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Fresh subsets of ultrapowers

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Abstract In [8], Shelah and Stanley constructed a κ^+-Aronszjan tree with an ascent path using $\square_\kappa$. We show that $\square_{\kappa,2}$ does not imply the existence of Aronszajn trees with ascent paths. The proof goes through an intermediate combinatorial principle, which we investigate further.

Keywords Aronszajn trees · Square principles · Forcing · Fresh subsets

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1 Introduction

Definition. *(Devlin [4], Shelah-Stanley [8]) Let $\lambda < \kappa$ be regular cardinals. Suppose T is a tree of height κ. A λ-ascent path through T is a sequence $x = \left\langle x^\alpha_\xi ;\, \alpha < \kappa,\, \xi < \lambda \right\rangle$ satisfying:*

- $\forall \alpha < \kappa\, \forall \xi < \lambda \left(x^\alpha_\xi \in T_\alpha \right)$;
- $\forall \alpha < \beta < \kappa\, \exists \delta < \lambda\, \forall \xi > \delta \left(x^\alpha_\xi <_T x^\beta_\xi \right)$.

The main interest in Aronszajn trees with an ascent path is due to the following fact:

Theorem. *(See [8]) If T is a κ^+-Aronszajn tree with a λ-ascent path, and $\lambda \neq \mathrm{cf}\kappa$, then T is not special.*

Aronszajn trees with ascent paths were first introduced in [4], where $\aleph_2$-Aronszajn trees with ω-ascent paths were constructed in L. The main construction is due to Shelah and Stanley:

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Theorem. *(Shelah - Stanley [8]) Let κ be a cardinal, for any $\lambda < \kappa$ with $\lambda \neq \mathrm{cf}\kappa$, $\square_\kappa$ implies the existence of a κ^+-Aronszajn tree with a λ-ascent path.*

The aim of this note is to show that a $\square_{\kappa,2}$ sequence is not enough:

Theorem 1. *We construct a model in which $\square_{\kappa,2}$ holds, yet for any $\lambda \neq \mathrm{cf}\kappa$, there is no κ^+-Aronszajn tree with a λ-ascent path. (Starting with a measurable cardinal, for regular κ, and infinitely many supercompacts, for singular κ.)*

Remark. *The principle $\square^{ta}_{\kappa,\delta}$, a weakening of $\square_\kappa$, was introduced by Neeman [7], where he shows that $\square^{ta}_{\omega_1,\omega}$ suffices to construct an $\aleph_2$-Aronszajn tree with an ω-ascent path. From theorem 1 it then follows that $\square^{ta}_{\omega_1,\omega}$ is not a consequence of $\square_{\omega_1,2}$. This fact was proven in [1], as well as other independence results about $\square^{ta}_{\kappa,\delta}$.*

The proof of theorem 1 will use an intermediate combinatorial principle, a fresh subset of an ultrapower. Recall the following definition:

Definition. *(Hamkins [5]) Let M be a model of set theory. Suppose $A \subset \kappa$ where κ is an ordinal in M. A is said to be fresh over M if*

$$A \notin M \text{ and } \forall \alpha \in \kappa \, (A \cap \alpha \in M) \, .$$

We will also say that A is an M-fresh subset of κ, or that A is fresh over M (when κ is implicit).

Specifically, we will be interested in fresh subsets over ultrapowers of V:

Definition 2. *Let λ be a regular cardinal, $\mathcal{U}$ an ultrafilter over λ. Let $j \colon V \longrightarrow \mathrm{Ult}\,(V,\mathcal{U})$ be the corresponding ultrapower embedding.*

A is said to be a $\mathcal{U}$-fresh subset of κ if $A \subset j\,(\kappa)$ and A is fresh over $\mathrm{Ult}\,(V,\mathcal{U})$.

(Note that $\mathcal{U}$ is not assumed to be countably complete, hence $\mathrm{Ult}\,(V,\mathcal{U})$ is not assumed to be well-founded.)

Let us first note that an Aronszajn tree with an ascent path yields a fresh subset:

Proposition 3. *Suppose $\lambda < \kappa$ are regular cardinals. Let T be a κ-Aronszajn tree with a λ-ascent path. Then for any uniform ultrafilter $\mathcal{U}$ over λ, there is a $\mathcal{U}$-fresh subset of κ.*

