A NOTE ON $\mathfrak{b}$ AND $\mathbf{add}(\mathcal{B})$

ASSAF SHANI

Cichon and Pawlikowsky [CP86] showed that after adding a single Cohen real, the bounding number $\mathfrak{b}$ in the generic extension $V[G]$ is collapsed to $\mathbf{add}(\mathcal{B})^V$, the additivity of the meager ideal as calculated in the ground model V.

Their proof in fact shows that in the Cohen extension, the covering number of the meager ideal $\mathbf{Cov}(\mathcal{B})$ increases above $\mathfrak{b}$. Then, using the equality $\mathbf{add}(\mathcal{B}) = \min(\mathfrak{b}, \mathbf{Cov}(\mathcal{B}))$, they conclude that $\mathbf{add}(\mathcal{B})$ and $\mathfrak{b}$ must agree in the extension. Finally, they show that $\mathbf{add}(\mathcal{B})$ is preserved, concluding that in the Cohen extension $\mathfrak{b} = \mathbf{add}(\mathcal{B}) = \mathbf{add}(\mathcal{B})^V$.

In this note we construct in a very concrete manner a sequence of length $\mathbf{add}(\mathcal{B})^V$, in the Cohen extension, witnessing that $\mathfrak{b} = \mathbf{add}(\mathcal{B})^V$ (see Corollary 1.9). This is done by presenting $\mathbf{add}(\mathcal{B})$ as a generic parametrized version of $\mathfrak{b}$, as follows.

1. $\mathbf{add}(\mathcal{B})$ as a generic parametrized version of $\mathfrak{b}$

Proposition 1.1. Let κ be a cardinal. The following are equivalent.

(1) $\kappa < \mathbf{add}(\mathcal{B})$;
(2) for any sequence of κ many Borel functions $f_\alpha \colon \mathbb{R} \to \omega^\omega$ there is a Borel function $f \colon \mathbb{R} \to \omega^\omega$ such that
$$\forall^* x \forall \alpha (f(x) \text{ dominates } f_\alpha(x)),$$
where $\forall^* x$ means "for comeager many x".

First, we recall the following characterizations of $\mathfrak{b}$ and $\mathbf{add}(\mathcal{B})$, in terms of interval partitions and chopped reals, respectively. These characterizations, in particular, illustrate that $\mathbf{add}(\mathcal{B}) \leq \mathfrak{b}$. We also sketch the proof that $\mathbf{add}(\mathcal{B})$ is preserved in the Cohen extension.

Definition 1.2 (See Blass [Bla10]). (1) Π is an interval partition (IP) if it partitions ω into finite intervals, which we always view in increasing order.
(2) Given two interval partitions Π and Σ, say that Π dominates Σ if for all but finitely many n there exists k such that $\Sigma(k) \subseteq \Pi(n)$.
(3) A Chopped real (CR) (x, Π) is a pair of $x \in 2^\omega$ and $\Pi \in IP$.
(4) A real $y \in 2^\omega$ matches a chopped real (x, Π) if there exists infinitely many intervals in Π on which y and x agree.
(5) (x, Π) engulfs (y, Σ) if for all but finitely many n there exists k such that $\Pi(n) \supseteq \Sigma(k)$ and x and y agree on $\Sigma(k)$.
(6) $\mathrm{Match}(x, \Pi)$ is the set of all reals y which match the chopped real (x, Π).

Lemma 1.3 ([Bla10]). (1) For any (x, Π), $\mathrm{Match}(x, \Pi)$ is comeager;
(2) any comeager set contains $\mathrm{Match}(x, \Pi)$ for some (x, Π);
(3) $\mathrm{Match}(x, \Pi) \subseteq \mathrm{Match}(y, \Sigma)$ if and only if (x, Π) engulfs (y, Σ).

Date: August 19, 2021.

Let $c \in 2^\omega$ be Cohen generic over V.

Claim 1.4 (see [Bla10, Chapter 11.3])**.** If $(x, \Pi) \in V[c]$ then there is (y, Σ) in V such that if z is in V and z matches (y, Σ), then z matches (x, Π).

