ON MOORE'S PARTITION

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1. Introduction

An S space is a regular space which is hereditarily separable but not Lindelöf. An L space is a regular space which is hereditarily Lindelöf but not separable. The S and L space problems ask for the existence of such spaces.

If the regularity requirement is weakened to Hausdorff, such spaces were constructed by Sierpiński [7] (see also [2]). Once regularity is assumed, as first asked by Hajnal and Juhasz in [2], the questions turned out to have much deeper ties to set theory.

An extensive account on the subject, as well as historical remarks, can be found in [10]. We mention here some results which are relevant to the questions considered below.

From the definitions, there is no clear relationship between the S and L space problems. The following result suggests that there is. A space is called a strong L (S) space if all of its finite powers are L (S) spaces.

Theorem 1 (Zenor [11])**.** There is a strong S space iff there is a strong L space.

Kunen showed that Martin's axiom implies that there are no strong S or L spaces, and asked whether it already implies the non existence of any S or L space.

Theorem 2 (Kunen [4] ($MA_{\aleph_1}$))**.** If X is a non-separable regular space, then X^k contains an uncountable discrete subset for some integer k.

Todorcevic later established the consistency of the non existence of S spaces, using proper posets. Szentmiklossy [8] showed that Martin's axiom does not suffice.

Theorem 3 (Todorcevic [9] (PFA))**.** For any partition $f\colon [\omega_1]^2 \to 2$, either there is a 0-homogeneous subset of size $\aleph_1$, or for some integer l there are uncountable sets $A \subset \omega_1$ and $\mathcal{B} \subset [\omega_1]^l$, where $\mathcal{B}$ is pairwise disjoint, such that for any $\alpha \in A$ and $b \in \mathcal{B}$ with $\alpha < b$, there is some $j < l$ for which $f(\alpha, b(j)) = 1$. Furthermore, this partition relation implies that there are no S spaces.

The partition relation above is abbreviated by $\omega_1 \to (\omega_1, \omega_1; \operatorname{fin}\omega_1)^2$. Subsets A and $\mathcal{B}$ as in the theorem will be called 1-homogeneous of tpye $(\omega_1; \operatorname{fin}\omega_1)$. If l is fixed, then such sets are 1-homogeneous of type $(\omega_1; [\omega_1]^l)$. Similar variations, such as type $(\operatorname{fin}\omega_1; [\omega_1]^l)$ are defined in the same manner.

Todorcevic concludes the paper [9] by mentioning that the dual partition relation, $\omega_1 \to (\omega_1, \operatorname{fin}\omega_1; \omega_1)^2$, would imply that there are no L spaces, and asks whether

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it is consistent. This again suggests a similarity between the S and L problems, especially with both partition relations having the common natural strengthening $\omega_1 \to (\omega_1, \omega_1; \omega_1)^2$, whose consistency also remained open.

In [10], Todorcevic shows that the two problems are not the same, by constructing a model with an L space but no S spaces (see also [1]). Finally, in a surprising turn of events, Moore gave a ZFC construction of an L space, thus giving a negative answer to the corresponding partition relation.

Theorem 4 (Moore [5])**.** There is a partition $o^*\colon [\omega_1]^2 \to \omega$ such that for any uncountable pairwise disjoint family $\mathcal{A} \subset [\omega_1]^k$ and uncountable $B \subset \omega_1$ and for any $\phi\colon k \to \omega$, there are $a \in \mathcal{A}$ and $\beta \in B$ such that $a < \beta$ and for all $i < k$, $o^*(a(i), \beta) = \phi(i)$.

In particular, $\omega_1 \not\to [\operatorname{fin}\omega_1; \omega_1]^2_\omega$, which is the statement above restricted to constant fucntions ϕ.

Theorem 5 (Moore [5])**.** There is an L space.

Moore showed that his L space satisfies the conclusion of Theorem 2 with $k = 2$ and made the following conjecture:

Conjecture 1 (Conjecture 4 in [5], assuming PFA)**.** If X is a non-separable regular Hausdorff space, then X^2 contains an uncountable discrete subspace.