Proof. Let $\mathcal{U}$ be a uniform ultrafilter over λ, $j \colon V \longrightarrow \mathrm{Ult}\,(V,\mathcal{U})$ the ultrapower embedding. Suppose $\langle x^\alpha ; \alpha < \kappa \rangle$ is a λ-ascent path in T, where $x^\alpha = \left\langle x^\alpha_\xi ; \xi < \lambda \right\rangle$. Note that for any $\alpha < \beta < \kappa$, $\left\{ \xi ; x^\alpha_\xi \lhd_T x^\beta_\xi \right\} \in \mathcal{U}$, hence $[x^\alpha]_\mathcal{U} \lhd_{j(T)} \left[x^\beta\right]_\mathcal{U}$. Let $b^* = \langle [x^\alpha]_\mathcal{U} \, ; \alpha < \kappa \rangle$. Since κ is regular, $j''\kappa$ is cofinal in $j\,(\kappa)$, so b^* generates a cofinal branch through $j\,(T)$. Denote this branch by b. W.l.o.g., we can assume that T is coded as a subset of κ by some $c \colon T \longrightarrow \kappa$, such that $z \lhd_T w \Longrightarrow c\,(z) < c\,(w)$, hence b is a subset of $j\,(\kappa)$. By the definition of a branch, each bounded section of b is in $\mathrm{Ult}\,(V,\mathcal{U})$, as it can be defined from one element in $j\,(T)$. Thus for any $\alpha < j\,(\kappa)$, $b \cap \alpha \in \mathrm{Ult}\,(V,\mathcal{U})$. Furthermore, by elementarity, $\mathrm{Ult}\,(V,\mathcal{U}) \vDash$ "$j\,(T)$ is an Aronszajn tree", so $b \notin \mathrm{Ult}\,(V,\mathcal{U})$. $\square$

In section 2 it is shown that having a fresh subset (over a small ultrafilter) is a combinatorial principle which exhibits incompactness. As customary for such principles, we study forcing notions to add and destroy it. The main results are in section 5, where we construct a model in which $\square_{\kappa,2}$ holds yet there are no λ-fresh subsets of κ^+ for any $\lambda \neq \mathrm{cf}\kappa$. We also separate the notion of a λ-fresh subset for different values of λ, by constructing a model in which κ has a λ-fresh subset, but no γ-fresh subsets for any $\gamma < \lambda$.

2 Fresh subsets of ultrapowers

Let λ be a regular cardinal, $\mathcal{U}$ an ultrafilter over λ. Let $j\colon V \longrightarrow \mathrm{Ult}\,(V,\mathcal{U})$ be the corresponding ultrapower embedding. For $x, y \in \mathcal{P}_\kappa\kappa$, we write $x \leq y$ to mean "y end extends x".
Suppose A is a $\mathcal{U}$-fresh subset of κ, for a regular cardinal $\kappa > \lambda$. Note that, since $\mathrm{cf}\kappa \neq \lambda$, $j''\kappa$ is cofinal in $j\,(\kappa)$.
For each $\alpha < \kappa$, by assumption, $A \cap j\,(\alpha) \in \mathrm{Ult}\,(V,\mathcal{U})$. Take some $f_\alpha\colon \lambda \longrightarrow \mathcal{P}_\kappa\kappa$ which represents it, i.e. $A \cap j\,(\alpha) = [f_\alpha]_{\mathcal{U}}$.
Let $\mathcal{F} = \{f_\alpha;\, \alpha < \kappa\}$. Then $\mathcal{F}$ is a family of functions satisfying:

1. $\forall f \in \mathcal{F}\,(f\colon \lambda \longrightarrow \mathcal{P}_\kappa\kappa)$;
2. $\forall f, g \in \mathcal{F}$, $[f]_{\mathcal{U}} \parallel [g]_{\mathcal{U}}$ (that is, either $[f]_{\mathcal{U}} \leq [g]_{\mathcal{U}}$ or $[g]_{\mathcal{U}} \leq [f]_{\mathcal{U}}$);
3. There is no function $F\colon \lambda \longrightarrow \mathcal{P}\,(\kappa)$ s.t. $[F]_{\mathcal{U}} = \bigcup_{f \in \mathcal{F}} [f]_{\mathcal{U}}$.

Conversely, given a family $\mathcal{F}$ of functions $f\colon \lambda \longrightarrow \mathcal{P}_\kappa\kappa$, which satisfies the conditions above, the set $A = \bigcup_{f \in \mathcal{F}} [f]_{\mathcal{U}}$ is a $\mathcal{U}$-fresh subset of κ.
A family $\mathcal{F}$ satisfying (1) and (2) above, will be called a $\mathcal{U}$-coherent family (relative to κ). If it also satisfies (3) then we say it is a fresh family.
A function F as in clause (3) will be said to uniformize the family $\mathcal{F}$.

Definition 4. *For a filter $\mathcal{U}$ over λ, define a $\mathcal{U}$-fresh subset of κ by the existence of a family $\mathcal{F}$ satisfying properties 1-3 above.*

For a regular cardinal $\lambda < \kappa$, we say that κ has a λ-fresh subset if it has a $\mathcal{U}$-fresh subset for some filter over λ.

