of $=_{\mathbb{R}}$ using ordinals. A similar argument works for random real forcing.

Proposition 7.6. Let $\mathbb{P}$ be the poset for adding a single random real in $[0, 1]$, with respect to Lebesgue measure. Then there is no $\{\mathbb{P}\}$-absolute classification of $=_{\mathbb{R}}$ with ordinal classifying invariants.

Proof. Let $\mathbb{P}_2$ be the poset to add a random real in $[0, 1]^2$, with the product measure. Then $\mathbb{P}$ and $\mathbb{P}_2$ are forcing equivalent. Moreover, there are $\mathbb{P}_2$-names σ_l, σ_r for reals in $[0, 1]$ so that it is forced that each of σ_l and σ_r are $\mathbb{P}$-generics, and $V[\sigma_l] \cap V[\sigma_r] = V$. The rest of the argument follows as above. □

Proposition 7.7. Assume that $\mathcal{P}$ contains either Cohen forcing or Random real forcing on 2^ω (with the usual product topology and product measure). Then there is no $\mathcal{P}$-absolute classification of E_0 with classifying invariants in 2^α, for any ordinal α.

Proof. Let $\mathbb{P} \in \mathcal{P}$ be either Cohen or Random real forcing, in 2^ω. Fix an ordinal α and assume for a contradiction that $c\colon 2^\omega \to 2^\alpha$ is a $\mathcal{P}$-absolute complete classification of E_0. Let $b \in V$ be the parameter used to define c. Let $x \in 2^\omega$ be $\mathbb{P}$-generic over V, and define $A = [x]_{E_0}$.

Both Cohen and Random forcing have the following property: For any two conditions $p, q \in \mathbb{P}$, there is an automorphism π of $\mathbb{P}$, swapping finitely many values of the generic $x \in 2^\omega$, such that $\pi(p)$ is compatible with q. In particular, $\pi(\dot{A}) = \dot{A}$. It follows that for any formula ϕ and parameter $v \in V$, the truth value of $\phi^{V[x]}(A, v)$ is decided in V.

Let $y = c(x)$, $y \subseteq \alpha$. y can be defined in $V[x]$ from $A = [x]_{E_0}$, as follows: for $\zeta < \alpha$, $\zeta \in y$ if and only if there is some $x' \in A$ for which $\zeta \in c(x')$. The latter can be written as a formula $\phi^{V[x]}(\zeta, A, b)$. It follows that y can be defined in V as the set of $\zeta < \alpha$ for which it is forced that $\phi(\zeta, \dot{A}, \check{b})$ holds, and so $y \in V$.

Fix a condition $p \in \mathbb{P}$ forcing that $c(\dot{x}) = \check{y}$. Then for any two $\mathbb{P}$-generics x_1, x_2 which extend p, $x_1 \; E_0 \; x_2$. We conclude that E_0 has a non-meager, or positive measure, equivalence class, a contradiction. □

For each countable ordinal α there is a natural Borel equivalence relation $\cong_\alpha$, with a natural complete classification using hereditarily countable sets in $\mathcal{P}^\alpha(\mathbb{N})$, the α'th-iterated powerset of $\mathbb{N}$ (see [HKL98, FS89]). For example, one can take $\cong_0$ to be $=_\mathbb{N}$, and define $\cong_{n+1}$ to be $\cong_n^+$. Following the approach in [HK96, Introduction (E)] and [HKL98], an equivalence relation is considered "classifiable using hereditarily countable sets of rank α" if it is Borel reducible to $\cong_\alpha$. As $\cong_{\alpha+1} \not\leq_B \cong_\alpha$, invariants of rank $\alpha+1$ are generally more complex than invariants of rank α.

To see that these intuitions are preserved in the context of generically absolute classifications, we would want to show that $\cong_{\alpha+1}$ does not admit a generically absolute classification using hereditarily countable invariants in $\mathcal{P}^\alpha(\mathbb{N})$. This follows from the results in [Sha21]. In fact, similarly to Proposition 7.7 above, this is true without demanding the invariants to be hereditarily countable, and with $\mathbb{N}$ replaced by any ordinal.

Theorem 7.8. Fix a countable ordinal α, $\beta < \alpha$, any ordinal ζ, and let $I = \mathcal{P}^\beta(\zeta)$, the β'th-iterated powerset of ζ. Then $\cong_\alpha$ does not admit a generically absolute classification with invariants in I.

For $\alpha < \omega$, this is proved in [Sha21, Section 4], by finding a model of the form $V(A)$, where A is an invariant for $\cong_{n+1}$, so that $V(A)$ cannot be presented as $V(B)$ for any set B in $\mathcal{P}^n(\zeta)$, for any ordinal ζ. The conclusion in [Sha21] is that there is no absolute classification of $\cong_{n+1}$ using invariants in $\mathcal{P}^n(\zeta)$. The same proof shows that there is no such generically absolute classification, since A is constructed as the $\cong_{n+1}$-invariant of a single Cohen-generic real. The modifications for countable $\alpha \geq \omega$ are explained in [Sha21, Section 8].

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