BORELL REDUCIBILITY AND VIRTUAL CLASSES (DRAFT - UNDER PREPARATION)

ASSAF SHANI

In set theory, a *virtual* object is one that exists in a generic extension. This note aims to introduce the study of virtual equivalence classes, and its applicability, particularly towards questions about Borel reducibility and Vaught's conjecture. The emphasis will be on the unifying aspect of this approach. For example, we will present proofs of the topological Vaught's conjecture for CLI groups (due to Becker) and Harrington's theorem on models of size $\aleph_1$ for counterexamples to the $\mathcal{L}_{\omega_1,\omega}$-Vaught conjecture, as well as a result on abelian torsion groups of size $\aleph_1$ due to Fuchs and Kulikov, and the well known existence of a sequence of Turing degrees satisfying $x'_{n+1} \leq_T x_n$. With respect to the last two results especially, this note is very much in the spirit of [Mil17], proving things "the hard way". Nevertheless, the given treatment of the topological Vaught conjecture is quite simple, and for several application to equivalence relations the only known proofs use virtual classes.

This note will be most attractive to a reader with some background and an interest in set theory. Such reader will find the details reasonable, and the general approach quite pleasing. I hope that any reader interested in Borel reducibility or Vaught's conjecture will also benefit from this note, in particular to get an intuition for what kind of questions may be susceptible to the techniques presented here.

The reader is referred to [Mil17] for background on the interaction between descriptive set theory and forcing. Particularly, we assume familiarity with Cohen forcing (in both the combinatorial and topological presentations), the Levy collapse, and absoluteness results.

The paper [Hjo98] is also a survey of applications of 'virtual Borel sets' to descriptive set theory. In a sense the present note can be seen as a continuation of that line of work. The focus is a little different and most applications presented here are not present in [Hjo98]. When there is overlap , the current presentation is cleaner thanks to recent developments such as [Kan08b, LZ20]. Chapter 2 of [LZ20, Chapter 2] is a systematic study of virtual equivalence classes, the most comprehensive to date. Virtual classes were also studied recently from various perspectives [KMS16, KS16, URL17, Sha21]. This note, while far from comprehensive, attempts to present a more unified storyline. Many of the proofs presented here are different than in other sources.

For an introduction to Borel reducibility the reader is referred to the books [Gao09, Kan08b] and the surveys [MR21, For18, KTD12, HK01, Kec99]. The book [Kan08b] particularly emphasizes forcing techniques. With regards to applications to Borel reducibility, we focus here on techniques rather than state-of-the-art results. In particular we use virtual classes to study well known benchmark equivalence relations such as $E_0, E_0^{\mathbb{N}}, =^+, E_{\omega_1}$. Some recent results are presented, in particular the study of isomorphism and bi-embeddability of torsion groups from [CT19], and the study of Archimedean ordered groups from [CMRS23].

1. Virtual classes and Vaught's conjecture

Let X be a Polish space and E an analytic equivalence relation on X. An important fact which we use below is that the space X and the equivalence relation E can be reasonably interpreted in any generic extension. The discussion below naturally extends beyond analytic equivalence relations, as long as this can be done.

Definition 1.1. Let $\mathbb{P}$ be a forcing poset and τ and $\mathbb{P}$-name so that $\mathbb{P} \Vdash \tau \in X$. We may think of τ as a *virtual* member of X, or of $[\tau]_E$ as a *virtual equivalence class*. (A closely related terminology is that of *virtual Borel sets* [Ste84, Hjo98].) Assume further that in any generic extension, given two filters $G_1, G_2 \subseteq \mathbb{P}$ which are generic over the ground model

$$\tau[G_1] \; E \; \tau[G_2].$$

In this case we say that the pair $(\mathbb{P}, \tau)$ is a *stable virtual E-class* [Kan08b, 17.1.2], or an *E-pin* [LZ20, 2.1.1].

A product forcing argument shows that the above condition is equivalent to asserting that

$$\mathbb{P} \times \mathbb{P} \Vdash \tau_l \; E \; \tau_r,$$

where τ_l, τ_r are $\mathbb{P}\times\mathbb{P}$ names so that, given a filter $G_l \times G_r \subseteq \mathbb{P}\times\mathbb{P}$, $\tau_l[G_l\times G_r] = \tau_l[G_l]$ and $\tau_r[G_l \times G_r] = \tau_r[G_r]$. See [LZ20, 2.1.2].

In this case $[\tau]_E$ is an equivalence class in a generic extension, which is "stable" or "pinned" in the sense that its interpretation does not depend on the generic filter. We will often omit the adjective 'stable' and call $(\mathbb{P}, \tau)$ a virtual E-class.

Given $x \in X$ and a poset $\mathbb{P}$, let $\check{x}$ be the canonical $\mathbb{P}$-name for x. Then $(\mathbb{P}, \check{x})$ is a stable virtual E-class.

Definition 1.2. Say that a stable virtual E-class $(\mathbb{P}, \tau)$ is *pinned* [Kan08b, 17.1.2], or *trivial* [LZ20, 2.3.1], if there is some $x \in X$ so that

$$\mathbb{P} \Vdash \check{x} \; E \; \tau.$$

Non-trivial virtual classes arise naturally from definable violations of the continuum hypothesis. In the descriptive set theoretic context, the continuum hypothesis is often identified with a perfect set property: a set is either countable, or admits a definable injective image of $2^{\mathbb{N}}$. In this sense, a definable set of size $\aleph_1$ can be seen as a violation of the continuum hypothesis.

By classical descriptive set theoretic wisdom, all analytic sets satisfy the perfect set property [Kec95, 29.1]. The modern wisdom is that any reasonably definable subset of a Polish space satisfies the perfect set property (see [Kan03], page 145 as well as Theorem 27.9 and the following discussion). Considering sets defined as quotients of Polish spaces by definable equivalence relations, we have *Silver's dichotomy* [Gao09, 5.3.5]: for any co-analytic equivalence relation E (in particular, a Borel equivalence relation), either there are countably many E-classes or there is a perfect set of E-inequivalent elements.

Example 1.3. Let X be the Polish space of all linear orderings of $\mathbb{N}$. Define E_{ω_1} on X by $x \; E_{\omega_1} \; y$ if x, y are isomorphic, or if both are ill-founded. E_{ω_1} is an analytic equivalence relation with $\aleph_1$ many equivalence classes.

Fix an uncountable ordinal α, let $\mathbb{P}_\alpha = \mathrm{Col}(\omega, |\alpha|)$, and let τ_α be a $\mathbb{P}_\alpha$-name for a member of X of order-type α. Then $(\mathbb{P}_\alpha, \tau_\alpha)$ is a non-trivial virtual E_{ω_1}-class.

Lemma 1.4. Let $\mathbb{P}$ be a forcing poset and τ and $\mathbb{P}$-name so that $\mathbb{P} \Vdash \tau \in X$. Given $p \in \mathbb{P}$, let $\mathbb{P} \restriction p$ be the poset of all conditions stronger than p. Assume that there is no $p \in \mathbb{P}$ for which $(\mathbb{P} \restriction p, \tau)$ is a stable virtual E-class. Then there is a perfect set $Y \subseteq X$ of E-inequivalent elements.

Proof. Given $p \in \mathbb{P}$, by the assumption (p, p) does not force, with respect to $\mathbb{P} \times \mathbb{P}$, that $\tau_l \mathrel{E} \tau_r$, so there is a condition (q_1, q_2) extending (p, p) and forcing that $\tau_l \not\mathrel{E} \tau_r$. So for any $p \in \mathbb{P}$ we may find q_1, q_2 extending p so that $(q_1, q_2) \Vdash \tau_l \not\mathrel{E} \tau_r$. We can iterate this fact to build a 'binary tree of conditions'.

Let $(D_n : n \in \omega)$, $(C_n : n \in \omega)$ be sequences of dense open subsets of $\mathbb{P}$ and $\mathbb{P} \times \mathbb{P}$ respectively. We may recursively define conditions $p_s \in \mathbb{P}$ for each s in the full binary tree $2^{<\omega}$ so that for each $s \in 2^{<\omega}$,

- $p_s \in D_{|s|}$, where $|s|$ is the length of s,
- $p_{s^\frown 0}, p_{s^\frown 1}$ both extend p_s;
- $(p_{s^\frown 0}, p_{s^\frown 1})$ force that $\tau_l \not\mathrel{E} \tau_r$.
- $(p_{s^\frown 0}, p_{s^\frown 1}) \in C_{|s|+1}$

A branch in the binary tree $b \in 2^\omega$ corresponds to a sequence of conditions $(p_{b \restriction n} : n \in \omega)$. The idea is that this sequence of conditions decide enough information about τ to determine a unique point in $x_b \in X$. This can be achieved by a sufficiently rich family $(D_n : n \in \omega)$. Furthermore, a sufficiently rich family $(C_n : n \in \omega)$ will ensure that for distinct branches b, b', $x_b \not\mathrel{E} x_{b'}$.

One way to do this is as follows. Fix a 'sufficiently elementary' countable model M and let $(D_n : n \in \omega)$, $(C_n : n \in \omega)$ be an enumeration of all dense open subsets in M of $\mathbb{P}$, $\mathbb{P} \times \mathbb{P}$ respectively. ☐

For a countable first order language $\mathcal{L}$ consider $\mathrm{Mod}(L)$, the Polish space of all $\mathcal{L}$-structures with universe $\mathbb{N}$, and $\cong_L$ the isomorphism relation on $\mathrm{Mod}(L)$ (see [Gao09, 3.6]). Given a theory T we let $\mathrm{Mod}(T)$ be the subspace of all $\mathcal{L}$-structures which are models of T, and $\cong_T$ the isomorphism relation restricted to $\mathrm{Mod}(T)$.

The equivalence relation $\cong_T$ is generally analytic, so is not covered by Silver's dichotomy. *Vaught's conjecture* asserts that a violation of CH cannot be defined in this way, that is, either $\cong_T$ has countably many countable models, up to isomorphism, or there is a perfect set of pairwise non-isomorphic models of T (see [Gao09, p.261]).

Proposition 1.5. Suppose T is a counterexample to Vaught conjecture. That is

1. $\mathrm{Mod}(T)$ has uncountably many equivalence classes;
2. there is no perfect set of $\cong_T$-in-equivalent elements.

Then there is an unpinned stable virtual $\cong_T$-class.

Proof. We use the fact that item (1) above is absolute, that is, true in any generic extension. Work in a $\mathbb{P} = \mathrm{Col}(\omega, \mathbb{R})$ generic extension. As there are uncountably many $\mathrm{Mod}(T)$-equivalence classes, and only countably many from the ground model, there is some member of $\mathrm{Mod}(T)$ which is not equivalent to any ground model member. That is, we may find a $\mathbb{P}$-name τ so that for any $x \in \mathrm{Mod}(T)$ in the ground model, $\mathbb{P} \Vdash \tau \not\cong_T \check{x}$. By Lemma 1.4 and item (2), there is some $p \in \mathbb{P}$ so that $(\mathbb{P} \restriction p, \tau)$ is a stable virtual $\cong_T$-class, which is not trivial by the choice of τ.