Remark. *If A is a $\mathcal{U}$-fresh subset of κ where $\mathcal{U}$ is a filter over λ, then for any filter $\tilde{\mathcal{U}}$ extending $\mathcal{U}$, A is a $\tilde{\mathcal{U}}$-fresh subset as well. It is sometimes convenient to take some ultrafilter $\tilde{\mathcal{U}}$, as done in proposition 3.*

Next we show that, for small filters $\mathcal{U}$, having $\mathcal{U}$-fresh subsets of κ is an incompactness property of κ.

Definition 5. *For a filter $\mathcal{U}$, we say that κ reflects $\mathcal{U}$ if there are no $\mathcal{U}$-fresh subsets of κ.*
For inaccessible κ, we say that κ reflects (small) filters, if for any $\lambda < \kappa$ and any filter $\mathcal{U}$ over λ, κ reflects $\mathcal{U}$.

Theorem 6. *If κ is weakly compact, then κ reflects (small) filters.*

Proof. Let $\mathcal{U}$ be a filter over a cardinal $\lambda < \kappa$, and let $\mathcal{F} = \langle f_\alpha;\, \alpha < \kappa \rangle$ be a $\mathcal{U}$-coherent family over κ.
Define the following partition $h : [\kappa]^2 \longrightarrow \mathcal{P}(\lambda)$. For any $\alpha < \beta < \kappa$,

$$h(\alpha, \beta) = \{\eta < \lambda;\, f_\alpha(\eta) \parallel f_\beta(\eta)\} \in \mathcal{U}.$$

By $\kappa \longrightarrow (\kappa)^2_{2^\lambda}$, there exists a homogenous subset $X \subset \kappa$, $|X| = \kappa$ with some fixed color $A \in \mathcal{U}$.
Define $g : \lambda \longrightarrow \mathcal{P}(\kappa)$ by $g(\eta) = \bigcup_{\alpha \in X} f_\alpha(\eta)$, $\eta < \lambda$. By the definition of X and A, for any $\alpha \in X$ we have $\{\eta < \lambda;\, f_\alpha(\eta) \leq g(\eta)\} \supset A \in \mathcal{U}$. Therefore,

$$\forall \alpha \in X\, ([f_\alpha] \leq [g])\,, \text{ hence } \bigcup_{\alpha \in X} [f_\alpha] \parallel [g]\,.$$

Since $|X| = \kappa$, $\bigcup_{\alpha \in X} [f_\alpha]$ is cofinal in $j(\kappa)$. So $\bigcup_{\alpha \in X} [f_\alpha]$ and $[g]$ are cofinal subsets of $j(\kappa)$ and are compatible, hence $\bigcup_{\alpha \in X} [f_\alpha] = [g]$. Finally, by the coherence of $\mathcal{F}$, $\bigcup_{\alpha < \kappa} [f_\alpha] = \bigcup_{\alpha \in X} [f_\alpha] = [g]$. Thus g uniformizes $\mathcal{F}$. $\square$

In L, reflecting (small) filters is equivalent to weak compactness for inaccessible cardinals (see section 3).
Similarly we have:

Theorem 7. *Suppose κ is strongly compact, $\theta \geq \kappa$. Then θ reflects ultrafilters of size $< \kappa$.*

For the purpose of constructing models with small successors cardinals that reflect ultrafilters, we need the following variation of the theorem above, allowing generic elementary embeddings.

Theorem 8. *Suppose $\rho \leq \kappa \leq \theta$ are regular cardinals, and there is a poset P which is ρ-closed and the following holds:*
In V^P there is a transitive model M and an elementary embedding $k\colon V \longrightarrow M$ with critical point κ, s.t. $k(\theta) > \sup k''\theta > \theta$.
Then, in V^P, θ has no λ-fresh subsets from V, for any $\lambda < \rho$.

Note that it is important the ultrafilter $\mathcal{U}$ under consideration is over some cardinal below ρ. So that not only it is below the critical point of k, but also forcing with P does not change the structure of $\mathrm{Ult}(V, \mathcal{U})$.