Proof. Enumerate all pairs (p, n) where p is a condition in the Cohen forcing $\mathbb{P}$ and $n \in \omega$. We construct (y, Σ) recursively. At stage m, we define the m'th interval of Σ and the values of y in it.

For each (p, n) in the list so far, we extend y so that there is some q extending p which forces y agrees with $\dot{x}$ on some interval of $\dot{\Pi}$ beyond the n'th interval (which will be then contained in the m'th interval of Σ.

Suppose z matches (y, Σ). Then by construction, for any condition p it is not the case that there is and n such that p forces z does not match (x, Π) past n. This is because, by construction, for any such p and any n there is an extension of p which forces z to agree with x on an interval of Π past n. So z agrees with x on infinitely many intervals in Π. □

Remark 1.5. If Π is the trivial partition of ω into singletons, then there is no (y, Σ) in V which engulfs (c, Π).

Corollary 1.6. If Y is non meager in V then Y is non meager in $V[c]$.

Proof. If $D = \mathrm{Match}(x, \Pi)$ is comeager in $V[c]$, we need to show D is not disjoint from Y. Let (y, Σ) be as in the claim above. Since Y is non meager in V there is $z \in \mathrm{Match}(y, \Sigma)^V \cap Y$, and so $z \in Y \cap D$ in $V[c]$. □

Fact 1.7 (Cichon-Pawlikowsky [CP86])**.** If $D \subseteq \mathbb{R}$ is comeager in $V[c]$, there is a comeager subset of the plane $C \subseteq \mathbb{R} \times \mathbb{R}$ in V such that $D = C_c$.

Corollary 1.8 (Cichon-Pawlikowsky [CP86])**.** $\mathbf{add}(\mathcal{B})$ is preserved by a Cohen real extension.

Proof. First, in V there is a sequence $\left\langle C_\alpha :\, \alpha < \mathbf{add}(\mathcal{B})^V \right\rangle$ where each C_α is meager, yet $C = \bigcup C_\alpha$ is not meager. C_α remains meager, and C remains non meager, so $\mathbf{add}(\mathcal{B})^{V[c]} \leq \mathbf{add}(\mathcal{B})^V$.

Suppose D_α, $\alpha < \kappa$, are comeager sets in $V[c]$, where $\kappa < \mathbf{add}(\mathcal{B})^V$. We wish to show that $D = \bigcap_{\alpha<\kappa} D_\alpha$ is comeager. For each α there is a comeager set C_α in the plane such that $D_\alpha = (C_\alpha)_c$. Let $C = \bigcap_{\alpha<\kappa} C_\alpha$. Since $\kappa < \mathbf{add}(\mathcal{B})^V$, C is comeager in the plane, and so $D = (C)_c$ is comeager. □

Proof of Proposition 1.1. Fix $\kappa < \mathbf{add}(\mathcal{B})$ and Borel functions $f_\alpha \colon \mathbb{R} \to \mathrm{IP}$, $\alpha < \kappa$. We need to find a Borel $f \colon \mathbb{R} \to \mathrm{IP}$ such that $\forall^* x \in \mathbb{R}$, $f(x)$ dominates $f_\alpha(x)$ for all $\alpha < \kappa$. Let $\bar{0} \in 2^\omega$ be the constant 0 sequence. Define $f'_\alpha \colon \mathbb{R} \to \mathrm{CR}$ by $f'_\alpha(x) = (\bar{0}, f_\alpha(x))$. Let

$$D_\alpha = \{(x, y) :\ \text{y matches } f'_\alpha(x)\}.$$

For each $\alpha < \kappa$, D_α is a comeager subset of the plane. Since $\kappa < \mathbf{add}(\mathcal{B})$, the intersection $\bigcap_{\alpha<\kappa} D_\alpha$ is comeager. Fix dense open sets C_n such that $\bigcap_{\alpha<\kappa} D_\alpha = \bigcap_{n<\omega} C_n$.