A closely related question was asked earlier in the paper:

Question 1 (Question 5.7 in [5], assuming PFA)**.** If $c\colon [\omega_1]^2 \to 2$, are there $A \subset \omega_1$ and $\mathcal{B} \subset [\omega_1]^2$ which are uncountable with $\mathcal{B}$ being pairwise disjoint and a $\phi\colon 2 \to 2$ such that for all $\alpha \in A$ and $b \in \mathcal{B}$ with $\alpha < b$ there is an $i < 2$ such that $c(\alpha, b(i)) \neq \phi(i)$?

The statement in the question is a formal weakening of $\omega_1 \to (\omega_1; [\omega_1]^2)^2_2$, which we get by only allowing a constant function ϕ.

A quick proof of $\omega_1 \not\to (\omega_1; [\omega_1]^2)^2_2$ is given in section 2 below. A negative solution to Question 1 and Conjecture 1 is established in section 4.

It is well known how to construct S and L spaces from strong failures of partition relations, as was originally done by Hajnal and Juhasz [3]. See also chapter 6 and the beginning of chapter 8 in [10]. The following construction of an L space is given in section 7 of [5] for $l = 1$, the general proof is the same. These arguments can also be found in proposition 4.3 of [6].

Theorem 6. Let l be an integer. Suppose there is a coloring $c\colon [\omega_1]^2 \to 2$ satisfying the following: whenever $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ are uncountable families of pairwise disjoint sets and $\phi\colon k \times l \to 2$, there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a < b$ and for all $i < k$ and $j < l$,

$$c(a(i), b(j)) = \phi(i, j).$$

For $\beta < \omega_1$, define $w_\beta\colon \omega_1 \to 2$ by $w_\beta(\xi) = c(\xi, \beta)$ if $\xi < \beta$ and $w_\beta(\xi) = x_0$ otherwise. Let $\mathcal{L} = \{w_\beta;\, \beta < \omega_1\}$ viewed as a subspace of 2^{ω_1}.

Then $\mathcal{L}$ is a regular non separable subspace of 2^{ω_1} whose l'th power is hereditarily Lindelöf.

Our focus will be on constructing colorings with properties as above, and the corresponding L spaces will follow.

In section 5 we show that the theorems of Kunen and Todorcevic are optimal, by proving the following theorem, which is the main result in this paper.

Theorem 7. For any integer l, there is a partition $c\colon [\omega_1]^2 \to \omega$ such that whenever $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ are uncountable families of pairwise disjoint sets and $\phi\colon k \times l \to \omega$, there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a < b$ and for all $i < k$ and $j < l$,

$$c(a(i), b(j)) = \phi(i, j).$$

Together with Theorem 6 we get:

Theorem 8. For any integer l there is an L space whose l'th power is also an L space.

The proof presented in section 2 directly appeals to Theorem 4.3 from Moore's [5]. This theorem summarizes the main technical results developed there about the oscillation function. A stronger version of this theorem, which already follows from Moore's proof, will be necessary. The strengthened version is formulated in section 3, and the results in the paper will be derived directly from it. Other than that, no technical knowledge from [5] is necessary. A sketch is given of how the proof from [5] readily gives the stronger version. The reader is referred to [5] for the definitions and details behind the oscillation function.

Note that the results of section 5 subsume those in section 4, which subsume those in section 2.

2. A quick proof of $\omega_1 \not\to (\omega_1; [\omega_1]^2)^2_2$

Using the method of minimal walks, Moore defines a function $\mathrm{osc}\colon [\omega_1]^2 \to \omega$ with the following property:

Theorem 9 (Theorem 4.3 in [5])**.** For every $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ which are uncountable families of pariwise disjoint sets and every $n < \omega$, there are $a \in \mathcal{A}$ and b_m $(m < n)$ in $\mathcal{B}$ such that for all $i < k$, $j < l$ and $m < n$:

$$a < b_m, \text{ and } \mathrm{osc}(a(i), b_m(j)) = \mathrm{osc}(a(i), b_0(j)) + m.$$

Note that the partition $c\colon [\omega_1]^2 \to 2$, defined by $c(\alpha, \beta) = \mathrm{osc}(\alpha, \beta) \mod 2$, readily gives the failure of $\omega_1 \to (\omega_1; \omega_1)^2_2$.