The absoluteness of (1) follows as it can be presented as a $\mathbf{\Pi}^1_2$ statement. The most obvious attempt to phrase it does not quite work: 'for any sequence $x \in$

$(\mathrm{Mod}(T))^{\mathbb{N}}$ there is $y \in \mathrm{Mod}(T)$ so that for all $n \in \mathbb{N}$ y is not isomorphic to $x(n)$', is a $\mathbf{\Pi}^1_3$ formula. An equivalent $\mathbf{\Pi}^1_2$ statement is 'for any countable list of of $\mathcal{L}_{\omega_1,\omega}$ sentences $(\psi_1, \psi_2, \dots)$ there is $y \in \mathrm{Mod}(T)$ with Scott sentence different from ψ_i for each $i \in \mathbb{N}$'. See also [Lar17, Remark 10.4]

□

Given an $\mathcal{L}_{\omega_1,\omega}$ sentence ϕ, let $\mathrm{Mod}(\phi)$ the Polish space of all $\mathcal{L}$-structures with universe $\mathbb{N}$, which satisfy ϕ, and $\cong_\phi$ the isomorphism relation on $\mathrm{Mod}(\phi)$. The $\mathcal{L}_{\omega_1,\omega}$-Vaught conjecture asserts that $\cong_\phi$ either has countably many equivalence classes, or else there is a perfect set of $\mathrm{Mod}(\phi)$-inequivalent members of $\mathrm{Mod}(\phi)$.

The isomorphism relation on models with universe $\mathbb{N}$ may be viewed as induced by an action of S_∞, the group of all permutations of $\mathbb{N}$. Equipped with the pointwise convergence topology, this is a Polish group. Given a Polish group G and a continuous action $a\colon G \curvearrowright X$ on a Polish space X, let E_a be the induced *orbit equivalence relation* on X,

$$x \mathrel{E} y \iff \exists g \in G(g \cdot x = y).$$

The *topological Vaught conjecture* asserts that E_a either has countably many equivalence classes, or else there is a perfect set of E_a-inequivalent members of X. Say that a group G *satisfies Vaught's conjecture* if this holds for any continuous action of G.

Proposition 1.6. Suppose $a\colon G \curvearrowright X$ is a counterexample to the topological Vaught conjecture. Then there is an unpinned stable virtual E_a-class.

The proof proceeds as Proposition 1.5, after proving the absoluteness of "E_a is a counterexample to the topological Vaught conjecture". See [Hjo01, Claim (2) on p.133].

1.1. **Pinned equivalence relations.**

Definition 1.7. Say that E is *pinned* if every stable virtual E-class is pinned.

We begin with a few simple examples.

Example 1.8. The equality relation on $\mathbb{R}$, $=_{\mathbb{R}}$, is pinned.

Proof. If $(\mathbb{P}, \tau)$ is a stable virtual $=_{\mathbb{R}}$-class, G_1, G_2 are mutually $\mathbb{P}$-generics over the ground model, then

$$x \coloneqq \tau[G_1] = \tau[G_2] \in V[G_1] \cap V[G_2] = V.$$

We see that $\mathbb{P} \Vdash \tau \mathrel{E} \check{x}$. □

Example 1.9. Let Γ be a countable group, $a\colon \Gamma \curvearrowright X$ a Borel action, and $E = E_a$ the induced orbit equivalence relation. Then E is pinned.

Proof. If $(\mathbb{P}, \tau)$ is a stable virtual E-class, G_1, G_2 are mutually $\mathbb{P}$-generics over V, then

$$A \coloneqq [\tau[G_1]]_E = [\tau[G_2]]_E \in V[G_1] \cap V[G_2] = V.$$

In this calculation we used the fact that, as Γ is a countable group, the orbit $[\tau[G_1]]_E = \Gamma \cdot \tau[G_1]$ is the same whether calculated in $V[G_1]$ or $V[G_1 \times G_2]$. Fix some $x \in A$. Then $x \in V$ and $\mathbb{P} \Vdash \tau \mathrel{E} \check{x}$. □

Example 1.10. Let Γ_n, $n \in \mathbb{N}$, be countable groups, $a_n\colon \Gamma_n \curvearrowright X_n$ Borel actions. Consider the pointwise product action $a\colon \prod_{n\in\mathbb{N}} \Gamma_n \curvearrowright \prod_{n\in\mathbb{N}} X_n$, and let $E = E_a$ be the induced orbit equivalence relation on $\prod_{n\in\mathbb{N}} X_n$. Then E is pinned.

Proof. Fix a stable virtual E-class $(\mathbb{P}, \tau)$, and fix G_1, G_2 mutually $\mathbb{P}$-generics over V. Define

$$A_n \coloneqq \Gamma_n \cdot \tau[G_1]_n = \Gamma_n \cdot \tau[G_2]_n,$$

where $\tau[G_i]_n \in X_n$ is the n'th coordinate of the sequence $\tau[G_i]$. Then

$$A \coloneqq (A_n : n \in \mathbb{N}) \in V[G_1] \cap V[G_2] = V.$$

Working in V, fix a sequence $x \in \prod_n A_n$. Then $\mathbb{P} \Vdash \tau\ E\ \check{x}$. □

As we have seen in Example 1.3, E_{ω_1} is not pinned. By Proposition 1.5, a counterexample to Vaught's conjecture gives rise to an unpinned equivalence relation. Another central example is the following.

Example 1.11. Consider the natural action $a\colon S_\infty \curvearrowright \mathbb{R}^{\mathbb{N}}$, defined by $g\cdot x = x \circ g^{-1}$. Let $E = E_a$ be the induced orbit equivalence relation on $\mathbb{R}^{\mathbb{N}}$. Then E is not pinned.

Proof. Let $\mathbb{P} = \mathrm{Col}(\omega, \mathbb{R})$, and let τ be a $\mathbb{P}$-name which is forced to be a member of $\mathbb{R}^{\mathbb{N}}$ enumerating the ground model reals. Working in a generic extension, if G_1, G_2 are $\mathbb{P}$-generics over V, then $\tau[G_1], \tau[G_2]$ are two enumerations of the same set, so there is a permutation $g \in S_\infty$ so that $\tau[G_1] \circ g = \tau[G_2]$. Therefore $(\mathbb{P}, \tau)$ is a stable virtual E-class. For any $x \in \mathbb{R}^{\mathbb{N}}$ in the ground model, a countable set of reals, it is forced that $\{x_n : n \in \mathbb{N}\}$ is a strict subset of $\{\tau_n : n \in \mathbb{N}\}$, and so $\tau \not\!E\ \check{x}$. So $(\mathbb{P}, \tau)$ is not trivial. □

A Polish group G is *CLI* if it *admits a complete left invariant metric.* See [Gao09, Chapter 2] for a thorough discussion on CLI Polish groups. We mention here that:

- any abelian Polish group is CLI;
- given countable groups Γ_n, $n \in \mathbb{N}$, the product group $\prod_{n\in\mathbb{N}} \Gamma_n$ is CLI.
- the group S_∞ is not CLI.

A big success towards the topological approach to Vaught's conjecture is a theorem of Sami [Sam94] that all abelian Polish groups satisfy the Vaught conjecture. This was strengthened by Becker, who proved that all CLI groups satisfy the Vaught conjecture.

Theorem 1.12 (Hjorth [Hjo99], see [Kan08b, 17.4])**.** Let G be a CLI Polish group, $a\colon G \curvearrowright X$ a continuous action. Then E_a is pinned.

The theorem, together with Proposition 1.6, implies Becker's theorem.

Corollary 1.13 (Becker)**.** Every CLI group satisfies the Vaught conjecture.

2. Different virtual classes

So far we focused on whether unpinned stable virtual classes exist or not. Next, following [LZ20], we study the space of all stable virtual classes: "the virtual realm".

Definition 2.1 (Larson-Zapletal [LZ20, 2.1.4])**.** Let $(\mathbb{P}, \tau)$ and $(\mathbb{Q}, \sigma)$ be stable virtual E-classes. Assume further that in any generic extension, given two filters $G \subseteq \mathbb{P}$, $H \subseteq \mathbb{Q}$ which are generic over the ground model

$$\tau[G]\ E\ \sigma[H].$$

In this case say that $(\mathbb{P},\tau)$ and $(\mathbb{Q},\sigma)$ are *E-equivalent*[1], denoted $(\mathbb{P},\tau)\ E\ (\mathbb{Q},\sigma)$.

Again a product forcing argument shows that $(\mathbb{P},\tau)\ E\ (\mathbb{Q},\sigma)$ if and only if

$$\mathbb{P}\times\mathbb{Q}\Vdash \tau\ E\ \sigma,$$

where we identify τ,σ with the corresponding $\mathbb{P}\times\mathbb{Q}$-names which interpret the left and right generics, respectively. See [LZ20, 2.1.5].

Example 2.2.
- If $(\mathbb{P},\tau)$ is pinned, $x\in X$ in the ground model is such that $\mathbb{P}\Vdash\tau\ E\ \check{x}$, then $(\mathbb{P},\tau)\ E\ (\mathbb{Q},\check{x})$, where $\mathbb{Q}$ is the trivial (or any) forcing.
- In Example 1.3, $(\mathbb{P}_\alpha,\tau_\alpha),(\mathbb{P}_\beta,\tau_\beta)$ are not equivalent for $\alpha\neq\beta$. On the other hand, if $\mathbb{P}=\mathrm{Col}(\omega,\mathbb{R})$, and τ is a $\mathbb{P}$-name for linear order of $\mathbb{N}$ whose order type is ω_1 of the ground model, then $(\mathbb{P}_{\omega_1},\tau_{\omega_1})\ E_{\omega_1}\ (\mathbb{P},\tau)$.

We now revisit Proposition 1.5. By collapsing more and more, we may get many different virtual classes, similar to Example 1.3.

Lemma 2.3. Suppose $E=E_a$ is a counterexample to the topological Vaught conjecture. Let $\mathcal{S}$ be a set of stable virtual E-classes. Then there is a stable virtual E-class $(\mathbb{P},\tau)$ which is not equivalent to any $(\mathbb{Q},\sigma)\in\mathcal{S}$.

Proof. Fix a cardinal κ larger than $|\mathcal{S}|$ and $|\mathcal{P}(\mathbb{Q})|$ for any $\mathbb{Q}\in\mathcal{S}$, and let $\mathbb{P}=\mathrm{Col}(\omega,\kappa)$. Working in a $\mathbb{P}$-generic extension, for each $(\mathbb{Q},\sigma)\in\mathcal{S}$ we may find a filter $G_{(\mathbb{Q},\sigma)}\subseteq\mathbb{Q}$ which is generic over the ground model. Since E has uncountably many classes, in the generic extension, we may find $x\in X$ so that $x\not\mathrel{E}\sigma[G_{(\mathbb{Q},\sigma)}]$ for any $(\mathbb{Q},\sigma)\in\mathcal{S}$.