3 Aronszajn trees with ascent path

In order to construct a κ^+-Aronszajn tree with a λ-ascent path, the Shelah-Stanley proof only uses the following consequence of $\square_\kappa$: a $\square(\kappa^+)$-sequence $\mathcal{C} = \langle C_\alpha;\, \alpha < \kappa^+ \rangle$ together with a stationary subset $S \subset \kappa^+$, s.t. $S \subset \mathrm{cof}(\lambda)$ and S is disjoint to the limit points of $\mathcal{C}$, i.e. for any $\alpha < \kappa^+$, $S \cap C'_\alpha = \emptyset$.
Furthermore, the proof can be adapted for any regular cardinal, so we have:

Theorem. *(See [8]) Let $\lambda < \kappa$ be regular cardinals. Suppose there is a $\square(\kappa)$-sequence $\mathcal{C}$ with a stationary subset S of κ, s.t. $S \subset \mathrm{cof}(\lambda)$ and S is disjoint to the limit points of $\mathcal{C}$. Then there is a κ-Aronszajn tree T with a λ-ascent path.*

Note that the hypothesis of the theorem (for any $\lambda < \kappa$) is satisfied in L, for all inaccessible cardinals κ s.t. κ is not weakly compact.
In particular, together with theorem 6 and proposition 3, we immediately get that in L, the following are equivalent for inaccessible κ:

- κ is weakly compact.
- κ reflects (small) filters.
- κ reflects filters over ω.

Recall the standard construction of a special κ^+-Aronszajn tree from the assumption $\kappa^{<\kappa} = \kappa$. A simple variation (adding to the usual proof an inductive construction of an ascent path, as in the Shelah-Stanley construction) gives a special κ^+-Aronszajn tree with a κ-ascent path.
Furthermore, if we only assume $\kappa^{<\kappa} \leq \kappa^+$, the same construction can be repeated, and it gives a tree of height κ^+, with levels of size $\leq \kappa^+$, no branch, and with a κ-ascent path. Now the same argument as in section 1 shows that the ascent path codes a κ-fresh subset of κ^+.
Thus for a regular κ s.t. $\kappa^{<\kappa} \leq \kappa^+$, there is a κ-fresh subset of κ^+. This means that having a κ-fresh subset is not an incompactness property for κ^+. (Compare to non-reflecting subsets of κ^+ with cofinalities κ.)

Definition 9. *Let κ be a cardinal. κ^+ is said to reflect (small) filters, if for every $\lambda < \kappa$, for every filter $\mathcal{U}$ over λ, κ reflects $\mathcal{U}$.*

4 Separation of weak squares

Weak squares were introduced by Schimmerling, and separated by Jensen. The proof presented here follows that in [6], but simplifies the arguments by using the following two lemmas, which isolate the difference between the threading poset for different weak squares. The separation proof will only use these abstract properties, so that the same arguments can be applied to separate squares and fresh subsets in section 5 below.
We denote $S(\kappa, \lambda)$ as the natural poset for adding a $\square(\kappa, \lambda)$-sequence by partial sequences of clubs approximating the square sequence, ordered by extension. In $V^{S(\kappa,\lambda)}$, $T(\kappa, \lambda)$ is the natural poset to thread the generic $\square(\kappa, \lambda)$-sequence by bounded approximations to the threading club, ordered by end extension. For definitions and basic properties of these posets, see [3], [2] or [6].
Similarly, we use the posets $S(\kappa, < \lambda)$ and $T(\kappa, < \lambda)$ to add and thread a $\square(\kappa, < \lambda)$ sequence, $S_{\kappa,\lambda}$ and $T_{\kappa,\lambda}$ for $\square_{\kappa,\lambda}$, and $S_{\kappa,<\lambda}$ and $T_{\kappa,<\lambda}$ for $\square_{\kappa,<\lambda}$.

Lemma 10. *Suppose κ is a regular cardinal and $\eta < \kappa$, $\mathcal{C} = \langle C_\alpha ; \alpha < \kappa \rangle$ is a $\square(\kappa, < \eta)$-sequence and P is a poset satisfying*

$$P^\eta \text{ is } \kappa\text{-distributive.}$$

Then P does not thread $\mathcal{C}$. (By P^η we mean the full support power.)

Proof. Suppose otherwise, that there is a P-name τ s.t. P forces τ is a thread through $\mathcal{C}$. Let $\prod_{\alpha<\eta} G_\alpha$ be a P^η-generic over V and denote $D_\alpha = \tau^{G_\alpha}$. Since $\mathcal{C}$ has no thread in V, and $\{G_\alpha;\, \alpha < \eta\}$ are pairwise mutually generic, then $\{D_\alpha;\, \alpha < \eta\}$ is a sequence of pairwise distinct clubs threading $\mathcal{C}$. However, as P^η is κ-distributive, κ is regular in the generic extension, thus $D = \bigcap_{\alpha<\eta} D_\alpha$ is a club. Furthermore, we can find a $\beta \in D$ s.t. all the D_α's disagree below β. Now

$$\forall \alpha < \eta \,(\beta \in D_\alpha \Longrightarrow D_\alpha \cap \beta \in C_\beta)\,,$$

in contradiction to $|C_\beta| < \eta$. $\square$

Lemma 11. *Let $S = S(\kappa, <\lambda)$ and $T = T(\kappa, <\lambda)$. Then*

$$\Vdash_S \forall \eta < \lambda\, (T^\eta \text{ is } \kappa\text{-distributive})\,.$$

The lemma follows from the following proposition, which states that for $\eta < \lambda$, $S * T^\eta$ is forcing isomorphic to a κ-directed closed poset.