In this section we work with the structure $6 = \{0, 1, 2, 3, 4, 5\}$, where $+$ is taken modulo 6. If X is a subset of 6 and k an integer, $X + k = \{x + k \mod 6;\ x \in X\}$.

Let $B = \{0, 1, 3\} \subset 6$.

Lemma 10. For any integer k, $B \cap (B + k) \neq \emptyset$.

Note that then $B \cup (B + k) \neq 6$. Since $6 \smallsetminus (B + k) = (6 \smallsetminus B) + k$ it follows that the lemma also holds for the complement, i.e., for every k, $(6 \smallsetminus B) \cap ((6 \smallsetminus B) + k) \neq \emptyset$.

Define $f\colon [\omega_1]^2 \to 2$ by $f(\alpha, \beta) = 0$ iff $\mathrm{osc}(\alpha, \beta) \mod 6 \in B$.

Proposition 11. f is a counterexample to $\omega_1 \to (\omega_1; [\omega_1]^2)^2_2$.

Proof. Suppose $A \subset \omega_1$, $\mathcal{B} \subset [\omega_1]^2$ are uncountable and $\mathcal{B}$ is pairwise disjoint. We show that there are $\alpha \in A$ and $b \in \mathcal{B}$ such that $\alpha < b$ and $f(\alpha, b(0)) = f(\alpha, b(1)) = 0$. The same argument, replacing B with $6 \setminus B$, would produce $\alpha \in A$ and $b \in \mathcal{B}$ such that $\alpha < b$ and $f(\alpha, b(0)) = f(\alpha, b(1)) = 1$.

Apply Theorem 9 for A and $\mathcal{B}$ with $n = 6$, to get $\alpha \in A$ and $b_m (m < 6)$ in $\mathcal{B}$ such that for any $j < 2$, and $m < n$:

$$\alpha < b_m \text{ and } \operatorname{osc}(\alpha, b_m(j)) = \operatorname{osc}(\alpha, b_0(j)) + m$$

Let $k = \operatorname{osc}(\alpha, b_0(1)) - \operatorname{osc}(\alpha, b_0(0))$. Then for each m, $\operatorname{osc}(\alpha, b_m(0)) = \operatorname{osc}(\alpha, b_m(1)) - k$. By the lemma above, there is some $t \in B \cap (B + k)$. Let $m = t - \operatorname{osc}(\alpha, b_0(1)) \mod 6$. Then

$$\operatorname{osc}(\alpha, b_m(1)) \mod 6 = \operatorname{osc}(\alpha, b_0(1)) + m \mod 6 = t \in B.$$

Also,

$$\operatorname{osc}(\alpha, b_m(0)) \mod 6 = t - k \mod 6 \in B.$$

Thus $f(\alpha, b_m(0)) = f(\alpha, b_m(1)) = 0$ as required. □

Remark 12. It follows from Proposition 7.13 of [5] that there are uncountable sets $A, \mathcal{B}$ such that for any $\alpha < b$ from $A, \mathcal{B}$ respectively, $\operatorname{osc}(\alpha, b(0)) = \operatorname{osc}(\alpha, b(1))$. That is, k above is always 0. So the partition above is not a counterexample to Question 1.

To solve Question 1 we first need to extract more information from Moore's proof.

3. Moore's theorem

One thing we need, but do not get from the statement of Theorem 9, is to fix the differences $\operatorname{osc}(a(i), b_0(j)) - \operatorname{osc}(a(i), b_0(0))$ before choosing n (and so before choosing a). However, this already follows from Moore's proof of Theorem 9 in [5]. The set Ψ is added for technical reasons needed later.

Theorem 13 (as 4.3 in [5])**.** For every $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ which are uncountable families of pariwise disjoint sets, and any countable set X, there are $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ and $d \in \omega^k \times \omega^l$ such that for any finite set of formulas Ψ over X and for every $n < \omega$, there are $a \in \mathcal{A}$ and b_m $(m < n)$ in $\mathcal{B}$ such that for all $i < k$, $j < l$, and $m < n$: $a < b_m$, $\operatorname{type}^\Psi(a) = \operatorname{type}^\Psi(a_0)$, $\operatorname{type}^\Psi(b_m) = \operatorname{type}^\Psi(b_0)$,

$$\begin{aligned} &\operatorname{osc}(a(i), b_m(j)) = \operatorname{osc}(a(i), b_0(j)) + m \text{ and} \\ &\operatorname{osc}(a(i), b_0(j)) = \operatorname{osc}(a(i), b_0(0)) + d(i,j). \end{aligned} \tag{1}$$

We sketch below why the additional conclusion is satisfied in Moore's proof, see [5] for details and the relevant definitions.