In the ground model, we may find a name τ so that

$$\mathbb{P}\Vdash\tau\not\mathrel{E}\sigma[\dot{G}_{(\mathbb{Q},\sigma)}],$$

where, for $(\mathbb{Q},\sigma)\in\mathcal{S}$, $\dot{G}_{(\mathbb{Q},\sigma)}$ is a $\mathbb{P}$-name for a filter in $\mathbb{Q}$ generic over the ground model. It follows that $\mathbb{P}\times\mathbb{Q}\Vdash\tau\not\mathrel{E}\sigma$. Finally, as in Proposition 1.5, there must be some $p\in\mathbb{P}$ so that $(\mathbb{P}\restriction p,\tau)$ is a stable virtual class. $(\mathbb{P}\restriction p,\tau)$ is not equivalent to any $(\mathbb{Q},\sigma)$ from $\mathcal{S}$, as required. □

Given a virtual E-class $(\mathbb{P},\tau)$, say that $\mathbb{P}$ is the *support* of $(\mathbb{P},\tau)$.

Definition 2.4 ([LZ20, 2.5.1])**.** Define $\kappa(E)$ to be the smallest cardinal κ so that every stable virtual E-class is equivalent to one supported by a poset of cardinality $<\kappa$. If no such κ exists, $\kappa(E)=\infty$. $\kappa(E)$ is called the *pinned cardinal* of E.

We may rephrase Lemma 2.3 as follows.

Corollary 2.5. If $E=E_a$ is a counterexample to the topological Vaught conjecture, then $\kappa(E)=\infty$.

For Borel equivalence relations, the pinned cardinal is bounded.

Theorem 2.6 ([LZ20, 2.5.6])**.** If E is a Borel equivalence relation, then $\kappa(E)<\beth_{\omega_1}$. More precisely, if E is $\mathbf{\Pi}^0_\alpha$, then $\kappa(E)\leq\beth_\alpha^+$.

Together with Proposition 1.6, we recover a corollary of Silver's dichotomy.

Corollary 2.7. If $E=E_a$ is a Borel orbit equivalence relation, then E satisfies Vaught's conjecture.

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[1] In [LZ20] the extension of E to virtual classes is denoted by $\bar{E}$. Furthermore, there the pairs $(\mathbb{P},\tau)$ are called *pins*, and a *virtual E-class* refers to the $\bar{E}$-equivalence class of $(\mathbb{P},\tau)$.

An ultimate extension of the above discussion is the following theorem of Hjorth.

Theorem 2.8 (Hjorth [Hjo97]). Suppose $E = E_a$ is a counterexample to the topological Vaught conjecture, $a\colon G \curvearrowright X$. Then there is a closed subgroup G' of G and a continuous homomorphism from G' onto S_∞.

It is known that if G is CLI then it does not satisfy the conclusion of the theorem. Hjorth took this further and characterized when the topological Vaught conjecture fails on analytic sets. The proof of the theorem relies heavily on the existence of many inequivalent stable virtual equivalence classes (Corollary 2.5).

2.1. **Some set theoretic questions about pinned cardinals.**

Pinned cardinals. From a set theoretic point of view, understanding which cardinals are "definable", in some reasonable sense, is of interest. As equivalence relations on Polish spaces are of particular interest, and are considered natural objects, we consider $\kappa(E)$ as interesting cardinals.

Question 2.9. Which cardinals κ are equal to $\kappa(E)$ for some analytic equivalence relation E?

For example, for $0 < \alpha < \omega_1$, the cardinals $\aleph_\alpha$ and $\beth_\alpha^+$ are realized as pinned cardinals of Borel equivalence relations. See [LZ20, 2.5.15] and [LZ20, 2.5.18].

Larson and Zapletal [LZ20, 2.5.10] proved that a measurable cardinal κ reflects the statement "$\kappa(E) < \infty$" for analytic equivalence relations E. That is, if E is analytic and $\kappa(E) < \infty$, then $\kappa(E) < \kappa$.

Question 2.10. What is the least cardinal κ so that for any analytic equivalence relation E, if $\kappa(E) < \infty$ then $\kappa(E) < \kappa$?

Larson and Zapletal proved that the answer is between the first ω_1-Erdos and the first measurable cardinal. Moreover, if one restricts to orbit equivalence relations E, the answer is precisely the first ω_1-Erdos cardinal. See [LZ20, 2.5.8] and [LZ20, 2.5.9].

Supports for stable virtual classes. All the virtual classes above came from collapsing some cardinal to be countable.

Question 2.11. Which forcing notions $\mathbb{P}$ can support a non-trivial stable virtual class for an analytic equivalence relation E?

For orbit equivalence relations, such supporting poset must collapse.

Theorem 2.12 ([LZ20, 2.6.6]). If $(\mathbb{P}, \tau)$ is an unpinned stable virtual E_a-class, for an orbit equivalence relation E_a, then $\mathbb{P}$ collapses $\aleph_1$ to be countable.

For non-orbit equivalence relations the situation is more subtle. For example, Namba forcing supports a non-trivial stable virtual class for a (necessarily non-orbit) analytic equivalence relation [LZ20, 2.6.8]. On the other hand, no *reasonable forcing*[2], in particular no proper forcing, can support a non-trivial stable virtual class for any analytic equivalence relation [LZ20, 2.6.2].

[2] *Reasonable* is a technical notion, introduced by Foreman and Magidor [FM95], while studying definable counterexamples to the continuum hypothesis.

3. Virtual classes for isomorphism relations

Let us focus on isomorphism relations for countable structures: $\cong_\phi$ on $\mathrm{Mod}(\phi)$ where ϕ is an $\mathcal{L}_{\omega_1,\omega}$ sentence (or a countable first order theory). These are the orbit equivalence relations which are induced by S_∞. Much of the early work on analytic equivalence relations was focused on isomorphism relations, and was deeply connected to model theory.

Example 3.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, on some domain M. Let $\mathbb{P}_\mathcal{M} = \mathrm{Col}(\omega, M)$ and let $\tau_\mathcal{M}$ be a $\mathbb{P}_\mathcal{M}$-name for a structure on $\mathbb{N}$ isomorphic to $\mathcal{M}$. Then $(\mathbb{P}_\mathcal{M}, \tau_\mathcal{M})$ is a stable virtual $\cong_\mathcal{L}$-equivalence class.

If the set M is countable, we may find $x \in \mathrm{Mod}(\mathcal{L})$ which is isomorphic to $\mathcal{M}$, and therefore $(\mathbb{P}_\mathcal{M}, \tau_\mathcal{M})$ is pinned. The converse is not necessarily true. Let $\mathcal{L}$ be $\{=\}$ and M some uncountable set. Then $(\mathbb{P}_\mathcal{M}, \tau_\mathcal{M})$ is pinned, as τ is forced to be isomorphic to the structure $\mathbb{N}$.

Example 3.2. In the language of linear orders, the stable virtual classes from Example 1.3 are of the form $(\mathbb{P}_\mathcal{M}, \tau_\mathcal{M})$ where $\mathcal{M} = (\alpha, \in)$, for an ordinal α. For uncountable α, $(\mathbb{P}_\mathcal{M}, \tau_\mathcal{M})$ is unpinned.

Example 3.3. Let $\mathcal{L} = \{U_n : n \in \mathbb{N}\}$, where each U_n is an unary predicate. Given a subset $A \subseteq 2^\mathbb{N}$, consider the structure $\mathcal{M}(A)$ with domain A so that $\mathcal{M}(A) \models U_n(x) \iff x(n) = 1$. Then $(\mathbb{P}_{\mathcal{M}(A)}, \tau_{\mathcal{M}(A)})$ is a stable virtual $\cong_\mathcal{L}$-class, which is pinned if and only if A is countable.

An $\mathcal{L}$-structure $\mathcal{M}$ with universe $\mathbb{N}$ may be identified with a sequence $x \in (2^\mathbb{N})^\mathbb{N}$, where $x(n)(m) = 1 \iff \mathcal{M} \models U_m(n)$. The isomorphism relation $\cong_\mathcal{L}$ is induced by the natural S_∞ action on $(2^\mathbb{N})^\mathbb{N}$, permuting the indices, which is identified with Example 1.11.

Definition 3.4 (see [URL17])**.** Let $(\mathbb{P}, \tau)$ be a stable virtual $\cong_\mathcal{L}$-class. Say that $(\mathbb{P}, \tau)$ is *grounded* if there is some (possibly uncountable) model $\mathcal{M}$ (in the ground model) so that

$$\mathbb{P} \Vdash \tau \cong \mathcal{M}.$$

Note that in this case $(\mathbb{P}, \tau)$ and $(\mathbb{P}_\mathcal{M}, \tau_\mathcal{M})$ are $\cong_\mathcal{L}$-equivalent. Say that an equivalence relation E is *grounded* if every stable virtual E-class is grounded.

Note that a pinned equivalence relation is grounded.

Example 3.5. In Example 3.3, it can be shown that for any stable virtual $\cong_\mathcal{L}$-class $(\mathbb{P}, \tau)$, there is a set $A \subseteq 2^\mathbb{N}$ so that $\mathbb{P} \Vdash \tau \simeq \mathcal{M}(A)$. So $\cong_\mathcal{L}$ is grounded.

In Example 1.3 it can be shown that for any stable virtual E_{ω_1}-class $(\mathbb{P}, \tau)$, either τ is forced to be ill-founded, or there is an ordinal α so that $\mathbb{P} \Vdash \tau \simeq (\alpha, \in)$. So E_{ω_1} is grounded. More generally, isomorphism relations for rigid structures are grounded [LZ20, 2.4.5].

More subtle instances of grounded isomorphism relations are given in [URL17], such as the isomorphism relations for the theories REF(bin) and REF(inf).

3.1. A theorem of Kaplan and Shelah and some applications.

For an isomorphism relation $\cong_\phi$, as it is an orbit equivalence relation, a support $\mathbb{P}$ for an unpinned stable virtual E-class must collapse $\aleph_1$, by Theorem 2.12.

Theorem 3.6 ([KS16, Corollary 4.3])**.** Let $(\mathbb{P}, \tau)$ be a stable virtual class for an isomorphism relation. If $(\mathbb{P}, \tau)$ is not grounded, then $\mathbb{P}$ collapses $\aleph_2$ to be countable.

The proof goes through a translation between the question of a stable virtual isomorphism class being grounded and the following question: let T be a first order theory with dense isolated types, must T have an atomic model. The answer is positive if $|T| \leq \aleph_1$. This was proven in the 1970's independently by Knight, Kueker, and Shelah (see [KS16, Proposition 4.2]). Laskowski and Shelah [LS93] constructed a theory of size $\aleph_2$ with dense isolated types but no atomic model.

We present a few applications of Theorem 3.6. The theorem will be applied in the following form.

Corollary 3.7. If $(\mathrm{Col}(\omega, \omega_1), \tau)$ is a stable virtual isomorphism class then there is a structure M (of size $\leq \aleph_1$) so that $\mathrm{Col}(\omega, \omega_1) \Vdash \tau \cong \check{M}$.

3.1.1. *Harrington's theorem.* Going back to Vaught's conjecture, we recover the following theorem due to Harrington. See also [Lar17] and [KMS16].

Theorem 3.8 (Harrington)**.** Suppose $\cong_\phi$ is a counterexample to the $\mathcal{L}_{\omega_1,\omega}$-Vaught conjecture. Then ϕ has models of size $\aleph_1$ with arbitrary high Scott rank below ω_2.

