The important technical lemma:

Lemma 1. *(Magidor) Let ρ be a regular cardinal, κ,λ cardinals s.t $\rho<\kappa<\lambda$. Suppose that in $V^{\mathrm{Col}(\rho,<\kappa)}$, P is a ρ-closed poset and $|P|<\lambda$. Let i be the identity complete embedding of $\mathrm{Col}(\rho,<\kappa)$ into $\mathrm{Col}(\rho,<\lambda)$. Then i can be extended to a complete embedding j of $\mathrm{Col}(\rho,<\kappa)*P$ into $\mathrm{Col}(\rho,<\lambda)$ such that the quotient forcing ${}^{\mathrm{Col}(\rho,<\lambda)}/_{j[\mathrm{Col}(\rho,<\kappa)*P]}$ is ρ-closed in $V^{j[\mathrm{Col}(\rho,<\kappa)*P]}$.*

The use of this lemma is by establishing indestructibility, under closed enough posets, to certain reflection properties, when dealing with non-large cardinals.

Lemma 2. *Suppose κ is a regular cardinal, $\kappa<\mu$ is a measurable cardinal, and we force with $\mathrm{Col}(\kappa,<\mu)$. In the resulting model, $\mu=\kappa^+$ and the following statement is true, even after a further forcing extension by $\mathrm{Col}(\kappa,\mu)$:*

For any $\lambda<\kappa$, every coherent sequence $\mathscr{C}=\langle\mathscr{C}_\alpha:\alpha<\mu\rangle$, where $|\mathscr{C}_\alpha|<\lambda$, has a thread.

Claim 3. Suppose Q is a poset and τ is a Q-name s.t $Q\Vdash\tau\notin\check{V}$. Let $G\times H$ be a $Q\times Q$-generic over V, let

$$a=\tau^G,\quad b=\tau^H.$$

Then in $V[G\times H]$, $a\neq b$.

Main lemma for separating squares:

Lemma 4. *Suppose $\lambda<\kappa$, $\mathscr{C}=\langle C_\alpha;\alpha<\kappa^+\rangle$ is a $\square_{\kappa,<\lambda}$-sequence and P is a poset satisfying*

$$P^\lambda \text{ is } \lambda^+\text{-distributive}.$$

Then P does not thread $\mathscr{C}$.

Proof. Suppose otherwise, that there is a P-name τ s.t P forces τ is a thread through $\mathscr{C}$. Let $\prod_{\alpha<\lambda}G_\alpha$ be a P^λ-generic over V and denote $D_\alpha=\tau^{G_\alpha}$. Since $\mathscr{C}$ has no thread in V, by claim 3 we have that $\langle D_\alpha;\alpha<\lambda\rangle$ is a sequence of distinct clubs threading $\mathscr{C}$. However, as P^λ is λ^+-distributive, $\mathrm{cf}(\kappa)\geq\lambda^+$ in the generic extension. Thus $D=\bigcap_{\alpha<\lambda}D_\alpha$ is a club, and we can find a $\beta\in D$ s.t below β all the D_α's are distinct. Now

$$\forall\alpha<\lambda\left(\beta\in D_\alpha\implies D_\alpha\cap\beta\in C_\beta\right),$$

in contradiction to $\left|C_\beta\right|<\lambda$. □

Remark. In particular, if P is κ-closed, then P does not thread $\mathscr{C}$.

Now we can prove lemma 2:

Proof. (Lemma 2) Let $j\colon V\longrightarrow M$ be an elementary embedding corresponding to the measurable cardinal μ. Let $G\subset\mathrm{Col}(\kappa,<\mu)$ be generic over V and $H\subset Q=\mathrm{Col}(\kappa,\mu)$ generic over $V[G]$. Write

$$j(\mathrm{Col}(\kappa<\mu)*Q)=\mathrm{Col}(\kappa,<j(\mu))^M*j(Q)=\mathrm{Col}(\kappa,<\mu)*\mathrm{Col}(\kappa,[\mu,j(\mu)))^M*j(Q).$$

By lemma 1, we have

$$j(\mathrm{Col}(\kappa<\mu)*Q)=\mathrm{Col}(\kappa,<\mu)*Q*R*j(Q),$$

where R is κ-closed (and so is $j(Q)$). The definition of R can be carried out inside M - this is possible since $M^{\mu} \subset M$ and we can assume, without loss of generality, that Q is coded as a partial order on μ. Note that all those forcings are κ-closed in M, hence also κ-closed as posets in V.

In $M[G][H]$, we have $H = j''H \subset j(Q)$ and $j(Q) = \mathrm{Col}(\kappa, j(\mu)) = \mathrm{Col}(\mu, j(\mu))$. Hence we can take $q = \bigcup H \in \mathrm{Col}(\mu, j(\mu))$ as a master condition. Let K be $R * j(Q)$-generic over $V[G][H]$ containing q. Let $V^* = V[G*H*K]$. In V^* we can define an embedding

$$\tilde{j}\colon V[G*H] \longrightarrow M[G*H*K]$$

which extends j.