Suppose $\mathcal{A} \subset [\omega_1]^k$, $\mathcal{B} \subset [\omega_1]^l$ and X are as in the statement. Take a countable $M \prec H(\aleph_2)$ containing all the relevant objects, let $\delta = M \cap \omega_1$. Take $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ both above δ. Define $d \in \omega^k \times \omega^l$ by

$$d(i,j) = \operatorname{osc}(a(i), b_0(j); L(\delta, b_0(j))) - \operatorname{osc}(a(i), b_0(0); L(\delta, b_0(0)))$$

for $i < k$ and $j \in l$. Suppose Ψ is a finite set of formulas over X. Let $\mathcal{A}' = \{a \in \mathcal{A};\ \mathrm{type}^{\Psi}(a) = \mathrm{type}^{\Psi}(a_0)\}$ and $\mathcal{B}' = \{b \in \mathcal{B};\ \mathrm{type}^{\Psi}(b) = \mathrm{type}^{\Psi}(b_0)\}$. Since $X \in M$ and X is countable then $X \subset M$, and therefore $\mathcal{A}', \mathcal{B}' \in M$.

Let n be an integer. Moore's construction gives $a_m \in \mathcal{A}'$ and $b_m \in \mathcal{B}'$ for $m < n$, all outside of M, by repeated applications of Lemma 4.4 from [5], which satisfy $\mathrm{osc}(a_m(i), b_m(j); \mathrm{L}(\delta, b_m(j))) = \mathrm{osc}(a_0(i), b_0(j); \mathrm{L}(\delta, b_0(j))) + m$.

At the end a_n is reflected to some $a \in M$, and a is chosen so that for any $i < k$, $e_{b_m(j)} \mid \mathrm{L}(a(i), \delta)$ does not depend on m, j. Thus the addition to the oscillation given by $\mathrm{osc}(a(i), b(j); L(a(i), \delta))$ is the same for all m and j. From this the additional conclusion in Theorem 13 follows.

4. A solution to Question 1

In [5] Moore considers a partition involving raising elements of the unit circle to powers given by the oscillation function. Then an essential use of Kronecker's Theorem is what enables him to deal with arbitrary integers k in Theorem 4. Kronecker's Thoerem will be used, for the same purpose, in this section and in section 5 below.

For notational convenience, we will work with the additive structure on $I = [0, 1)$ instead of the multiplicative structure on the unit circle. For any integer m and $r, w \in I$, the operations $r + w$, $m \cdot r$ and the distance $|r - w|$ are calculated modulo 1. In this context, Kronecker's Theorem is:

Theorem (Kronecker's Theorem)**.** Suppose that r_i $(i < k)$ are elements of I which are rationally independent. For every $\epsilon > 0$ there is a natural number n_ϵ such that if u, v are in I^k, there is an $m < n_\epsilon$ such that for all $i < k$,

$$|u_i + m \cdot r_i - v_i| < \epsilon.$$

Fix a sequence $\langle r_\alpha;\ \alpha < \omega_1\rangle$ of rationally independent elements of I. For $a \in [\omega_1]^{<\omega}$ and an integer N, let $n(N, a)$ be the n_ϵ given by Kronecker's Theorem for $r_i = r_{a(i)}$ and $\epsilon = \frac{1}{N}$. Let $Q = [0, 1) \cap \mathbb{Q}$. Define $f\colon [\omega_1]^2 \to I$ by

$$f(\alpha, \beta) = \mathrm{osc}(\alpha, \beta) \cdot r_\alpha + r_\beta \mod 1.$$

Define a coloring $c\colon [\omega_1]^2 \to 2$ by

$$c(\alpha, \beta) = 0 \text{ iff } f(\alpha, \beta) \in [0, \frac{1}{2}).$$

It follows from the next theorem that c witnesses the failure of Question 1.