Proof. Fix an ordinal $\alpha < \omega_2$. Let $\mathcal{P} = \mathrm{Col}(\omega, \omega_1)$. We repeat the arguments from Proposition 1.5, relying on the absoluteness of the statements involved.

In a $\mathbb{P}$-generic extension, α is countable, and ϕ has uncountably many models up to isomorphism, so we may find one with Scott rank greater than α. Let τ be a $\mathbb{P}$-name for such model. By Lemma 1.4, there is some $p \in \mathbb{P}$ so that $(\mathbb{P} \restriction p, \tau)$ is a stable virtual $\cong_\phi$-class.

By Theorem 3.6, $(\mathbb{P} \restriction p, \tau)$ is grounded, so there is a structure $\mathcal{M}$ so that $p \Vdash \tau \simeq \mathcal{M}$. As $\mathcal{M}$ is forced to be countable by $\mathbb{P}$, then $|\mathcal{M}| \leq \aleph_1$. Finally, the Scott rank of $\mathcal{M}$ is greater than α after forcing with $\mathbb{P}$, and therefore also in the ground model. □

3.1.2. *Abelian torsion groups and Ulm invariants.*

Countable Abelian torsion groups are classified by their Ulm-invariants. We focus on reduced torsion abelian p-groups for a fixed p. Specifically, the map

$$C \mapsto \sigma(C) = (\sigma(\lambda, k) : \lambda < \kappa(C), k \in \mathbb{N}),$$

is a complete classification. Here $\sigma(\lambda, k)$ is the number of groups of order p^k appearing in a decomposition of $C_\lambda / C_{\lambda+1}$ as a direct sum of cyclic groups.

Observation 3.9. The map $C \mapsto \sigma(C)$ is absolute for generic extensions. That is, $\sigma(C)$ remains the same when calculated in a forcing extension.

Given a sequence $A = (\sigma(\lambda, k) : \lambda < \kappa, k \in \mathbb{N})$, where κ is an ordinal and $\sigma(\lambda, k) \leq \omega$, we may ask if A is a classifying invariant for the classification above, that is, if there is a reduced abelian p-group C so that $\sigma(C) = A$. The countable case is answered by a theorem of Zippin.

Theorem 3.10 (Zippin)**.** Assume that

- κ is countable;
- for any $\lambda < \kappa$ there is some k so that $\sigma(\lambda, k) \neq 0$;
- for $\lambda + 1 < \kappa$ there are infinitely many k so that $\sigma(\lambda, k) \neq 0$.

Then there is a countable reduced abelian p-group C so that $\sigma(C) = (\sigma(\lambda, k) : \lambda < \kappa, k \in \mathbb{N})$.

Zippin's result was extended by Kulikov and Fuchs to groups of size $\aleph_1$.

Theorem 3.11. Assume that

- $\kappa < \omega_2$;
- for any $\lambda < \kappa$ there is some k so that $\sigma(\lambda, k) \neq 0$;
- for $\lambda + 1 < \kappa$ there are infinitely many k so that $\sigma(\lambda, k) \neq 0$.

Then there is a reduced abelian p-group C of size $\leq \aleph_1$ so that $\sigma(C) = (\sigma(\lambda, k) : \lambda < \kappa, k \in \mathbb{N})$.

We recover this result directly from Zippin's theorem and Theorem 3.6.

Proof. Let $\mathbb{P} = \mathrm{Col}(\omega, \omega_1)$. Since κ is countable after forcing with $\mathbb{P}$, it follows from Zippin's theorem that there is a $\mathbb{P}$-name τ for a countable reduced abelian p-group so that $\mathbb{P} \Vdash \sigma(\tau) = (\sigma(\lambda, k) : \lambda < \kappa, k \in \mathbb{N})$. That is, $(\mathbb{P}, \tau)$ is a stable virtual $\cong_p$-class. By Theorem 3.6, there is a reduced abelian group C so that $\mathbb{P} \Vdash \tau \cong C$. By the absoluteness of the Ulm invariants, it follows that $\sigma(C) = (\sigma(\lambda, k) : \lambda < \kappa, k \in \mathbb{N})$, as desired. □

3.1.3. *Scott sets and models of PA.* Given a model M of Peano Arithmetic, seen as an end extension of the standard model, its *standard system* $\mathrm{SSy}(M)$ is the collection of sets of the form $Y \cap \mathbb{N}$ where Y is definable in M (with parameters). The standard system of a model of PA always satisfies the following properties.

Definition 3.12. A set $S \subseteq \mathcal{P}(\mathbb{N})$ is a *Scott set* if

- If $A \in S$ and B is Turing reducible to A then $B \in S$;
- S is a Boolean algebra;
- If $T \in S$ codes an infinite binary tree then there is $b \in S$ coding an infinite branch of T.

It is an open question whether any Scott set can be realized as the standard system of a model of PA. For countable sets, Scott proved that it is the case.

Theorem 3.13 (Scott)**.** Any countable Scott set can be realized as the standard system of a model of PA.

Knight and Nadel extended the result to sets of size $\aleph_1$.

Theorem 3.14 (Knight and Nadel [KN82b])**.** Any Scott set of size $\aleph_1$ can be realized as the standard system of a model of PA.

See [Git08] for more background on this problem, and results beyond $\aleph_1$. Here we provide a quick proof of Knight and Nadel's theorem, using Scott's theorem and Theorem 3.6. First, we introduce a related question.

Fix a countable language. Fix a recursive coding of formulas as natural numbers. We therefore identify theories and types as subsets of $\mathbb{N}$. Similarly, via a fixed recursive coding of binary sequences, we may identify binary trees as subsets of $\mathbb{N}$.

Definition 3.15. Given a Scott set S, say that a model M is *S-saturated* if

- For any $a_1, \ldots, a_n \in M$, $\mathrm{tp}(a_1, \ldots, a_n) \in S$;
- For any $a_1, \ldots, a_n \in M$ and any type $p(x, y_1, \ldots, y_n) \in S$, if $p(x, a_1, \ldots, a_n)$ is consistent then it is realized in M.

Fact 3.16. Given a Scott set S and an S-saturated model M of PA, the standard system of M is precisely S.

Scott's Theorem 3.13 follows from the following.

Theorem 3.17 (Scott, see [Wil80])**.** If S is a countable Scott set and T is a first order theory coded in S, then there is an S-saturated model of T.

This immediately extends to size $\aleph_1$, using Theorem 3.6.

Theorem 3.18. If S is a Scott set of size $\leq \aleph_1$ and T is a first order theory coded in S, then there is an S-saturated model of T.

Proof. We will use the fact that a countable S-saturated model is unique up to isomorphism. This is because S-saturated models are ω-homogeneous. Let $\mathbb{P} = \mathrm{Col}(\omega, \omega_1)$. Since S is a countable Scott set after forcing with $\mathbb{P}$, there is a $\mathbb{P}$-name τ for a countable S-saturated model of T. It follows that $(\mathbb{P}, \tau)$ is a stable virtual isomorphism class. By Theorem 3.6, there is a model M so that $\mathbb{P} \Vdash \tau \cong M$. Now M is the desired S-saturated model of T. □

Theorem 3.18 was proved in the context of models of PA by Knight and Nadel [KN82a]. Note that if T is PA, the theorem produces a model of PA whose standard system is S, concluding the proof of Theorem 3.14.

4. Borel reducibility

Given equivalence relations E and F on Polish spaces X and Y, say that E is *Borel reducible* to F, denoted $E \leq_B F$, if there is a Borel measurable $f\colon X \to Y$ so that

$$x_1 \mathrel{E} x_2 \iff f(x_1) \mathrel{F} f(x_2), \text{ for any } x_1, x_2 \in X.$$

In this case, f induces a well defined injective map on the quotient spaces $X/E \hookrightarrow Y/F$. So '$E \leq_B F$' is often thought of as 'the "Borel cardinality" of X/E is less than or equal to the "Borel cardinality" of Y/F.

For an equivalence relation E so that all E-classes are Borel (this includes all Borel equivalence relations and all orbit equivalence relations), the statement 'E has countably many equivalence classes' is equivalent to '$E \leq_B =_{\mathbb{N}}$', where $=_{\mathbb{N}}$ is the equality relation on $\mathbb{N}$. The statement 'E has perfectly many classes' is equivalent to '$=_{\mathbb{R}} \leq_B E$'. The study of Borel reducibility is an extension of the study of cardinalities of quotients of Polish spaces, beyond questions about the continuum hypothesis.

The celebrated E_0-dichotomy, due to Harrington, Kechris, and Louveau, shows that for Borel equivalence relations there is an "next cardinality" after $=_{\mathbb{R}}$. A classification problem E is said to be **concretely classifiable** if $E \leq_B =_{\mathbb{R}}$. That is, if there is a Borel measurable complete classification using real numbers as invariants.

Theorem 4.1 ([HKL90])**.** Let E be a Borel equivalence relation. If E is not concretely classifiable, then $E_0 \leq_B E$.

For an introduction to Borel reducibility, the reader is referred to the books [Gao09, Kan08b], as well as the surveys [MR21, For18, KTD12, HK01, Kec99]. Here we focus on applications of virtual classes to Borel reducibility.

Lemma 4.2. Let E and F be analytic equivalence relations on Polish spaces X and Y respectively. Assume that $f\colon X \to Y$ is a Borel reduction of E to F. Let $(\mathbb{P}, \tau)$ be a stable virtual E-class. Let σ be a $\mathbb{P}$-name forced to be equal to $f(\tau)$. Then $(\mathbb{P}, \sigma)$ is a stable virtual F-class. Moreover, $(\mathbb{P}, \tau)$ is pinned if and only if $(\mathbb{P}, \sigma)$ is pinned.

Corollary 4.3. If F is pinned and $E \leq_B F$ then E is pinned.

More generally:

Corollary 4.4 ([LZ20, 2.5.4])**.** If $E \leq_B F$ then $\kappa(E) \leq \kappa(F)$.

Remark 4.5. For Lemma 4.2, and the corollaries, the assumption that the reduction is Borel was only used to conclude that this map is well defined, and is still a reduction, in any generic extension. We will consider reductions with this property which may not be Borel.

4.1. **The Friedman-Stanley jump operation.** Given an equivalence relation E on X, the **Friedman-Stanley jump** of E is the equivalence relation E^+ on $X^{\mathbb{N}}$ defined by

$$x \; E^+ \; y \iff \forall n \exists m (x(n) \; E \; y(n)) \text{ and } \forall n \exists m (y(n) \; E \; x(m)),$$

equivalently, if $\{[x(n)]_E : n \in \mathbb{N}\} = \{[y(n)]_E : n \in \mathbb{N}\}$. The quotient $E^+/X^{\mathbb{N}}$ may be identified with $\mathcal{P}_{\aleph_0}(E/X)$, the countable powerset of E/X.

Friedman and Stanley [FS89] proved that this is a *jump operator* on Borel equivalence relations. That is, $E <_B E^+$ for any Borel equivalence relation E.

