$0^\sharp$ AND SPECIALIZING TREES IN L

In this note we prove:

Theorem 1. *Assume $0^\sharp$ exists. Suppose κ is a cardinal, and κ^+ its successor (both as calculated in V). Let T be a tree in the constructible universe L of height κ^+ and size $\leq \kappa^+$, and assume that T has no branch (in L). Then T is special in V.*

Under these assumption, κ^+ has the tree property in L. The point is that we allow the trees to be fat.

Theorem 1 has the following strong converse, which follows from a result of Shelah and Stanley.

Theorem 2. *If $0^\sharp$ does not exist, there is a cardinal κ and a tree T in L such that T is a non special κ^+-Aronszajn tree in V.*

We mention the following theorem:

Theorem (Foreman, Magidor and Shelah [1])**.** *Assume $0^\sharp$ exists. For any non trivial poset $P \in L$, forcing with P adds a real.*

It is an open question whether this statement is equivalent to $0^\sharp$.

Assume $0^\sharp$ exists. We will use the following standard facts about the canonical class of indiscernibles I.

Fact 1 (See 9.8 in Kanamori)**.** *Let σ be an n-ary canonical term and $\alpha_0 < ... < \alpha_n$ ordinals from I. If $\sigma(\alpha_0, ..., \alpha_{n-1})$ is an ordinal then $\sigma(\alpha_0, ..., \alpha_{n-1}) < \alpha_n$.*

Fact 2 (The remarkable condition. See 9.10 in Kanamori)**.** *Let σ be an $(m+n+1)$-ary term, $\alpha_0 < ... < \alpha_{m+n}$ and $\alpha_{m-1} < \beta_m < ... < \beta_{m+n}$ all in I. Then $\sigma(\alpha_0, ..., \alpha_{m+n}) = \sigma(\alpha_0, ..., \alpha_{m-1}, \beta_m, ..., \beta_{m+n})$.*

Lemma 1 (See Jech)**.** *Assume $0^\sharp$ exists, and I is the canonical class of indiscernibles. Let κ be an ordinal such that $V \models \mathrm{cf}\kappa > \omega$ and $L \models \kappa$ is regular. Then κ is a limit point of I.*

Lemma 2 (See Jech 18.3)**.** *Assuming $0^\sharp$ exists, if κ is any ordinal such that there is some L-fresh subset of κ, then $\mathrm{cf}\kappa = \omega$. (By an L-fresh subset of κ we mean a set $X \subset \kappa$ such that $X \cap \alpha \in L$ for each $\alpha < \kappa$, yet $X \notin L$.)*

We will also use the following two simple lemmas:

Lemma 3. *Suppose $T = \bigcup_{\alpha \in \kappa} T_\alpha$ and for each α there is a function $f_\alpha \colon T_\alpha \longrightarrow \kappa$ which is specializing on T_α. Then there is a sepcializing function $f \colon T \longrightarrow \kappa$.*

Lemma 4. *Suppose $T = \bigsqcup_{x \in X} T_x$ is a disjoint union, where X is a set. Assume that given two different $x, y \in X$, for any $s \in T_x$ and $t \in T_y$, $s \perp t$. Assume also that for each x there is a function $f_x \colon T_x \longrightarrow \kappa$ which is specializing on T_x. Then there is a specializing function $f \colon T \longrightarrow \kappa$.*

We now begin the proof of theorem 1.

Let $T \in L$ be a tree on κ^+ of size $\leq \kappa^+$, with no branch in L. We may assume that T is a subset of the ordinal κ^+. By lemma 1, κ^+ is a limit point of I. Note that, since $T \in L$, a branch in T is an L-fresh subset of κ^+. By lemma 2, T has no branch in V as well.

Fix a term τ and sequences of ordinals $\bar{u}, \bar{v}$ such that $T = \tau(\bar{u}, \bar{v})$, where $\bar{u} < \kappa^+$ and $\min \bar{v} = \kappa^+$. Let $u = \sup \bar{u} < \kappa^+$.