Theorem 14. Suppose $\mathcal{A} \subset [\omega_1]^k$, $\mathcal{B} \subset [\omega_1]^2$ are uncountable families of pairwise disjoint sets, and $\phi\colon k \times 2 \to 2$. There are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a < b$ and for any $i < k$ and $j < 2$,

$$c(a(i), b(j)) = \phi(i, j).$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$ and ϕ be as in the statement. Take $X = \mathbb{N} \cup Q$ and let $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ and $d \in \omega^k \times \omega^2$ be given by Theorem 13, applied to $\mathcal{A}, \mathcal{B}$ and X. For $i < k$ and $j < 2$, let

$$\theta(i, j) = d(i, j) \cdot r_{a_0(i)} + r_{b_0(j)} - r_{b_0(0)}.$$

Note that since $\{r_\alpha;\, \alpha < \omega_1\}$ are rationally independent, then $\theta(i,1) \neq 0, \frac{1}{2}$ for each $i < k$. ($\theta(i,0) = 0$ for each $i < k$ and is there just for notational uniformity.) Take some $\epsilon > 0$ such that $|\theta(i,1)|, |\frac{1}{2} - \theta(i,1)| > \epsilon$ for each $i < k$. Let $d = \sup\{d(i,1);\, i < k\}$. Take N such that $\frac{1}{N} < \frac{\epsilon}{10 \cdot d}$, and let $n = n(N, a_0)$.

By applying the conclusion of Theorem 13, with a suitable Ψ, we get a, $b_m (m < n)$ satisfying (1) such that $|r_{a(i)} - r_{a_0(i)}| < \frac{1}{N}$, $|r_{b_m(j)} - r_{b_0(j)}| < \frac{1}{N}$ and $n(N,a) = n(N,a_0)$. Define

$$u_i = \mathrm{osc}(a(i), b_0(0)) \cdot r_{a(i)} + r_{b_0(0)}.$$

Let

$$v_i' = \begin{cases} \frac{\epsilon}{2} & \phi(i,0) = 0, \phi(i,1) = 0; \\ \frac{1}{2} - \frac{\epsilon}{2} & \phi(i,0) = 0, \phi(i,1) = 1; \\ 1 - \frac{\epsilon}{2} & \phi(i,0) = 1, \phi(i,1) = 0; \\ \frac{1}{2} + \frac{\epsilon}{2} & \phi(i,0) = 1, \phi(i,1) = 1. \end{cases}$$

Define $v_i = v_i'$ if $0 < \theta(i,1) < \frac{1}{2}$, and $v_i = \frac{1}{2} - v_i'$ if $\frac{1}{2} < \theta(i,1) < 1$. The point is that, as can be verified, for any x,

$$\text{if } |x - (v_i + \theta(i,j))| < \frac{\epsilon}{2} \text{ then } x \in [0, \frac{1}{2}) \text{ iff } \phi(i,j) = 0. \tag{2}$$

Since $n(N,a) = n$, there is some $m < n$ such that for each $i < k$,

$$|u_i + m \cdot r_{a(i)} - v_i| < \frac{1}{N}. \tag{3}$$

It is now left to verify that a and b_m satisfy the conclusion of the theorem:

$$\begin{aligned} f(a(i), b_m(j)) &= \mathrm{osc}(a(i), b_m(j)) \cdot r_{a(i)} + r_{b_m(j)} \\ &= (\mathrm{osc}(a(i), b_0(0)) + m + d(i,j)) \cdot r_{a(i)} + r_{b_0(0)} + r_{b_m(j)} - r_{b_0(0)} \\ &= u_i + m \cdot r_{a(i)} + d(i,j) \cdot r_{a(i)} + (r_{b_m(j)} - r_{b_0(j)}) + (r_{b_0(j)} - r_{b_0(0)}) \\ &= u_i + m \cdot r_{a(i)} + \theta(i,j) + d(i,j)(r_{a(i)} - r_{a_0(i)}) + (r_{b_m(j)} - r_{b_0(j)}). \end{aligned}$$