Example 4.6. Let $=^+$ be $(=_{\mathbb{R}})^+$, defined on $\mathbb{R}^{\mathbb{N}}$ by $x =^+ y \iff \{x(n) : n \in \mathbb{N}\} = \{y(n) : n \in \mathbb{N}\}$. $=^+$ is Borel bireducible with orbit equivalence relation from Example 1.11, and the isomorphism relation from Example 3.3. By Theorem 1.12 and Corollary 4.3, we have:

Corollary 4.7. $=^+$ is not Borel reducible to any orbit equivalence relation induced by a CLI group.

The iterated Friedman-Stanley jumps, $=^{+\alpha}$, are defined recursively, where $=^{+1}$ is $=^+$, $=^{+(\alpha+1)}$ is $(=^{+\alpha})^+$, and we take products at limit stages (see [Gao09, 12.2.6]). The Friedman-Stanley jumps play a central role in the theory of equivalence relations. A classification problem is considered *classifiable using countable sets of reals as complete invariants* if it is Borel reducible to $=^+$; "classifiable using countable sets of countable sets of reals as complete invariants" if it is Borel reducible to $=^{+2}$; and so on.

5. Potential invariants

For our treatment of equivalence relations which are classifiable by countable structures, their central property is the existence of an absolute complete classification. Recall that, for an equivalence relation E on X, a *complete classification* is a map $c \colon X \to I$ so that

$$x_1 \; E \; x_2 \iff c(x_1) = c(x_2), \text{ for all } x_1, x_2 \in X.$$

Definition 5.1. Say that c is an **absolute complete classification** if

(1) The map $c \colon X \to I$ is defined in some set theoretic way ($c(x) = A \iff \psi(x, A)$, for some set theoretic formula ψ.).
(2) In any ZF extension $V \subseteq N$, ψ still defines a map which is a complete classification of E.
(3) Given ZF models $V \subseteq N \subseteq W$, if x, A are in N and $N \models \psi(x, A)$, then $W \models \psi(x, A)$.

Similarly, a map $c \colon X \to I$ is an **absolute E-invariant map** if the it satisfies the above with 'complete classification' replaced by 'E-invariant', that is, $x_1 \; E \; x_2 \implies c(x_1) = c(x_2)$.

The third condition says that the calculation of the invariant of x, $A = c(x)$, does not change as we move to a generic extension. This is the crucial aspect making such classifications "reasonable". For example, it prevents the classification $x \mapsto [x]_E$ from being "reasonable" when E is not a countable equivalence relation. In a sense, it is a generous way of saying that the invariant $c(x)$ can be *computed from* x. That is, the computation is local, and does not depend on things unrelated to x, like some set theoretic truths in the universe.

For an isomorphism relation $\cong_{\mathcal{L}}$, the map

$$M \mapsto \varphi_M,$$

sending a model to its Scott sentence, is an absolute complete classification as above. Furthermore, in this case the invariants φ_M can be coded by hereditarily countable sets.

In more concrete scenarios, there is often a nice absolute complete classification, using hereditarily countable sets as invariants, of a more simple combinatorial nature. The seemingly technical conditions (2) and (3) above are always trivial to verify.

Example 5.2. The map $\mathbb{R}^{\mathbb{N}} \to \mathcal{P}_{\aleph_0}(\mathbb{R})$, $x \mapsto \{x(n) : n \in \mathbb{N}\}$, is an absolute complete classification of $=^+$.

Consider the second Friedman-Stanley jump $=^{++}$ defined on the space $(\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$. The map $(\mathbb{R}^{\mathbb{N}})^{\mathbb{N}} \to \mathcal{P}_{\aleph_0}\mathcal{P}_{\aleph_0}(\mathbb{R})$,

$$x \mapsto \{\{x(n)(m) : m \in \omega\} : n \in \omega\}$$

is an absolute complete classification of $=^{++}$.

Similarly, there is an absolute complete classification of $=^{+\alpha}$ using invariants in $\mathcal{P}^{\alpha}_{\aleph_0}(\mathbb{R})$. See also [Sha23, 7.4].

Remark 5.3. Let E and F be analytic equivalence relation on Polish spaces X and Y respectively. Suppose F admits an absolute complete classification $y \mapsto B_y$ as above. Assume further that E is Borel reducible to F. Then E admits an absolute complete classification as well (with the same type of invariants).

Definition 5.4. Let E be an equivalence relation on a Polish space X and $x \mapsto A_x$ an absolute complete classification of E. Say that a set A is a **potential E-invariant** if in some forcing extension there is an x in X such that $A = A_x$. If A is a potential invariant for E, say that A is **trivial** if there is an x in the ground model such that $A = A_x$. In a given model in which there is some x so that $A = A_x$, we will say that A is **realized**.

Example 5.5. In Example 5.2, the potential invariants for $=^+$ (with this fixed absolute classification) are precisely all sets of reals. The potential invariants for $=^{+\alpha}$ are all sets in $\mathcal{P}^{\alpha}(\mathbb{R})$.

Potential invariants directly correspond to stable virtual classes, as explained below. This approach to study virtual classes is essentially equivalent to the one taken by Ulrich, Rast, and Laskowski [URL17]. See [PS24, Section 4].

Lemma 5.6. For E as above, there is a one-to-one correspondence between

- stable virtual E-classes $(\mathbb{P}, \tau)$, and
- potential invariants A,

such that $(\mathbb{P},\tau)$ is pinned if and only if A is trivial. More specifically, a potential invariant A corresponds to $(\mathbb{P},\tau)$ if and only if $\mathbb{P}\Vdash A_\tau=\check{A}$.

Proof. Let $(\mathbb{P},\tau)$ be a stable virtual E-class. Let $G_l\times G_r$ be $\mathbb{P}\times\mathbb{P}$-generic and let x_l,x_r be the interpretations of τ according to G_l,G_r respectively. Since x_l and x_r are E-related, it follows that $A=A_{x_l}=A_{x_r}$. Furthermore, this set A is in the intersection $V[G_l]\cap V[G_r]$, which is equal to V by mutual genericity. If $(\mathbb{P},\tau)$ is pinned there is $x\in V$ such that $x\mathrel{E}x_l$. In particular, $A_x=A_{x_l}=A$, so A is a trivial. Conversely, if there is $x\in V$ with $A=A_x$ then x witnesses that $(\mathbb{P},\tau)$ is pinned: given any $\mathbb{P}$-generic G over V, $A_{\tau[G]}=A=A_x$, so $\tau[G]$ is E-related to x.

Now let A be a potential invariant for E. By assumption there is a poset $\mathbb{Q}$, a generic G and $x\in V[G]$ such that $A=A_x$. Let τ be a $\mathbb{Q}$-name such that $\tau[G]=x$. Fix a condition $q\in\mathbb{Q}$ such that q forces that $A_\tau=A$ and define $\mathbb{P}=\mathbb{Q}\restriction p$. Now $(\mathbb{P},\tau)$ is a stable virtual E-class such that $\mathbb{P}\Vdash A_\tau=A$.

□

A corollary of the proof is that the map $(\mathbb{P},\tau)\mapsto A$ described there is a complete classification of all stable virtual E-classes, up to E-equivalence, where the classifying invariants are precisely the potential invariants for E.

Corollary 5.7. For E as above, the pinned cardinal $\kappa(E)$ is the smallest cardinal κ such that any potential E-invariant is trivial in a generic extension by a poset of size $<\kappa$.

Example 5.8. For $\alpha<\omega_1$, $\kappa(=^{+\alpha})=\beth_\alpha^+$. In particular, for $\alpha<\beta$, $=^{+\beta}\not\leq_B=^{+\alpha}$.

See [LZ20, 2.5.5] for a proof of the full Friedman-Stanley theorem using these ideas. The original proof of Friedman and Stanley used Borel determinacy.

6. Cardinal arithmetic considerations

In this section we consider some applications of Corollary 4.4 where the desired inequality between the pinned cardinals is not proved outright, but is consistent. This will suffices to conclude Borel irreducibility, as the latter is absolute.

6.1. **Below $=^+$.** The first example of an unpinned Borel equivalence relation was $=^+$, specifically, appearing in Hjorth's proof that $=^+$ is not Borel reducible to a CLI group action (see Example 4.6). Kechris asked if $=^+$ is the minimal example.

Question 6.1 (see [Kan08b, Question 17.6.1])**.** If E is an unpinned Borel equivalence relation, must $=^+$ be Borel reducible to E?

This was refuted by Zapletal in [Zap11]. We present a counterexample. Consider the graph $\mathcal{G}$ defined on $2^{\mathbb{N}}$ by[3]

$$x\mathrel{\mathcal{G}}y\iff x\leq_T y\text{ or }y\leq_T x,$$

where $\leq_T$ is the Turing reducibility relation on $2^{\mathbb{N}}$. One can verify that:

(1) $\mathcal{G}$ has a clique of size $\aleph_1$;
(2) $\mathcal{G}$ does not have cliques of size $\aleph_2$.

[3]Zapletal used a different graph, see [Zap11, Fact 2.2]. This version was suggested by Clinton Conley.

Let $\mathcal{C} \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ be the set of all $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\{x(n) : n \in \mathbb{N}\}$ is a $\mathcal{G}$-clique. Consider the Borel equivalence relation E defined as $(=_{2^{\mathbb{N}}})^+ \restriction \mathcal{C}$, the restriction of $(=_{2^{\mathbb{N}}})^+$ to the (invariant) Borel set $\mathcal{C}$.

Note that the map $x \mapsto \{x(n) : n \in \omega\}$ remains an absolute complete classification of E. The potential invariants are precisely all sets of reals $A \subseteq 2^{\mathbb{N}}$ which form a $\mathcal{G}$-clique. In particular, all the potential invariants are of size $\leq \aleph_1$, and so $\kappa(E) \leq \aleph_1^+$. Also, any uncountable $\mathcal{G}$-clique $A \subseteq 2^{\mathbb{N}}$ is a non-trivial potential invariant. It follows that E is not pinned, and $\kappa(E) = \aleph_1^+$. We claim that E is a counterexample to Question 6.1.

Claim 6.2. $=^+$ is not Borel reducible to E.

Proof. This follows from Corollary 4.4. More specifically, move to a forcing extension in which CH fails: $\beth_1 > \aleph_1$. In this extension, $\kappa(E) = \aleph_1^+ < \beth_1^+ = \kappa(=^+)$, and therefore $=^+ \not\leq_B E$. Finally, the statement '$=^+ \leq_B E$' is absolute for forcing extensions, and therefore $=^+$ is not Borel reducible to E in the ground model as well. □

6.2. **A comic relief.** For $x \in 2^{\mathbb{N}}$, let x' be its Turing jump. Recall the following fact from computability theory.

Theorem 6.3. There exists a sequence of reals $x_0, x_1, \dots$ so that $x_n \geq_T x'_{n+1}$.

Define $x \prec y \iff x' \leq_T y$. Define a graph $\mathcal{G}$ on $2^{\mathbb{N}}$ by

$$x \; \mathcal{G} \; y \iff x \prec y \text{ or } y \prec x.$$

Define $\mathcal{C} \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ as all $\mathcal{G}$-cliques (a Borel set), and E as $(=_{2^{\mathbb{N}}})^+ \restriction \mathcal{C}$, as above.