Note that if C is dense open in the plane then C_x is dense open for almost all x. Furthermore, given dense open sets $E_n \subseteq \mathbb{R}$ we can definably, in a Borel way, construct a chopped real (y, Σ) such that $\mathrm{Match}(y, \Sigma)$ is contained in $\bigcap_n E_n$.

For any $x \in \mathbb{R}$ such that $(C_n)_x$ is dense open (comeager many x) let $f'(x)$ be such chopped real (y, Σ), so that $\mathrm{Match}(y, \Sigma)$ is contained in $\bigcap_n (C_n)_x$, and let $f(x) = \Sigma$.

For each such x and for any $\alpha < \kappa$, $f'(x)$ engulfs $f'_\alpha(x)$, thus $f(x)$ dominates $f_\alpha(x)$, as desired.

It remains to show that there is a sequence of Borel functions f_α, $\alpha < \mathbf{add}(\mathcal{B})$, with $f_\alpha \colon 2^\omega \to \mathrm{IP}$, so that there is no Borel function f for which $\forall^* x \forall \alpha (f(x)$ dominates $f_\alpha(x))$ holds. Fix chopped reals (x_α, Π_α), $\alpha < \mathbf{add}(\mathcal{B})$ which cannot be engulfed simultaneously. Define $f_\alpha \colon 2^\omega \to \mathrm{IP}$ as follows.

$f_\alpha(x) = \tilde{\Pi}^x_\alpha$ is a result of gluing together segments of interval in Π_α so that for each interval in $\tilde{\Pi}^x_\alpha$ there is a subinterval from Π_α on which x and x_α agree. This is done if, and as long as possible. If not possible, just leave it as Π_α. (For a sufficiently generic x this will be possible.)

Note that f_α is a Borel function. We show that there is no Borel f such that $\forall^* x \forall \alpha (f_\alpha(x) <^* f(x))$.

Suppose to the contrary that f is such a function. Let $c \in 2^\omega$ be a Cohen generic. Let $\Pi = f(c)$ and $\tilde{\Pi}_\alpha = \tilde{\Pi}^c_\alpha$. Then Π dominates $\tilde{\Pi}_\alpha$ for each α. Since c is Cohen generic, the definition of $\tilde{\Pi}^c_\alpha$ succeeds to find intervals on which x_α and c agree. It follows that (c, Π) engulfs (x_α, Π_α) for each α.

By Claim 1.4 there is a chopped real (y, Σ) in V such that for any $z \in V$, if z matches (y, Σ) then z matches (c, Π). Therefore, working in V: for any α, for any z matching (y, Σ), z matches (x_α, Π_α). By Lemma 1.3 (y, Σ) engulfs (x_α, Π_α) for each $\alpha < \mathbf{add}(\mathcal{B})$, in contradiction. □

Corollary 1.9 (Cichon-Pawlikowsky [CP86])**.** In $V[c]$, $\mathbf{add}(\mathcal{B})^V = \mathbf{add}(\mathcal{B})^{V[c]}$.

Proof. The second part of the proof above precisely showed that $\tilde{\Pi}_\alpha$, $\alpha < \mathbf{add}(\mathcal{B})^V$, are interval partitions which cannot be dominated in $V[c]$. So indeed $\mathfrak{b}^{V[c]} \leq \mathbf{add}(\mathcal{B})^V$. Also, $\mathbf{add}(\mathcal{B})^V = \mathbf{add}(\mathcal{B})^{V[c]} \leq \mathfrak{b}^{V[c]}$. □

References

[Bla10] A. Blass, Combinatorial Cardinal Characteristics of the Continuum, Handbook of Set Theory, Vols. 1, 2, 3, Springer, Dordrecht (2010), pp. 395-489

[CP86] J. Cichon , J. Pawlikowski, On ideals of subsets of the plane and on Cohen reals, J. Symbolic Logic 51 (1986) 560-569.

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
Email address: shani@math.harvard.edu