Proposition. *Define $E \subset S * T^\eta$,*

$$\begin{aligned} E = \{&(p, \langle d_\xi;\, \xi < \eta\rangle)\,;\, \exists \beta,\, p = \langle C_\alpha;\, \alpha \le \beta,\, \alpha \text{ limit ordinal}\rangle \in S \\ &\wedge \forall \xi < \eta\, (p \Vdash (d_\xi \in T) \wedge \max d_\xi = \beta)\}\,. \end{aligned}$$

Then E is dense and κ-directed closed.

Proof. E is κ-directed closed: Suppose $\left\langle \left(p^\zeta, \overrightarrow{d^\zeta}\right);\, \zeta < \mu \right\rangle$ are pairwise compatible, for some $\mu < \kappa$, where $\overrightarrow{d^\zeta} = \left\langle d^\zeta_\xi;\, \xi < \eta \right\rangle$. Define for $\xi < \eta$

$$d_\xi = \bigcup_{\zeta<\mu} d^\zeta_\xi, \quad q = \{d_\xi;\, \xi < \eta\}\,, \quad p = \bigcup_{\zeta<\mu} p^\zeta \frown q.$$

Then $\left(p, \overrightarrow{d}\right)$ is a lower bound of the sequence, and in E.
E is dense: Take a condition $\left(p, \overrightarrow{d}\right) \in S{*}T^\eta$, $\overrightarrow{d} = \langle d_\xi;\, \xi < \eta\rangle$, $p = \langle C_\alpha;\, \alpha \le \beta\rangle$. W.l.o.g., $\beta \ge \max d_\xi$ for all $\xi < \eta$. Define

$$e_\xi = d_\xi \cup (\beta + \omega \setminus \beta + 1)\,.$$

Let $C_{\beta+\omega} = \{e_\xi;\, \xi < \eta\}$, and let $q = \langle C_\alpha;\, \alpha \le \beta + \omega,\, \alpha \text{ limit}\rangle$, $\overrightarrow{e} = \langle e_\xi;\, \xi < \eta\rangle$. Then $(q, \overrightarrow{e}) \in E$ and $(q, \overrightarrow{e}) \le \left(p, \overrightarrow{d}\right)$. $\square$

The separation of weak squares is now immediate:
Suppose κ in V is weakly compact and indestructible to κ-directed closed posets of size κ. Fix some $\eta < \kappa$. Let $G \subset S(\kappa, \eta)$ be generic over V. Clearly, $V[G] \vDash \square(\kappa, \eta)$. Thus it remains to show that $V[G] \vDash \neg\square(\kappa, <\eta)$.

Indeed, assume by contradiction that $\mathcal{C}$ is a $\square(\kappa, <\eta)$-sequence in $V[G]$. Let $T = T(\kappa, \eta)$ be the threading poset corresponding to $S(\kappa, \eta)$, and let $H \subset T$ be generic over $V[G]$. Since $S(\kappa, \eta) * T$ is κ-directed closed and of size κ, then κ is weakly compact in $V[G][H]$, hence $V[G][H] \vDash \neg\square(\kappa, <\eta)$. So it must be that T added a thread through $\mathcal{C}$. However, by lemma 11, T^{η} is κ-distributive. This contradicts lemma 10.

These arguments can also be applied to smaller successor cardinals, say $\aleph_2$ or $\aleph_{\omega+1}$, using only these two lemmas. All that is necessary is to start in a model where these cardinals have enough reflection properties, and these are indestructible under nice enough forcings. Such models were constructed by Magidor, see [6] for successors of regulars and [3] for successors of singulars. We elaborate more in the proofs of theorems 13 and 14 below.

Remark. *The proof of lemma 10 above shows that if $P^{\lambda} \Vdash \mathrm{cf}\kappa > \lambda$, then P cannot thread a $\square(\kappa, <\lambda)$-sequence, but does not work if we weaken this assumption. The following strengthening is due to Yair Hayut: If $P^{\lambda} \Vdash \mathrm{cf}\kappa > \omega$ then P cannot thread a $\square(\kappa, <\lambda)$-sequence.*

5 Separation for fresh subsets

We now wish to get separation results for the notion of a fresh subset of κ. The proofs will follow the same outline as for the separation of squares.