Therefore

$$|f(a(i), b_m(j)) - (v_i + \theta(i,j))| \leq |u_i + m \cdot r_{a(i)} - v_i| + d(i,j)|(r_{a(i)} - r_{a_0(i)})| + |r_{b_m(j)} - r_{b_0(j)}|.$$

So by (3) and the choice of N,

$$|f(a(i), b_m(j)) - (v_i + \theta(i,j))| \leq \frac{2+d}{N} < \frac{3d \cdot \epsilon}{10d} < \frac{\epsilon}{2}.$$

Finally, by (2), $f(a(i), b_m(j)) \in [0, \frac{1}{2})$ iff $\phi(i,j) = 0$, thus $c(a(i), b_m(j)) = \phi(i,j)$. □

Corollary 15. There is an L space whose square is also an L space.

The corollary follows directly from Theorem 6 and Theorem 14 above.

The method above does not seem to generalize to solve the problem with arbitrary exponent l. A general solution is given in section 5 below.

5. Higher exponents

The following proposition shows the equivalence of $\omega_1 \to [\operatorname{fin}\omega_1; [\omega_1]^l]^2_\omega$ (which is the negation of (a) below), and the weaker statement, when we ask to avoid some pattern $\phi\colon k \times l \to \omega$. In this case, as shown later in this section, both relations are false. We formulate it in terms of an equivalence since the proof works in more general settings, for similar partition relations which are not provably false, such as $\omega_1 \to [[\omega_1]^k; \operatorname{fin}\omega_1]^2_\omega$. The failure of such partition relation with general patterns ϕ is often the one necessary for certain constructions, as it is in our case to construct the relevant L spaces.

Proposition 16. Fix an integer l. The following are equivalent:

(a) There is a coloring $\chi\colon [\omega_1]^2 \to \omega$ such that if $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ are uncountable families of pairwise disjoint sets and $t \in \omega$, then there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a < b$ and for any $i < k$ and $j < l$, $\chi(a(i), b(j)) = t$.

(b) There is a coloring $c\colon [\omega_1]^2 \to \omega$ such that if $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ are uncountable families of pairwise disjoint sets and $\phi\colon k \times l \to \omega$, then there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a < b$ and for any $i < k$ and $j < l$, $c(a(i), b(j)) = \phi(i, j)$.

Proof. The first statement is the restriction of the second to constant functions ϕ, so the converse direction is clear.

For the forward implication, we first construct a function $\varphi\colon \mathbb{R}^+ \to \omega$ satisfying the following: For any finite sequence of pairwise disjoint rational intervals $q(0), ..., q(k-1)$ and any $\phi\colon k \to \omega$ there is some $t \in \mathbb{N}$ such that

$$\text{for any } i < k \text{ if } x \in q(i) \text{ then } \varphi(t + x) = \phi(i). \tag{4}$$

This can be done by fixing an enumeration of all pairs $\langle q, \phi\rangle$ as above, and defining φ by ω many steps where at each step it is only defined on a bounded segment of $\mathbb{R}^+$.

Let $\chi\colon [\omega_1]^2 \to \omega$ witness that $\omega_1 \not\to [\operatorname{fin}\omega_1; [\omega_1]^l]^2_\omega$, as in (a). Define $c\colon [\omega_1]^2 \to \omega$ by

$$c(\alpha, \beta) = \varphi(\chi(\alpha, \beta) + r_\alpha + r_\beta).$$

Suppose $\mathcal{A} \subset [\omega_1]^k$, $\mathcal{B} \subset [\omega_1]^l$ and $\phi\colon k \times l \to \omega$ are as in (b). By thinning out we may assume that $a \cap b = \emptyset$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Let M be a countable elementary submodel of $H(\aleph_2)$ containing $\mathcal{A}$, $\mathcal{B}$, χ, Q and the sequence $\langle r_\alpha;\, \alpha < \omega_1\rangle$. Take some $a_0 \in \mathcal{A} \setminus M$ and $b_0 \in \mathcal{B} \setminus M$.