Now assume that $\mathscr{C}$ is a coherent sequence of length μ in $V[G*H]$. By the above, $\mathscr{C}$ has a thread in V^* (since any element of $\tilde{j}(\mathscr{C})_{\mu}$ will thread $\mathscr{C}$). But V^* is a κ-closed extension of $V[G*H]$, in contradiction to lemma 4. □

Henceforth we assume we work in a model as described in lemma 2, that is, after collapsing the measurable.

Lemma 5. *Suppose $S_{<\lambda}$ is the poset for adding a $\square_{\kappa,<\lambda}$-sequence, T is the threading poset. Then*

$$\forall \eta < \lambda \, (T^{\eta} \text{ is } \kappa\text{-distributive}).$$

(clearly by the lemma before T^{λ} is not κ-distributive.)

This follows from the following proposition, which states that for $\alpha < \lambda$, $S * T^{\alpha}$ is forcing isomorphic to a κ-closed poset.

Proposition. *Define $E \subset S * T^{\alpha}$,*

$$E = \left\{ \left(p, \langle d_{\xi}; \xi < \alpha \rangle\right); p = \langle C_{\alpha}; \alpha \leq \beta \rangle \in S \wedge \forall \xi < \alpha \left(p \Vdash \left(d_{\xi} \in T\right) \wedge \max d_{\xi} = \beta\right)\right\}.$$

Then E is dense and κ-closed.

Proof. E is κ-closed: Suppose $\left\langle \left(p^{\eta}, \overrightarrow{d^{\eta}}\right); \eta < \mu \right\rangle$ is a descending sequence of length $\mu < \kappa$, where $\overrightarrow{d^{\eta}} = \left\langle d^{\eta}_{\xi}; \xi < \alpha \right\rangle$. Define

$$d_{\xi} = \bigcup_{\eta<\mu} d^{\eta}_{\xi}, \quad q = \left\{d_{\xi}; \xi < \alpha\right\}, \quad p = \bigcup_{\eta<\mu} p^{\eta} \frown q.$$

Thus $\left(p, \overrightarrow{d}\right)$ is a lower bound of the sequence, and in E.

E is dense: Take a condition $\left(p, \overrightarrow{d}\right) \in S * T^{\alpha}$, $\overrightarrow{d} = \langle d_{\xi}; \xi < \alpha \rangle$, $p = \langle C_{\alpha}; \alpha \leq \beta \rangle$. W.l.o.g., $\beta \geq \max d_{\xi}$ for all $\xi < \alpha$. Define

$$e_{\xi} = d_{\xi} \cup (\beta + \omega \setminus \beta + 1).$$

Let $C_{\beta+\omega} = \left\{e_{\xi}; \xi < \alpha\right\}$, and let $q = \langle C_{\alpha}; \alpha \leq \beta + \omega \rangle$, $\overrightarrow{e} = \langle e_{\xi}; \xi < \alpha \rangle$. Then $(q, \overrightarrow{e}) \in E$ and $(q, \overrightarrow{e}) \leq \left(p, \overrightarrow{d}\right)$. □

Remark. The lemma for $\lambda = \nu^{+}$ says that if S and T add and destroy a $\square_{\kappa,\nu}$-sequence, then T^{ν} is κ-distributive (and $T^{\nu^{+}}$ is not).

A quick corollary is:

Corollary 6. *Separating squares*

Proof. We force with $S_{\kappa,\lambda}$, thus $\square_{\kappa,\lambda}$ holds. We want to show that $\square_{\kappa,<\lambda}$ fails. Assume by contradiction that there is some $\square_{\kappa,<\lambda}$ seuquence $\mathscr{C}$. Note that after forcing with T we get a $S_\lambda * T$-generic extension of V, and $S_\lambda * T$ is κ closed and collapses κ^+ to κ, thus $S_\lambda * T$ is forcing isomorphic to $\mathrm{Col}(\kappa,\kappa^+)$. Hence, by lemma 2, after forcing with T, $\mathscr{C}$ must have a thread. Thus forcing with T threaded $\mathscr{C}$, in contradiction to lemma 4, as T satisfies "T^λ is κ-distributive". □

Similarly, another easy application:

Proposition 7. *(The fly swatter) A non reflecting stationary subset of κ does not imply $\square(\kappa)$.*

Proof. Start with κ indestructible (under κ-Cohen forcing, for the property $\neg\square(\kappa)$). Let A and D be the forcing for adding and destroying a non reflecting stationary subset, $S \subset A$ generic. Then in $V[S]$ we have D^2 is κ-distributive, hence cannot destroy a $\square(\kappa)$-sequence, thus there is no $\square(\kappa)$-sequence in $V(S)$. Similarly, a non reflecting stationary subset does not imply $\square(\kappa,\lambda)$ for any $\lambda < \kappa$. □

And, for example:

Proposition 8. $\square_{\kappa,2}$ *does not imply* $\square(\kappa^+)$.

Proof. A $\square(\kappa^+)$-sequence cannot be destroyed by a forcing s.t P^2 is κ-distributive. □

And also

References

[1] On the strengths and weaknesses of weak squares