Lemma 12. *Suppose $\mathcal{U}$ is an ultrafilter over λ and A is a $\mathcal{U}$-fresh subset of κ. Suppose Q is a forcing notion such that $Q \times Q$ is λ^{+}-distributive. Then in V^{Q}, A remains a $\mathcal{U}$-fresh subset of κ.*

Proof. First, by λ^{+}-distributivity of Q, $\mathcal{U}$ remains an ultrafilter over λ, and the calculation of $\mathrm{Ult}(V,\mathcal{U})$ is the same in V and in V^{Q}. In particular, A is still a subset of $j(\kappa)$, and it is only necessary to show that $A \notin \mathrm{Ult}\left(V^{Q},\mathcal{U}\right)$. Assume otherwise. Let $\dot{g}$ be a Q-name s.t.

$$Q \Vdash \dot{g} \colon \lambda \longrightarrow \mathcal{P}(\kappa) \text{ and } [\dot{g}]_{\mathcal{U}} = A.$$

Since A is $\mathcal{U}$-fresh in V, we must have that $Q \Vdash \{\eta < \lambda;\, \dot{g}(\eta) \notin V\} \in \mathcal{U}$. W.l.o.g we can assume $Q \Vdash \forall \eta < \lambda\, (\dot{g}(\eta) \notin V)$.
Let $G_1 \times G_2$ be $Q \times Q$-generic over V. In particular G_1 and G_2 are both Q-generic over V. For $i \in \{1,2\}$, let $g_i = \dot{g}^{G_i}$.
Note that, by λ^{+}-distributivity of $Q \times Q$, we still have a well defined ultrapower map $[\,]_{\mathcal{U}} : V[G_1 \times G_2]^{\lambda} \longrightarrow \mathrm{Ult}(V[G_1 \times G_2],\mathcal{U})$, and the calculation of $\mathrm{Ult}(V,\mathcal{U})$ remains the same.
Therefore, by the definition of $\dot{g}$,

$$[g_1]_{\mathcal{U}} = A = [g_2]_{\mathcal{U}}\,.$$

However, by mutual genericity of G_1 and G_2, and the fact that $Q \Vdash \forall \eta < \lambda\, (\dot{g}(\eta) \notin V)$, we must have

$$\forall \eta < \lambda\, (g_1(\eta) \neq g_2(\eta))\,, \text{ thus } [g_1]_{\mathcal{U}} \neq [g_2]_{\mathcal{U}}\,,$$

in contradiction. □

Corollary 13. *(From a weakly compact cardinal) there is a model where κ is regular, $\square(\kappa, 2)$ holds, yet κ reflects (small) filters. In particular, for any $\lambda < \kappa$, there is no κ-Aronszajn tree with a λ-ascent path.*

Proof. The proof is now the same as described in section 4. Start in V where κ is weakly compact and indestructible to κ-directed closed posets of size κ. In particular, by theorem 6, κ reflects filters after any forcing extension by a κ-directed closed poset of size κ. Let G be generic for $S = S(\kappa, 2)$ (adding a $\square(\kappa, 2)$-sequence), and let T be the corresponding threading poset in $V[G]$. Recall that T^2 is κ-distributive by lemma 11.

Now $\square(\kappa, 2)$ holds in $V[G]$. It remains to show that there cannot be any λ-fresh subsets of κ, for $\lambda < \kappa$. Assume that there is such a fresh subset, then by lemma 12 it remains so in $V[G][H]$, where $H \subset T$ is generic. This is a contradiction, since $S * T$ is forcing isomorphic to a κ-directed closed poset of size κ.

Recall that by proposition 3, any κ-Aronszajn tree with a λ-ascent path, $\lambda < \kappa$, gives a $\mathcal{U}$-fresh subset of κ for any uniform ultrafilter $\mathcal{U}$ over λ. So the final statement follows. $\square$

Similar results can be obtained for small successor cardinals by incorporating the techniques from [3]:

Theorem 14. *(From a measurable cardinal) there is a model where $\square_{\omega_1,2}$ holds, yet ω_2 reflects all filters over ω. In particular, there are no ω_2-Aronszajn trees with an ω-ascent path.*

(From infinitely many supercompacts) there is a model where $\square_{\aleph_\omega,2}$ holds, yet $\aleph_{\omega+1}$ reflects (small) filters. In particular, there are no $\aleph_{\omega+1}$-Aronszajn trees with an ω_n-ascent path, for any $n \in \omega$.

Proof. For the first statement, start in V with a measurable cardinal κ, $j\colon V \longrightarrow M$ the ultrapower embedding. Let $G_0 \subset \mathrm{Col}(\omega_1, < \kappa)$ be generic. We use the following fact, due to Magidor (see [6]): In $V[G]$, for any ω_1-closed poset R and generic $I \subset R$, there is some further generic J to an ω_1-closed poset Q, such that in $V[G_0][I][J]$, there is an extension of j to $V[G_0][I]$. It now follows from theorem 8 (with $\lambda = \omega$, $\rho = \kappa = \theta = \omega_1$) that in $V[G_0][I][J]$ there are no ω-fresh subsets of ω_2 from $V[G][I]$.