For each $t \in T$, fix a term σ_t and a sequence of ordinals $\bar{x}$ such that $t = \sigma(\bar{x})$. Write $\bar{x} = \bar{a}_t, \bar{b}_t, \bar{c}_t$ where $\bar{a}_t \leq u < \bar{b}_t < \kappa^+ \leq \bar{c}_t$. Since $t \in \kappa^+$, by fact 2 we may assume that $\bar{c}_t$ are consecutive elements of I and $\min \bar{c}_t = \kappa^+$. Let Γ_t be the order type of $\bar{b}_t$ (i.e. the relationship between the order in the sequence and the order as ordinals).

There are countably many terms σ, countably many possible order types Γ, and countably many possible lengths of the sequence $\bar{c}$. There are at most κ many options for the sequence $\bar{a}$, as it is bounded by $u < \kappa^+$. So we can partition T into at most κ many subsets on which $\sigma, \bar{a}, \bar{c}$ and Γ are constants. By lemma 3, it suffices to work on each such subset. Thus the following lemma will conclude the theorem:

Lemma 5. *Let $S \subset T$, σ a term, $\bar{d}, \bar{c} \subset I$ such that $\bar{d} < \kappa^+ \leq \bar{c}$, k an integer and Γ an order type of k elements. Assume that all the elements of S are of the form $\sigma(\bar{d}, \bar{x}, \bar{c})$ where $\bar{x} \subset I$, $u \leq \max \bar{d} < \bar{x} < \kappa^+$ and the order type of $\bar{x}$ is Γ. Then S can be partitioned into κ many antichains.*

Proof. By induction, assume the claim is true for k and that the conditions of the lemma are satisfied with $k + 1$.

Let $Z \subset S$ be the set of all the minimal elements with respect to the tree relation. For $z \in Z$, define $S_z = \{t \in S;\, t \geq_T z\}$. By lemma 4, it suffices to partition each S_z to κ many antichains, so fix a z and assume $S = S_z$. i.e. for each $y \in S$, $z \leq_T y$. Let $\bar{x}$ be such that $z = \sigma(\bar{d}, \bar{x}, \bar{c})$.

Claim 1. *If $y = \sigma(\bar{d}, \bar{y}, \bar{c}) \in S$ then $\min \bar{y} \leq \max \bar{x}$.*

Proof. Assume otherwise. Fix a sequence $\langle \bar{y}_\xi;\, \xi < \kappa^+ \rangle$ such that $\bar{y}_\xi \subset I$ has the same order type of $\bar{y}$ and for $\xi < \zeta < \kappa^+$, $\sup \bar{y}_\xi < \min \bar{y}_\zeta$. By assumption,

$$\sigma(\bar{d}, \bar{x}, \bar{c}), \sigma(\bar{d}, \bar{y}, \bar{c}) \in \tau(\bar{u}, \bar{v}),\ \sigma(\bar{d}, \bar{x}, \bar{c}) \leq_{\tau(\bar{u}, \bar{v})} \sigma(\bar{d}, \bar{y}, \bar{c})$$

where $\tau(\bar{u}, \bar{v}) = T$. Also, $\bar{x}, \bar{y}$ have the same order type and $\sup \bar{x} < \min \bar{y}$.

Since $\bar{y}, \bar{y}_\xi$ are all above $\bar{u}$ and below $\bar{v}$, then by indiscernibility we get: for every $\xi < \kappa^+$, $\sigma(\bar{d}, \bar{y}_\xi, \bar{c}) \in T$, and $\sigma(\bar{d}, \bar{y}_\xi, \bar{c}) \leq_T \sigma(\bar{d}, \bar{y}_\zeta, \bar{c})$ for every $\xi < \zeta < \kappa^+$. Thus the set $\left\{\sigma(\bar{d}, \bar{y}_\xi, \bar{c});\, \xi < \kappa^+\right\}$ generates a branch in T of order type κ^+, a contradiction. □

