UNPINNED ACTIONS VIA METRIC SCOTT ANALYSIS

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Abstract. We prove Thompson's theorem, that any non-CLI Polish group induces an unpinned orbit equivalence relation, using the metric Scott analysis.

Theorem 0.1 (Hjorth-Thompson)**.** Let G be a Polish group. The following are equivalent.

(1) All orbit equivalence relations induced by G are pinned.
(2) G admits a complete left invariant metric. ("G is CLI".)

The direction (2) $\Longrightarrow$ (1) is due to Hjorth [Hjo99] (see also [Kan08, 17.4]). The direction (1) $\Longrightarrow$ (2) is due to Thompson [Tho06]. One corollary of Thompson's theorem is that for a Polish group G, if G is not CLI, then there is an orbit equivalence relation induced by G which is not essentially countable. This established a partial answer to a question of Kechris (see [KMPZ20, Problem 1.2]).

We give here a short proof of Thompson's theorem, relying on the metric Scott analysis of Ben Yaacov, Doucha, Nies, and Tsankov [BYDNT17], and their generalization of Gao's theorem to automorphism groups of metric structures.

Let E be an equivalence relation on a (Polish) space X. Recall that $(\mathbb{P}, \tau)$ is an **E-pin** if $\mathbb{P}$ is a forcing poset, τ is a $\mathbb{P}$-name forced to be in X and $\mathbb{P} \times \mathbb{P}$ forces that $\tau_{\text{left}}\ E\ \tau_{\text{right}}$. Equivalently, for any two generics G, H over V, $\tau[G]$ and $\tau[H]$ are in the same E-class (in any model that contains them both). An E-pin is **trivial** if there is some $x \in X$ in the ground model so that $\mathbb{P} \Vdash \tau\ E\ \check{x}$. An equivalence relation is said to be **pinned** if all of its pins are trivial.

Fix a separable metric structure M in a countable relational language L, and consider the group $G = \mathrm{Aut}(M)$. We define an equivalence relation E induced by a continuous action of G on a Polish space.

Let X_0 be the space of all 1-Lipschitz function from M to $[0, 1]$, and consider the product Polish space $X = X_0^\omega$. $G = \mathrm{Aut}(M)$ acts naturally on X_0 by $g \cdot \varphi(x) = \varphi(g^{-1} \cdot x)$ for any $\varphi \in X_0$. Note that $g \cdot \varphi$ is Lipschitz since g is an isometry of M. Let $a\colon G \curvearrowright X$ be the diagonal action, and let $E = E_a$ be the induced orbit equivalence relation.

Lemma 0.2. Let M be a separable metric structure. Assume that there is a non-separable model N for the Scott sentence of M. Then E is not pinned.

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Proof. Fix a dense set $D \subseteq N$. Let $\mathbb{P}$ be the poset adding a countable enumeration of D by finite approximations. In the $\mathbb{P}$-generic extension, the completion of N is a separable complete metric structure satisfying the Scott sentence of M, and is therefore isomorphic to M. Fix a $\mathbb{P}$-name $\dot{f}$ for an isomorphism from (the completions of) M to N.

Since N is not separable we may find $\epsilon > 0$ and $\{a_i : i < \omega_1\} \subseteq N$ so that $d(a_\alpha, a_\beta) \geq \epsilon$ for any $\alpha \neq \beta$. Fix a sequence $\langle r_\alpha : \alpha < \omega_1 \rangle$ of distinct members of 2^ω. As in [Tho06, p.1121], define $\varphi_i \colon N \to \mathbb{R}$, for $i \in \omega$, by

$$\varphi_i(x) = \inf \{ d(x, a_\alpha) : \alpha < \omega_1 \text{ and } r_\alpha(i) = 1 \}.$$

(Assume for convenience that the metric d is bounded by 1.) The important property of these functions is the following. Suppose $a \in N$ and $d(a, a_\alpha) < \frac{\epsilon}{4}$. Then for any $i \in \omega$, $\varphi_i(a) < \frac{\epsilon}{4}$ if and only if $r_\alpha(i) = 1$.

Each φ_i is a 1-Lipschitz function defined on N. Let τ be the name for the sequence $\left\langle \varphi_i \circ \dot{f} : i < \omega \right\rangle$, a member of X. We show first that $(\mathbb{P}, \tau)$ is an E-pin.

Suppose $G \times H$ is a generic filter for $\mathbb{P} \times \mathbb{P}$. Then in $V[G \times H]$ there is an automorphism g of M so that $\dot{f}[G] \circ g = \dot{f}[H]$. Now $g^{-1} \cdot \tau[H] = \tau[G]$, so $\tau[H]$ and $\tau[G]$ are E-related, as required.

It remains to show that this pin is not trivial. Assume towards a contradiction that there is $x = \langle \psi_i : i < \omega \rangle \in X$ so that $\mathbb{P}$ forces that $\check{x} \; E \; \tau$. Working in the ground model, let $\langle q(n) : n < \omega \rangle$ be a dense subset of M, and define a sequence $\langle s_n : n < \omega \rangle$ of members of 2^ω as follows.

$$s_n(i) = 1 \iff \psi_i(q(n)) < \tfrac{\epsilon}{4},$$

There must be some $\alpha < \omega_1$ so that $r_\alpha \neq s_n$ for all $n < \omega$.

Fix a generic $G \subseteq \mathbb{P}$ and let $f = \dot{f}[G]$ be the isomorphism from M to N. There is some automorphism g of M so that $g \cdot x = \tau$, meaning $\varphi_i \circ f \circ g = \psi_i$ for all $i < \omega$.

Fix n so that $d^N(f(g(q(n))), a_\alpha) < \frac{\epsilon}{4}$and let $a = f(g(q(n)))$. Now for any $i < \omega$

$$s_n(i) = 1 \iff \psi_i(q(n)) < \tfrac{\epsilon}{4} \iff \varphi_i(a) < \tfrac{\epsilon}{4} \iff r_\alpha(i) = 1.$$

So $s_n = r_\alpha$, a contradiction. □

Recall that any Polish group G is isomorphic to a group of the form $\mathrm{Aut}(M)$ for some separable metric structure M (see [Gao09, Lemma 2.5.1] and [Gao09, Theorem 2.4.5.]). (1) $\implies$ (2) of Theorem 0.1 now follows. Fix a Polish group G, and some separable metric structure so that $G = \mathrm{Aut}(M)$. If G is not CLI, then by [BYDNT17, Theorem 9.2] the Scott sentence of M has some non-separable model N, and therefore G has a non-pinned action as above.

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