Let G be generic over $V[G_0]$ for $S = S_{\omega_1,2}$ (adding a $\square_{\omega_1,2}$-sequence), and let T be the corresponding threading poset in $V[G_0][G]$. Note that T^2 is ω_1-distributive (similar to the proof of lemma 11, it follows from the fact that $S * T^2$ is ω_1-closed. See [3]). Now $\square_{\omega_1,2}$ holds in $V[G_0][G]$. It remains to show that there are no ω-fresh subsets of ω_2. Assume there is such ω-fresh subset of ω_2 and let $H \subset T$ be generic. By lemma 12 there is still a fresh subset in $V[G_0][G][H]$. Let $R = S * T$, $I = G * H$. R is ω_1-closed, so let Q be as described above, J generic for Q. Since Q is ω_1-closed, then Q^2 is ω_1-distributive, so by lemma 12 there is still an ω-fresh subset of ω_2 in $V[G_0][I][J]$, which is a contradiction.

The proof of the second statement follows similarly, if we start with a model in which $\aleph_{\omega+1}$ satisfies reflection properties after forcing with any $\aleph_n$-closed poset, for any n. Such models were constructed in [3]). We will not go further into those details here, but only mention that the reflection achieved in these models is in the form of elementary embeddings which exists in further generic extension under closed enough forcing. That is where we can apply theorem 8 to get reflection of small filters. $\square$

Next, we wish to separate the notion of 'having a λ-fresh subset' for different values of λ. For this, we first study a natural forcing notion for adding a fresh subset, and the complementary poset that 'uniformizes' the generic fresh subset.

Let $P = \text{Cohen}(\kappa)$, that is, elements in P are bounded subsets of κ, ordered by end extension. Let $\mathcal{U}$ be some filter over λ. The poset under consideration is the reduced power $Q = P^{\lambda}/\mathcal{U}$.
The natural projection $P^{\lambda} \longrightarrow Q$ defined by $f \mapsto [f]_{\mathcal{U}}$ is a projection of forcing notions. In fact $\forall f \in P^{\lambda} \forall q \in Q \left(q \leq [f] \Longrightarrow \exists g \leq_{P^{\lambda}} f \left([g] = q\right)\right)$.
An immediate consequence, since P^{λ} is κ-closed, is that Q is κ-strategically closed. Furthermore, there is a κ-distributive quotient poset $R = P^{\lambda}/Q$ such that $Q * R \simeq P^{\lambda}$.
More explicitly, given a Q-generic G, the quotient poset can be described as $R = \left\{f \in P^{\lambda};\ [f]_{\mathcal{U}} \in G\right\}$. Note that, for $\xi < \lambda$, the map $R \longrightarrow P$, $f \mapsto f(\xi)$ is a projection of forcing notions.

Proposition 15. *Q adds a $\mathcal{U}$-fresh subset of κ.*

Proof. Let $G \subset Q$ be generic over V. By definition, $G \subset V^{\lambda}/\mathcal{U}$. Take $A = \bigcup_{g \in G} g$, then $A \subset \kappa^{\lambda}/\mathcal{U}$, and any bounded segment of A is in fact equal to one $g \in G$. We want to show that A is a $\mathcal{U}$-fresh subset of κ in $V[G]$, so it only remains to show that $A \notin V[G]^{\lambda}/\mathcal{U}$.
Assume otherwise, that there is some $F\colon \lambda \longrightarrow \mathcal{P}(\kappa)$ in $V[G]$ such that $[F]_{\mathcal{U}} = A$.
Let $H \subset R$ be R-generic over $V[G]$. Define $\tilde{F}\colon \lambda \longrightarrow \mathcal{P}(\kappa)$ by $\tilde{F} = \bigcup_{f \in H} f$.
By the definition of R, $\left[\tilde{F}\right]_{\mathcal{U}} = A$. Furthermore, since $Q * R$ is κ-closed (hence in particular λ^{+}-distributive), the ultrapower calculations of V and $V[G]$ do not change when moving to $V[G][H]$.
However, H is generic over $V[G]$, so $\tilde{F}(\xi)$ is P-generic over $V[G]$ for each $\xi < \lambda$. Since $F \in V[G]$, we get

$$\forall \xi < \lambda \left(\tilde{F}(\xi) \neq F(\xi)\right),$$

in contradiction with $\left[\tilde{F}\right]_{\mathcal{U}} = A = [F]_{\mathcal{U}}$. $\square$