Let $\Xi = \left\{\min \bar{y};\, \sigma(\bar{d}, \bar{y}, \bar{c}) \in S\right\}$. By the claim, Ξ is bounded by $\max \bar{x}$, thus Ξ is of size $\leq \kappa$. For $\xi \in \Xi$, let $S_\xi = \left\{\sigma(\bar{d}, \bar{y}, \bar{c});\, \sigma(\bar{d}, \bar{y}, \bar{c}) \in S \wedge \min \bar{y} = \xi\right\}$. By lemma 3, it suffices to partition each S_ξ into κ many antichains. Let $\bar{d}' = \bar{d}^\frown \xi$. Then S_ξ satisfies the hypothesis of the lemma 5 with $\bar{d}'$ and k. By the inductive hypothesis, we can partition S_ξ into κ many antichains. □

This finishes the proof of theorem 1.

Recall the following result of Shelah and Stanley [3]:

Theorem ([3])**.** *Let $\mathcal{C} = \langle C_\alpha;\ \alpha < \kappa^+\rangle$ be a $\square(\kappa^+)$-sequence, and let $S \subset \kappa^+$ be stationary s.t. $S \subset \mathrm{cof}(\lambda)$ and S is disjoint to the limit points of $\mathcal{C}$, i.e. for any $\alpha < \kappa^+$, $S \cap C'_\alpha = \emptyset$. Then there is a κ^+-tree T with a λ-ascent path and a weakly specializing function f defined on the levels in S. i.e., f is defined on the levels T_α for $\alpha \in S$ and gives values below α, and f satisfies the specializing condition.*

Since T is weakly special, it is Aronszajn. Since T has an ascent path, it is not special (see [3]*).*

proof of theorem 2. Assume that $0^\sharp$ does not exists. Let κ be a strong limit singular cardinal of uncountable cofinality, so that V and L agree on κ^+. In L, $\square_\kappa$ holds, so, by Shelah-Stanley, there is a κ^+-aronszajn tree with an ω-ascent path. Note that the ascent path remains so in any outer model, thus the tree is not special in V as well. This gives a weak version of theorem 2 (i.e. only that T has no branch in L).

For the stronger version, we want a tree which is also Aronszajn in V. So we need a weakly specializing function defined on a set which is stationary in V.

Let $\mathcal{C}$ be the $\square_\kappa$ sequence in L. Recall the standard construction of a square sequence with a stationary set disjoint from its limit points (see [2]):

Consider the regressive function f defined by $f(\alpha) = \mathrm{otp}(C_\alpha)$ for α with $\mathrm{cf}\alpha = \omega$. Find, in V, a fixed value μ on a stationary $S \subset \{\alpha \in \kappa^+;\ \mathrm{cf}\alpha = \omega\}$. Since $\mathcal{C}$ and f are in L, the set $S' = \{\alpha;\ f(\alpha) = \mu\}$ is in L, and contains S, hence is stationary in V. Now $S' \cap \lim(C_\alpha)$ contains at most one element, for each $\kappa < \alpha < \kappa^+$. Let $D_\alpha = C_\alpha \setminus \gamma$ for the unique $\gamma \in \lim C_\alpha \cap S'$. Then S', $\mathcal{D}$ are in L, S' is stationary in V, $\mathcal{D}$ is a $\square(\kappa^+)$-sequence and S' is disjoint from the limit points of $\mathcal{D}$.

We can now apply the Shelah-Stanley construction to get a κ^+-tree in L with an ascent path which is weakly special on a V-stationary subset, thus has no branch in V. □

References

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[2] Magidor, M., Lambie-Hanson, C.: On the strengths and weaknesses of weak squares. Appalachian Set Theory: 2006-2012. Cambridge University Press, 2012.

[3] Shelah, S., Stanley, L.: Weakly compact cardinals and nonspecial Aronszajn trees. Proc. Amer. Math. Soc., 104(3):887-897, 1988.

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