Note that since $\{r_\alpha;\, \alpha < \omega_1\}$ are rationally independent, then $r_{a_0(i)} + r_{b_0(j)} \neq r_{a_0(i')} + r_{b_0(j')}$ whenever $(i, j) \neq (i', j')$. Let $q(i, j)\, (i < k, j < l)$ be pairwise disjoint rational intervals such that $r_{a_0(i)} + r_{b_0(j)} \in q(i, j)$. Take a rational $\epsilon > 0$ small enough such that for any $i < k$ and $j < l$, the 2ϵ-interval around $r_{a_0(i)} + r_{b_0(j)}$ is contained in $q(i, j)$. Fix sequences of rationals $x \in Q^k$ and $y \in Q^l$ such that $|r_{a(i)} - x(i)|, |r_{b(j)} - y(j)| < \epsilon$ for any $i < k$ and $j < l$.

Take an integer t such that (4) holds for $q(i, j)\, (i < k, j < l)$ and ϕ. Let $\mathcal{A}' = \big\{a \in \mathcal{A};\, (\forall i < k)\, |r_{a(i)} - x(i)| < \epsilon\big\}$ and $\mathcal{B}' = \big\{b \in \mathcal{B};\, (\forall j < l)\, |r_{b(j)} - y(j)| < \epsilon\big\}$. Note that $x, y, \epsilon \in M$, since $Q \subset M$. Therefore $\mathcal{A}'$ and $\mathcal{B}'$ are in M, and are uncountable since $a_0 \in \mathcal{A}'$ and $b_0 \in \mathcal{B}'$.

By the choice of χ, there are $a \in \mathcal{A}'$ and $b \in \mathcal{B}'$ such that $a < b$ and for all $i < k$ and $j < l$, $\chi(a(i), b(j)) = t$. Note that $|(r_{a(i)} + r_{b(j)}) - (r_{a_0(i)} + r_{b_0(j)})| < 2\epsilon$, thus by the choice of ϵ, $r_{a(i)} + r_{b(j)} \in q(i,j)$. Finally, by the choice of t,

$$c(a(i), b(j)) = \varphi(t + r_{a(i)} + r_{b(j)}) = \phi(i,j).$$

□

Question 2. Are the following two statements equivalent?

- There is a coloring $\chi\colon [\omega_1]^2 \to 2$ such that if $\mathcal{A} \subset \omega_1$ and $\mathcal{B} \subset [\omega_1]^l$ are uncountable families of pairwise disjoint sets and $t \in 2$, then there are $\alpha \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\alpha < b$ and for any $j < l$, $\chi(\alpha, b(j)) = t$.
- There is a coloring $c\colon [\omega_1]^2 \to 2$ such that if $\mathcal{A} \subset \omega_1$ and $\mathcal{B} \subset [\omega_1]^l$ are uncountable families of pairwise disjoint sets and $\phi\colon l \to 2$, then there are $\alpha \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\alpha < b$ and for any $j < l$, $c(\alpha, b(j)) = \phi(j)$.

As mentioned above, Proposition 16 shows that the two statements are equivalent if 2 is replaced by ω. A related question is:

Question 3. Are $\omega_1 \to [\omega_1; \operatorname{fin}\omega_1]^2_\omega$ and $\omega_1 \to (\omega_1; \operatorname{fin}\omega_1)^2_2$ equivalent?

Our main result, Theorem 7, states that clause (b) from Proposition 16 holds for every integer l. By Proposition 16, the following theorem will finish the proof.

Theorem 17. For any integer l, there is a coloring $\chi\colon [\omega_1]^2 \to \omega$ such that for every $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ which are uncountable families of pairwise disjoint sets and for any $t \in \omega$ there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a < b$ and $\chi(a(i), b(j)) = t$ for every $i < k$ and $j < l$.

Fix some integer l. The following auxiliary partitions will be used to prove the theorem.

Lemma 18. There is a sequence $\bar{\varphi} = \langle \varphi_\alpha;\, \alpha < \omega_1 \rangle$ of functions $\varphi_\alpha\colon I \to \omega$ satisfying that: For any $\alpha < \omega_1$, $d \in \omega^l$ and $t \in \omega$, there is some $\epsilon(\alpha, d, t) > 0$ and $q(\alpha, d, t)$ both in Q such that for any $x \in I$ and $j < l$,

$$\text{if } |x - (q(\alpha,d,t) + d(j) \cdot r_\alpha)| < \epsilon(\alpha,d,t) \text{ then } \varphi_\alpha(x) = t. \tag{5}$$

(Note that d could be constant.)