Remark. *Suppose $\mathcal{U}$ is an ultrafilter over λ, $j\colon V \longrightarrow \mathrm{Ult}(V,\mathcal{U})$. Let $G \subset Q = j(P)$ be generic over V. Then j extends to $j\colon V[G] \longrightarrow \mathrm{Ult}(V[G],\mathcal{U}) = \mathrm{Ult}(V,\mathcal{U})[j(G)]$. Proposition 15 above states that $G \notin \mathrm{Ult}(V,\mathcal{U})[j(G)]$. In fact, it can be shown that G is generic over $\mathrm{Ult}(V,\mathcal{U})[j(G)]$.*

Proposition 16. *Suppose $\mathcal{U}$ is a λ-closed filter, then R is λ-closed.*

Proof. Work in $V[G]$, for a V-generic $G \subset Q$. Suppose $\langle f_\alpha;\, \alpha < \eta\rangle$ for $\eta < \lambda$ is a descending chain in R. Let $\gamma = \sup\{\sup f_\alpha(\xi);\, \xi < \lambda,\, \alpha < \eta\} < \kappa$. Take some $g \in P^\lambda$ s.t. $[g] \in G$ and $\forall \xi < \lambda\, (\sup g(\xi) \geq \gamma)$. Let $f = \bigcup_\alpha f_\alpha$. Clearly f is a lower bound of $\langle f_\alpha;\, \alpha < \eta\rangle$. We claim that $f \in R$. For each $\alpha < \eta$, $[f_\alpha] \in G$, hence $[f_\alpha] \parallel [g]$, and in fact $[g] \leq_Q [f_\alpha]$, by the choice of g. Take $A_\alpha \in \mathcal{U}$ witnessing that, and let $A = \bigcap_{\alpha<\eta} A_\alpha \in \mathcal{U}$. For any $\xi \in A$, $\alpha < \eta$, $g(\xi) \leq_P f_\alpha(\xi)$. So for all $\xi \in A$, $g(\xi) \leq_P f(\xi)$, hence $[g] \leq_Q [f]$. It follows that $[f] \in G$, and so $f \in R$. □

We are now in position to separate the notion of having a λ-fresh subset for different values of λ.
Suppose κ is weakly compact, and indestructible under κ-directed closed forcings of size κ.
Take some uncountable regular $\lambda < \kappa$ and let $\mathcal{U}$ be the co-bounded filter over λ, then $\mathcal{U}$ is λ-closed.
Let $Q = \mathrm{Cohen}(\kappa)^\lambda / \mathcal{U}$, and R be the quotient $\mathrm{Cohen}(\kappa)^\lambda / Q$. Let $G \subset Q$ be generic over V.

Theorem 17. *In $V[G]$ there is a λ-fresh subset of κ, yet κ reflects all filters over η, for any $\eta < \lambda$.*

Proof. Assume otherwise, that there is some filter $\mathcal{V}$ over $\eta < \lambda$ and a $\mathcal{V}$-fresh subset of κ. After forcing with R, we get that κ is weakly compact again, hence reflects small filters. However, by the proposition above, R^2 is λ-distributive, so by lemma 12, R could not have destroyed any $\mathcal{V}$-fresh subsets of κ, in contradiction. □

Furthermore, these arguments can be extended for successors of regulars and successors of singulars, as described above, using the relevant models from [3]. For example:

Theorem 18. *(From a supercompact cardinal) for any n, we can construct a model in which there is an $\aleph_n$-fresh subset of $\aleph_{\omega+1}$, yet for any $m < n$, $\aleph_{\omega+1}$ reflects all filters over $\aleph_m$.*

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References

1. Chen, W., Neeman, I.: Square principles with tail-end agreement, Archive for Mathematical Logic, vol. 54 (2015), pp. 439–452.
2. Cummings, J.: Iterated forcing and elementary embeddings. In: Handbook of Set Theory, vol. 2. pp. 775–884. Springer, Berlin (2010).
3. Cummings, J., Foreman, M., Magidor, M.: Squares, scales and stationary reflection. Journal of Mathematical Logic, 01(01):35–98, 2001.

4. Devlin, K. J.: Reduced products of $\aleph_2$-trees, Fundamenta Mathematicae 118 (1983), pp 129-134.
5. Hamkins, J. D.: Small forcing makes any cardinal superdestructible. J. Symbolic Logic, vol. 63, iss. 1, pp. 51-58, 1998.
6. Magidor, M., Lambie-Hanson, C.: On the strengths and weaknesses of weak squares. Appalachian Set Theory: 2006-2012. Cambridge University Press, 2012.
7. Neeman, I.: Two applications of finite side conditions at ω_2. http://www.math.ucla.edu/ ineeman/
8. Shelah, S., Stanley, L.: Weakly compact cardinals and nonspecial Aronszajn trees. Proc. Amer. Math. Soc., 104(3):887-897, 1988.