Let $(x_\alpha : \alpha < \omega_1)$ be a $\leq_T$-increasing sequence so that $x_{\alpha+1} = (x_\alpha)'$. Then $\{x_\alpha : \alpha < \omega_1\}$ is a non-trivial potential invariant for E, and so E is not pinned. In particular, $E \not\leq_B =_{\mathbb{R}}$ (recall Example 1.8). Since E is Borel, it follows from Theorem 4.1 that $E_0 \leq_B E$.

Proof of Theorem 6.3. Assume for contradiction that the theorem fails, then the relation $\prec$ is wellfounded. For $x \in X$, let A_x be an enumeration of $\{x_n : n \in \mathbb{N}\}$ of order type o.t.$(\prec\restriction \{x_n : n \in \mathbb{N}\})$. Then $x \mapsto A_x$ is an absolute complete classification of E, using $(2^{\mathbb{N}})^{<\omega_1}$ as invariants. The latter may be coded as members of $2^{<\omega_1}$. We conclude that E_0 admits an absolute complete classification using sets of ordinals as invariants, which is a contradiction, as we will prove in Proposition 9.2 below.[4] □

6.3. **Analytic equivalence relations.** In the context of non-Borel analytic equivalence relation, especially non-orbit equivalence relations, the common notion of reducibility is *absolutely* Δ^1_2*-reducibility.* Corollary 4.4 still holds: $\kappa(E) \leq \kappa(F)$ if there is a is an absolute Δ^1_2-reduction from E to F. In fact, this is true if there is a reduction which is an absolute map as in Definition 5.1, where 'complete classification' is replaced with 'reduction'.

Calderoni and Thomas [CT19] studied the isomorphism relation $\cong_{\mathrm{TA}}$, and the bi-embeddability relation $\equiv_{\mathrm{TA}}$, for torsion abelian groups. They proved $\equiv_{\mathrm{TA}}$ is

[4] Another way of reaching the contradiction is to show that the classification $x \mapsto A_x$ is an 'Ulm-type classification' (see [HK95]), contradicting the fact that E_0 does not admit such classification.

strictly below $\cong_{\mathrm{TA}}$ with respect to absolutely Δ^1_2-reducibility.[5] Both $\equiv_{\mathrm{TA}}$ and $\cong_{\mathrm{TA}}$ are analytic equivalence relations with unbounded pinned cardinal. To distinguish between them, Calderoni and Thomas used the following local notion.

Definition 6.4 ([LZ20, 2.5.1])**.** Let $\mathbb{P}$ be a poset. Consider all stable virtual classes of the form $(\mathbb{P},\tau)$, up to E-equivalence. Let $\lambda(E,\mathbb{P})$ be the cardinality of this quotient.

From Lemma 4.2 we again see:

Corollary 6.5. If there is an absolute reduction from E to F, then $\lambda(E,\mathbb{P}) \leq \lambda(F,\mathbb{P})$

Suppose now E admits an absolute classification, $x \mapsto A_x$, as in Section 5. Recall Lemma 5.6, that there is a one-to-one correspondence taking a stable virtual E-class $(\mathbb{P},\tau)$ to a potential invariant A so that $\mathbb{P} \Vdash A_\tau = \check{A}$.

Corollary 6.6. $\lambda(E,\mathbb{P})$ is the cardinality of the set of all potential invariants for E which become trivial after forcing with $\mathbb{P}$.

Example 6.7.
(1) For E_{ω_1}, the potential invariants are ordinals, the trivial ones are countable ordinals. The ordinals which become countable after forcing with $\mathrm{Col}(\omega,\kappa)$ are precisely the ordinals below κ^+, so $\lambda(E_{\omega_1},\mathrm{Col}(\omega,\kappa)) = \kappa^+$.
(2) For $=^+$ the potential invariants are sets of reals, the trivial ones are the countable sets of reals. The sets of reals which become countable after forcing with $\mathrm{Col}(\omega,\omega_1))$ are precisely those of size $\leq \aleph_1$, so $\lambda(=^+,\mathrm{Col}(\omega,\omega_1)) = \beth_1^{\aleph_1}$.

We sketch here the proof from [CT19] that there is no absolute reduction from $\cong_{\mathrm{TA}}$ to $\equiv_{\mathrm{TA}}$. In fact, they show that there is no absolute reduction from $\cong_p$ to $\equiv_{\mathrm{TA}}$, where $\cong_p$ is the isomorphism relation restricted to torsion abelian p-groups. By the discussion above, it suffices to show that $\lambda(\mathbb{P},\cong_p) > \lambda(\mathbb{P},\equiv_{\mathrm{TA}})$, for some poset $\mathbb{P}$, in some forcing extension. We show below (Propositions 5.8 and 5.9 in [CT19]) that

(1) $\lambda(\cong_p,\mathrm{Col}(\omega,\omega_1)) \geq 2^{\aleph_1}$;
(2) $\lambda(\equiv_{\mathrm{TA}},\mathrm{Col}(\omega,\omega_1)) \leq \aleph_2^{\aleph_0}$.

Working in a generic extension in which $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$ we have

$$\lambda(\cong_p,\mathrm{Col}(\omega,\omega_1)) \geq 2^{\aleph_1} > \aleph_2 = \aleph_2^{\aleph_0} \geq \lambda(\equiv_{\mathrm{TA}},\mathrm{Col}(\omega,\omega_1)),$$

concluding the desired irreducibility.

6.3.1. *Isomorphism for torsion abelian groups.*

Corollary 6.8. Under the assumptions above, $A = (\sigma(\lambda,k) : \lambda < \kappa,\, k \in \mathbb{N})$ is a potential invariant for $\cong_p$, which is realized by any forcing which collapses κ to be countable.

For $\xi \in 2^{\omega_1}$, define $\sigma^\xi = \big(\sigma^\xi(\lambda,k) : \lambda < \omega_1,\, k \in \mathbb{N}\big)$, $\sigma^\xi(\lambda,k) \in \{0,1\}$, by

$$\text{if } \xi(\lambda) = 0, \text{ then } \sigma^\xi(\lambda,k) = 1 \iff k \text{ is even}$$

[5] In [CT19] they consider Δ^1_2-reductions, under the assumption that there exists a Ramsey cardinal, which ensures that such maps are absolute.

$$\text{if } \xi(\lambda) = 1, \text{ then } \sigma^{\xi}(\lambda, k) = 1 \iff k \text{ is odd}$$

Then $\{\sigma^{\xi} : \xi \in 2^{\omega_1}\}$ are distinct potential invariants for $\cong_p$ which become trivial after collapsing ω_1. We conclude that $\lambda(\cong_p, \mathrm{Col}(\omega, \omega_1)) \geq 2^{\aleph_1}$.

6.3.2. *Bi-embeddability for torsion abelian groups.*

7. Symmetric models

We saw that studying virtual equivalence classes in different models of set theory is useful to prove irreducibility results. For example, we were able to distinguish between various unpinned equivalence relations, by working in ZFC models with certain cardinal arithmetic assumptions, distinguishing their pinned cardinals.

Next we will apply similar ideas to distinguish between some equivalence relations which are pinned. The key is to work in choiceless models, where the equivalence relations may become unpinned.

Recall the examples from Section 1.1. The proof that $=_{\mathbb{R}}$ is pinned (Example 1.8), and the proof that a countable equivalence relation is pinned (Example 1.9) can be carried out in ZF. However, the proof in Example 1.10 used the axiom of choice. Similarly Theorem 1.12 used the axiom of choice, specifically DC. Below we use symmetric models in which DC, and various weak fragments of choice, fail.

In Section 7.2 we prove a Borel irreducibility result by considering the axiomatic strength (as a fragment of choice) of the statement 'E is pinned'. In Section 8 we consider finer questions of definability in symmetric models.

7.1. **Abstract nonsense.** Suppose A is a set in some generic extension of V. Let $V(A)$ be the minimal transitive model of ZF extending V and containing the set A. For example, this model can be written as the class directed union of $L(A, x)$ for $x \in V$. We will sometimes call this the model **generated by** A (over V).

There are similar ways of forming symmetric models: $L(A)$, $\mathrm{HOD}_{\mathrm{tc}\{A\}}$, $\mathrm{HOD}_{V,\mathrm{tc}\{A\}}$. For the A's we study here, all these models will have the same relevant properties. Also in our examples $V(A)$ and $\mathrm{HOD}_{V,\mathrm{tc}\{A\}}$ will coincide.

We view $V(A)$ as a (minimal) definable closure of A:

Fact 7.1. The following holds in $V(A)$. For any set X, there is some formula ψ, parameters $\bar{a}$ from the transitive closure of A and $v \in V$ such that X is the unique set satisfying $\psi(X, A, \bar{a}, v)$. Equivalently, there is a formula φ such that $X = \{x : \varphi(x, A, \bar{a}, v)\}$.

We will be particularly interested in sets definable from A and parameters in V alone.

We will primarily work with models $V(A)$ in which the axiom of choice fails.

Fact 7.2. If V satisfies AC and $A \subseteq V$ then $V(A)$ satisfies AC.

In particular, if A is a set of ordinals then $V(A)$ satisfies choice. If x is member of a Polish space, then x can be coded as a subset of $\mathbb{N}$, and therefore $V(x)$ satisfies choice as well.

7.2. **E_0^{ω} as an unpinned equivalence relation.** First let us see how pinned equivalence relations become unpinned. Let E_0^{ω} on $(2^{\omega})^{\omega}$ be defined by pointwise-equivalence. Note that the map

$$x \mapsto A_x = ([x(n)]_{E_0} : n < \omega)$$

is a complete classification for E_0^ω satisfying the required absoluteness properties. The invariants are countable sequences of E_0-classes. (Being an E_0-class is absolute. Same for other countable equivalence relations.)

Fix $x \in (2^\omega)^\omega$ a Cohen generic over V. Let $A_n = [x(n)]_{E_0}$ and $A = (A_n : n < \omega)$.

Remark 7.3. In $V(A)$, A is a potential invariant for E_0^ω.

Proposition 7.4. In $V(A)$, $\prod_{n\in\omega} A_n = \emptyset$.

It follows that in $V(A)$, A is a non-trivial potential invariant for E_0^ω.

Corollary 7.5. E_0^ω is not pinned in $V(A)$.

Recall that if E is a countable equivalence relation then "E is pinned" is provable in ZF (Example 1.9).

Corollary 7.6. E_0^ω is not Borel reducible to any countable Borel equivalence relation.

Proof. In $V(A)$, we see that E_0^ω is not Borel reducible to any countable Borel equivalence relation, by Corollary 4.3. By absoluteness, the same is true in V. □

We now prove Proposition 7.4.

Lemma 7.7. Suppose $Z \in V(A)$, $Z \subseteq V$ is definable using $x_0, ..., x_n$ and A. Then $Z \in V[x_0, ..., x_n]$.

Proof. Fix a formula ϕ and a parameter $v \in V$ such that, in $V(A)$,

$$z \in Z \iff \phi(z, A, x_0, ..., x_n, v).$$

Claim 7.8. Suppose p, q are conditions agreeing on $x_0, ..., x_n$, then p, q cannot force conflicting statements about $\phi^{V(\dot{A})}(\check{z}, \dot{A}, \dot{x}_0, ..., \dot{x}_n, \check{v})$.