Proof. Let $\langle d_n, t_n;\, n < \omega \rangle$ enumerate $\omega^l \times \omega$. Fix some $\alpha < \omega_1$. Define φ_α in ω stages, where at stage n it is defined on some set A_n of measure $< \frac{1}{l}$, where A_n is a finite union of intervals.

Suppose φ_α is defined on A_{n-1}. Let $B = \bigcup_{j<l}(A_{n-1} - d_n(j) \cdot r_\alpha)$, the union of l many shifts of A_{n-1}. Then the measure of B is < 1 and B is also a union of finitely many intervals. Thus there is some $q \in Q$ and $\epsilon > 0$ such that the interval $(q - \epsilon, q + \epsilon)$ is disjoint from B. So for each $j < l$ the ϵ-interval around $q + d_n(j) \cdot r_\alpha$ is disjoint from A_{n-1}.

We add these intervals, with some possibly smaller ϵ, to get A_n, so that the measure of A_n is $< \frac{1}{l}$. Define φ_α to take the value t_n on these intervals. This finishes the construction, and it can be verified that $\langle \varphi_\alpha;\ \alpha < \omega_1 \rangle$ satisfy the conclusion of the lemma. (For the remaining undefined points let φ_α take the value 0.) □

Proof of Theorem 17. Define $\chi\colon [\omega_1]^2 \to \omega$ by

$$\chi(\alpha, \beta) = \varphi_\alpha(\mathrm{osc}(\alpha, \beta) \cdot r_\alpha).$$

Suppose $\mathcal{A}$ and $\mathcal{B}$ are as in the statement of the theorem. Apply Theorem 13 with $\mathcal{A}, \mathcal{B}$ and $X = \mathbb{N} \cup Q \cup \{\bar{\varphi}\}$ to get $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$ and $d \in \omega^k \times \omega^l$. Fix some $t \in \omega$.

For each $i < k$ define $d_i \in \omega^l$ by $d_i(j) = d(i, j)$, let $q_i = q(a_0(i), d_i, t)$, $\epsilon_i = \epsilon(a_0(i), d_i, t)$ and take N such that $\frac{1}{N} < \min\{\epsilon_i;\ i < k\}$. Let $n = n(N, a_0)$ (as defined in the beginning of section 4).

Take a finite set of formulas Ψ, using the parameters $\{\bar{\varphi}, N, n, t, d_i, q_i, \epsilon_i;\ i < k\}$, such that the following holds: by applying the conclusion of Theorem 13 with n and Ψ, we get $a, b_0, ..., b_n$ satisfying (1) and such that $n(N, a) = n$, $q(a(i), d_i, t) = q_i$ and $\epsilon(a(i), d_i, t) = \epsilon_i$ for any $i < k$.

Let $u_i = \mathrm{osc}(a(i), b_0(0)) \cdot r_{a(i)}$. Take $m < n$ such that for each $i < k$,

$$|u_i + m \cdot r_{a(i)} - q_i| < \frac{1}{N}.$$

Let $x(i, j) = \mathrm{osc}(a(i), b_m(j)) \cdot r_{a(i)}$. It remains to show that a, b_m satisfy the conclusion of the theorem, that is, to show that $\varphi_{a(i)}(x(i, j)) = t$. For any $i < k$ and $j < l$,

$$\mathrm{osc}(a(i), b_m(j)) = \mathrm{osc}(a(i), b_0(0)) + m + d_i(j),$$

and so

$$x(i, j) = u_i + m \cdot r_{a(i)} + d_i(j) \cdot r_{a(i)}.$$

Thus

$$|x(i, j) - (q_i + d_i(j) \cdot r_{a(i)})| < \frac{1}{N} < \epsilon_i.$$

Finally, as $q(a(i), d_i, t) = q_i$ and $\epsilon(a(i), d_i, t) = \epsilon_i$, it follows from condition (5) that $\varphi_{a(i)}(x(i, j)) = t$, as required. □

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