Proof. Assume to the contrary, that $p \Vdash \phi^{V(\dot{A})}(z, ...)$ and $q \Vdash \neg\phi^{V(\dot{A})}(z, ...)$. Without loss of generality, assume that our generic $x \in (2^\omega)^\omega$ extends p. Let x' be the result of making finite changes to $x_{n+1}, x_{n+2}, ...$ so that x' extends q. Note that x' is generic over V. Furthermore, $\dot{A}[x] = \dot{A}[x'] = A$, and $x'_i = x_i$ for $i \leq n$.

Working in $V[x]$, we conclude that

$$\phi^{V(A)}(z, A, x_0, ..., x_n, v).$$

However, working in $V[x']$ we conclude that

$$\neg\phi^{V(A)}(z, A, x_0, ..., x_n, v),$$

a contradiction. □

Finally, we can define Z in $V[x_0, ..., x_n]$ as all $z \in V$ such that there is some p in the Cohen forcing which agrees with $x_0, ..., x_n$ and such that $p \Vdash \phi^{V(\dot{A})}(\check{z}, \dot{A}, \dot{x}_0, ..., \dot{x}_n, \check{v})$. □

Proof of Proposition 7.4. If $y \in \prod_{n\in\omega} A_n$, then $y \in (2^\omega)^\omega$ can be coded as a subset of $\omega \times \omega$, therefore as a subset of V. If additionally $y \in V(A)$, we conclude from the lemma that $y \in V[x_0, \dots, x_n]$ for some $n < \omega$. It follows that $x_{n+1} \in V[y_{n+1}] \subseteq V[y] \subseteq V[x_0, \dots, x_n]$. This is a contradition, as $x_{n+1} \in 2^\omega$ is Cohen-generic over $V[x_0, \dots, x_n]$. □

This direction is further explored in [Sha22]. For example, for countable Borel equivalence relations E, F so that E is (μ, F)-ergodic, for some probability measure μ, there is a symmetric model (a submodel of a random real extension) in which E^ω is unpinned, yet F^ω is pinned (see [Sha22, Proposition 3.11]). Also, there is a model in which E^ω is pinned, for any countable Borel equivalence relation E, yet the axiom 'countable choice for countable sets of reals' fails ([Sha22, Theorem 4.10]).

8. Definability of potential invariants

Let E and F analytic equivalence relations on Polish spaces X and Y respectively, and $f\colon X \to Y$ a Borel homomorphism from E to F, that is, $x \mathrel{E} y \implies f(x) \mathrel{F} f(y)$. Then for a stable virtual E-class $(\mathbb{P}, \tau)$, $(\mathbb{P}, f(\tau))$ is a stable virtual F-class. In other words, there is a map, definable using f as a parameter, sending stable virtual E-classes to stable virtual F-classes, and this holds uniformly in any generic extension.

Example 8.1. Assume further that E and F admit absolute classifications $x \mapsto A_x$ and $y \mapsto B_y$ respectively. Then stable virtual classes correspond to potential invariants (see Lemma 5.6). In this case, a Borel homomorphism f from E to F corresponds to a definable map, using as parameters f, E, F and the complete classifications, sending potential E-invariants to potential F-invariants.

For concreteness, let us write it here. The map sends a potential E-invariant A to the unique set B satisfying

'for some (equivalently, any) $x \in X$ in a generic extension,
if $A = A_x$ then $B = B_{f(x)}$'.

Furthermore, if $f\colon X \to Y$ is a reduction of E to F, then the map $A \mapsto B$ is injective. Moreover, in this case A can be defined from its image B as follows

'for some (equivalently, any) $y \in Y$ in a generic extension,
if $B = B_y$ and $x \in X$ is such that $f(x) \mathrel{F} y$, then $A = A_x$'.

Corollary 8.2. For E and F as above, if $E \leq_B F$ and A is a potential E-invariant in some generic extension, and B is the potential F-invariant corresponding to it, then $V(A) = V(B)$, where $V(A)$ is the minimal transitive extension of V containing A.

Example 8.3. Let us prove again that $E_0^{\mathbb{N}}$ is not Borel reducible to a countable Borel equivalence relation, using the symmetric model $V(A)$ from Section 7.2.

Assume for contradiction that there is a Borel reduction from $E_0^{\mathbb{N}}$ to a countable Borel equivalence relation F. Since F is countable, the map $x \mapsto [x]_F$ is an absolute complete classification. By Corollary 8.2 we conclude that $V(A) = V(B)$ where B is an F-class, that is, $B = [y]_F$ for some $y \in V(B)$. It follows that $V(A) = V(y)$, and therefore $V(A)$ satisfies choice, by Fact 7.2, a contradiction to Proposition 7.4.

Similar to Example 8.1, we can extend an absolute E-invariant map to be defined on potential invariants.

Definition 8.4. Assume that E on X admits an absolute complete classification $x \mapsto A_x$ and $c\colon X \to I$ is a an absolute E-invariant map. We define c on potential E-invariants as follows. Given a potential E-invariant A, define

$c(A)$ to be the unique $B \in I$ so that for some (equivalently, any) $x \in X$ in a generic extension, if $A = A_x$ then $B = c(x)$.

Note that the map on potential invariants $A \mapsto c(A)$ is defined uniformly, using the parameters used in the absolute definitions of $c\colon X \to I$ and of $x \mapsto A_x$.

9. Ergodicity and unclassifiability

Recall that E_0 is defined on $2^{\mathbb{N}}$ as eventual equality between binary sequences. E_0 can be seen as the orbit equivalence relation induced by an action of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2$. A fundamental result in the theory of classification is that an E_0 is not *concretely classifiable*, that is, $E_0 \not\leq_B =_{\mathbb{R}}$. The standard argument relies on a basic ergodic theoretic technique.

Fact 9.1. Let Γ be a countable group and X a Polish space. Assume that $a\colon \Gamma \curvearrowright X$ a generically ergodic action, that is, any Borel invariant set is either meager or comeager. Then any E_a-invariant Borel map $X \to \mathbb{R}$ is constant on a comeager set.

As each orbit is countable, we conclude that there is not Borel reduction from E_a to $=_{\mathbb{R}}$.

Similarly, in the measure theoretic context, if Γ is a countable group, $a\colon \Gamma \curvearrowright X$ is an ergodic action on a standard measure space (X, ν) (any invariant Borel set is either null or conull), then any E_a-invariant map $X \to \mathbb{R}$ is constant on a full measure set.

The action of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2$ on $2^{\mathbb{N}}$ described above is generically ergodic, and ergodic with respect to the coin-flipping measure on $2^{\mathbb{N}}$.

The goal of this section is to provide a generalized ergodicity criterion which

- implies unclassifiability by more complex invariants, such as sets of real numbers;
- is simple to apply, like the above ergodicity argument.

First, we view the unclassifiability of E_0 through an equivalent forcing point of view. Recall that a real number may be identified as a subset of ω. The unclassifiability of E_0 is true in greater generality. For example, in $L(\mathbb{R})$ there is no injective map from $2^{\mathbb{N}}/E_0$ into $\mathcal{P}(\alpha)$ for any ordinal α. (See [Hjo95], [CK11]. Also [Sha23, 7.7])

Proposition 9.2. There is no absolute complete classification of E_0 using invariants which are sets of ordinals. In fact, any absolute invariant map from E_0 to sets of ordinals must send a comeager subset of $2^{\mathbb{N}}$ to the same set of ordinals.

The proposition relies on the weak homogeneity of Cohen forcing $\mathbb{P}$: given any two conditions $p, q \in \mathbb{P}$ there is an automorphism γ of $\mathbb{P}$ so that $\gamma \cdot p$ is compatible with q. Let $x \in 2^{\mathbb{N}}$ be a Cohen generic real over L. Feferman showed that there is no definable well ordering of the reals in $L[x]$. Levy showed that in fact any hereditarily ordinal definable set in $L[x]$ is in L (see [Kan06, Section 7]). The following is a mild generalization, allowing the the parameter $[x]_{E_0}$ in definitions.

Claim 9.3. Let $x \in 2^{\mathbb{N}}$ be a Cohen generic real over V. If $Z \in V[x]$ is a set of ordinals which is definable using $[x]_{E_0}$ and parameters from V alone, then $Z \in V$.

Proof. Fix a name $\dot{Z}$ for Z so that it is forced that $\dot{Z}$ is definable from $[\dot{x}]_{E_0}$ and ground model parameters, where $\dot{x}$ is the canonical name for the generic real $x \in 2^{\mathbb{N}}$. We will simply define Z in V as the set of all ordinals α for which it is forced that $\check{\alpha} \in \dot{Z}$.

To see that it works, it suffices to show that no two conditions can disagree on a statement of the form $\check{\alpha} \in \dot{Z}$. For any two conditions p, q there is an automorphism

γ of Cohen forcing, flipping finitely many bits of x, so that $\gamma \cdot p$ is compatible with q. Since $[\gamma \cdot x]_{E_0} = [x]_{E_0}$, p and $\gamma \cdot p$ agree on the statement $\check{\alpha} \in \dot{Z}$. Since $\gamma \cdot p$ is compatible with q, they agree on the statement $\check{\alpha} \in \dot{Z}$ as well. □

Proof of Proposition 9.2. Note that the map $x \mapsto [x]_{E_0}$ is an absolute complete classification of E_0. Assume that $x \mapsto B_x$ is an absolute E_0-invariant map where B_x is always a set of ordinals. Let $x \in 2^{\mathbb{N}}$ be a Cohen generic real over V, and let $A = [x]_{E_0}$. As in Section 8 conclude that $B = B_x$ is definable from $[x]_{E_0}$ and parameters in V, and therefore $B \in V$, since B is a set of ordinals. We conclude that $B_{\dot{x}} = \check{B}$ is forced, and therefore there is a comeager set of $x \in 2^{\mathbb{N}}$ for which $B_x = B$. In particular, $x \mapsto B_x$ is not a complete classification. □

We will repeat these ideas in much greater generality below, which will allow us to prove generalized unclassifiability results.

Definition 9.4. Let $\mathbb{P}$ be a forcing poset, τ a $\mathbb{P}$-name. Say that τ is **$\mathbb{P}$-ergodic** if for any $p, q \in \mathbb{P}$ there is an automorphism γ of $\mathbb{P}$ so that $\gamma \cdot p$ is compatible with q and γ preserves τ, that is, for any $\mathbb{P}$-generic G, $\tau[G] = \tau[\gamma \cdot G]$.

Example 9.5. Let τ be a canonical $\mathbb{P}$-name for the emptyset. Then τ is $\mathbb{P}$-ergodic if and only if $\mathbb{P}$ is weakly homogeneous.

Definition 9.6. Say that a $\mathbb{P}$-name τ is **non-trivial** if there are two generic filters G, H so that $\tau[G] \neq \tau[H]$. Equivalently, τ is non-trivial if it is not forced to be a member of the ground model.

Example 9.7. Let (X, ν) be a standard measure space and $\Gamma \curvearrowright X$ an ergodic group action by measure preserving transformations. Let $\mathbb{P}$ be Random real forcing on X, $\dot{x}$ the name for the generic random real. Then $\Gamma \cdot \dot{x}$ is $\mathbb{P}$-ergodic.

Example 9.8. Let X be a Polish space and $\Gamma \curvearrowright X$ a generically ergodic action by homeomorphisms. Let $\mathbb{P}$ be Cohen forcing on X and $\dot{x}$ the name for the generic Cohen real. Then $\Gamma \cdot \dot{x}$ is $\mathbb{P}$-ergodic.

Theorem 9.9 (ZF)**.** Let $\mathbb{P}$ be a forcing poset, $\dot{A}$ a $\mathbb{P}$-name for a potential E-invariant. Assume that $\dot{A}$ is $\mathbb{P}$-ergodic. If $c\colon X \to I$ is an absolute E-invariant map, where in any $\mathbb{P}$-generic extension the members of I are subsets of V, then there is a fixed $B \in I$ in the ground model so that $c(\dot{A}) = \check{B}$ is forced by $\mathbb{P}$. (We may apply c to a potential invariant as in Section 8.)

Proof. Let σ be a $\mathbb{P}$-name so that it is forced by $\mathbb{P}$ that $c(\dot{A}) = \sigma$. By assumption, σ is forced to be a subset of V.

Claim 9.10. For any b in V, if some condition in $\mathbb{P}$ forces that $\check{b} \in \sigma$ then any condition in $\mathbb{P}$ forces that $\check{b} \in \sigma$.

Given the claim, we may define B to be the set of all b so that $\check{b} \in \sigma$ is forced, and conclude that $c(\dot{A}) = \sigma = \check{B}$ is forced by $\mathbb{P}$, as required.

To prove the claim, assume towards a contradiction that there are two conditions $p, q \in \mathbb{P}$ so that $p \Vdash \check{b} \in \sigma$ and $q \Vdash \check{b} \notin \sigma$. By assumption, there is an automorphism γ of $\mathbb{P}$ so that $\gamma \cdot p$ is compatible with q, and γ preserves $\dot{A}$. Since $\sigma = c(\dot{A})$ is forced, γ preserves σ as well. We conclude that $\gamma \cdot p \Vdash \check{b} \in \sigma$ and $q \Vdash \check{b} \notin \sigma$, contradicting that they are compatible. □

There are three types of generalizations here, beyond what is needed for Proposition 9.2, which will be crucial in the applications below:

(1) that A is a potential invariant (not necessarily realized in the given model);
(2) that invariants in I are arbitrary subsets of the ground model (not just ordinals);
(3) that the result will be applied in specifically designed models of ZF.

A typical application of the theorem will be to prove irreducibility to some $=^{+\alpha}$, as follows. Here $\mathcal{P}^\alpha$ denotes the iterated powerset operation, and $\mathcal{P}(\mathbb{N})$ is identified with $\mathbb{R}$.

Theorem 9.11. Let $\mathbb{P}$ be a forcing poset, $\dot{A}$ a non-trivial $\mathbb{P}$-name for a potential E-invariant. Assume that $\dot{A}$ is $\mathbb{P}$-ergodic. Assume further that forcing with $\mathbb{P}$ adds no new subsets of $\mathcal{P}^\alpha(\mathbb{N})$. Then there is no absolute complete classification of E using subsets of $\mathcal{P}^\alpha(\mathbb{N})$ as invariants. In particular, $E \not\leq_B =^{+\alpha+1}$.

For example, if $\mathbb{P}$ is adds no real numbers and $\dot{A}$ is an ergodic, non-trivial $\mathbb{P}$-name for a potential E-invariant, then $E \not\leq_B =^+$.

Proof. Assume for contradiction that $c\colon X \to I$ is a complete classification of E where I consists of subsets of $\mathcal{P}^\alpha(\mathbb{N})$. By assumption, the invariants in I are subsets of V in any $\mathbb{P}$-generic extension. By Theorem 9.9 there is a fixed set B in V so that $c(\dot{A}) = \check{B}$ is forced by $\mathbb{P}$. Finally, for any two $\mathbb{P}$-generics G, H over V, as $c(\dot{A}[G]) = c(\dot{A}[H])$ and c is a complete classification, we conclude that $\dot{A}[G] = \dot{A}[H]$, contradicting the assumption that $\dot{A}$ is not trivial. □

9.1. Irreducibility to $=^+$.

We present here two examples proving irreducibility to $=^+$ (unclassifiability by countable sets of reals).

9.1.1. *A proof of* $=^{++} \not\leq_B =^+$.

Let $\mathbb{P}$ be the poset of all countable partial functions $p\colon \mathbb{R} \to 2$, ordered by extension. We identify a generic filter with a 'generic function' $\mathbb{R} \to 2$, and in turn with a 'generic subset' $G \subseteq \mathbb{R}$. Let $\dot{B}$ be the $\mathbb{P}$-name for the set $B = \{H \subseteq \mathbb{R} : H \Delta G \text{ is countable}\}$, all subsets of $\mathbb{R}$ which have countable symmetric difference with G. Note that in the extension $V[G]$, B is a potential invariant for $=^{++}$, as it is a set of sets of reals. We will use the following basic facts:

- Forcing with $\mathbb{P}$ does not add reals;
- The generic subset G is not in the ground model.

Claim 9.12. $\dot{B}$ is $\mathbb{P}$-ergodic.

Proof. Given $p, q \in \mathbb{P}$, consider the automorphism of $\mathbb{P}$ acting on $r \in \mathbb{P}$ by flipping the value of $r(x)$ for any $x \in \operatorname{dom} p \cap \operatorname{dom} q$ for which $p(x) \neq q(x)$. Since only countably many values of G are changed, the automorphism fixes $\dot{B}$. □

We conclude from Theorem 9.9 that there is no absolute complete classification of $=^{++}$ using invariants which are subsets of V in any $\mathbb{P}$-generic extension. In any $\mathbb{P}$-generic extension, since no reals are added, a set of reals is a subset of V.

Corollary 9.13. There is no absolute complete classification of $=^{++}$ using invariants which are sets of reals. In particular, $=^{++} \not\leq_B =^+$.

9.1.2. *Archimedean groups.* Let $\mathcal{A} \subseteq \mathbb{R}^{\mathbb{N}}$ be all injective sequences $(x_n : n \in \mathbb{N})$ so that $x_0 = 0$ and $(\{x_n : n \in \mathbb{N}\}, +)$ is a subgroup of $(\mathbb{R}, +)$. $\mathcal{A}$ is a closed subset of $\mathbb{R}^{\mathbb{N}}$, equipped with the product topology.

Definition 9.14. Define the equivalence relation $\sim_{\mathcal{A}}$ on $\mathcal{A}$ as follows.

$$(x_n : n \in \mathbb{N}) \sim_{\mathcal{A}} (y_n : n \in \mathbb{N}) \iff$$
$\exists \lambda > 0$ so that the map $x \mapsto \lambda \cdot x$ is a bijection from $\{x_n : n \in \mathbb{N}\}$ to $\{y_n : n \in \mathbb{N}\}$.

This equivalence relation was studied in [CMRS23]. The main motivation is that $\sim_{\mathcal{A}}$ is equivalent to the classification problem for countable Archimedean groups. (See Proposition 3.1 in [CMRS23] and the discussion following it.) In particular, it is established in [CMRS23, Section 4] that

$$=^+ <_B \sim_{\mathcal{A}} <_B =^{++}$$

We provide here a proof of the lower bound.

Proposition 9.15. $\sim_{\mathcal{A}}$ is not Borel reducible to $=^+$.

This lower bound was recently used by [EGL24], where they study the complexity of classification for extremely amenable groups.

We identify a member $(x_n : n \in \mathbb{N}) \in \mathcal{A}$ with the subgroup $G = \{x_n : n \in \mathbb{N}\}$. Given an additive subgroup G of $\mathbb{R}$, define

$$G/a = \{g/a : g \in G\} \text{ and } A_G = \{G/a : a \in G \setminus \{0\}\}.$$

Fact 9.16 (See [CMRS23, Proposition 3.14])**.** The map

$$(x_n : n \in \mathbb{N}) \mapsto A_{\{x_n : n \in \mathbb{N}\}}$$

is an absolute complete classification of $\sim_{\mathcal{A}}$.

We will apply Theorem 9.9 using this classification. Note that a classifying invariant is a countable set of countable subgroups of $(\mathbb{R}, +)$.

Remark 9.17. We note that $\kappa(\sim_{\mathcal{A}}) = \kappa(=^+) = |\mathbb{R}|^+$, so the methods of sections 4 and 6 cannot be used to prove Proposition 9.15.

Suppose A is a potential $\sim_{\mathcal{A}}$-invariant, using the above classification. Then A is a set of sets of reals and there is $G \in A$ so that $A = \{G/a : a \in G \setminus \{0\}\}$ (though A and G may be uncountable). Since $G \subseteq \mathbb{R}$, after collapsing $|\mathbb{R}|$ to be countable we have some $(x_n : n \in \mathbb{N}) \in \mathcal{A}$ enumerating G, so that A is the classifying invariant of $(x_n : n \in \mathbb{N})$. That is, any potential $\sim_{\mathcal{A}}$-invariant is trivial after collapsing $|\mathbb{R}|$ to be countable. It follows from Corollary 5.7 that $\kappa(\sim_{\mathcal{A}}) \leq |\mathbb{R}|^+$. Note that $A = \{\mathbb{R}\} = \{\mathbb{R}/a : a \in \mathbb{R} \setminus \{0\}\}$ is a potential $\sim_{\mathcal{A}}$-invariant, and so $\kappa(\sim_{\mathcal{A}}) = |\mathbb{R}|^+$

We introduce some terminology from [CMRS23, 4.3].

Given a set $S \subseteq \mathbb{R}$, define $D(S)$ to be the set of all numbers of the form $a_1^{l_1} \cdot, \dots, \cdot a_n^{l_n}$, where $l_1, \dots, l_n$ are integers and $a_1, \dots, a_n$ are in S. Let Y be a perfect set of reals so that any $a_1, \dots, a_n \in Y$ are algebraic independent over $Y \setminus \{a_1, \dots, a_n\}$. In particular, for any two subsets $S_1, S_2 \subseteq Y$,

$$D(S_1) \cap D(S_2) = D(S_1 \cap S_2).$$

Definition 9.18. Let $\mathbb{P}$ be the poset of conditions $p = (S_p, G_p)$ where S_p is a countable subset of Y and G_p is a countable subset of $D(S_p)$. For $p, q \in \mathbb{P}$ say that q extends p if $S_q \supseteq S_q$ and $G_q \cap S(S_p) = G_p$.

$\mathbb{P}$ is a partial order with maximal element $(\emptyset,\emptyset)$. Given a generic filter $F \subseteq \mathbb{P}$, let G be the additive subgroup of $\mathbb{R}$ generated by $\bigcup_{p\in F} G_p$. Note that in the extension A_G is a potential invariant for $\sim_{\mathcal{A}}$: in any further generic extension in which G is countable, A_G is equal to A_x for any injective enumeration $x \in \mathcal{A}$ of G.

Claim 9.19. For any generic filter F, $G \notin V$. In particular $A_G \notin V$.

Proof. Fix a condition $p \in \mathbb{P}$. We need to find two extensions q_1, q_2 of p and some $x \in \mathbb{R}$ so that $q_1 \Vdash \check{x} \in \dot{G}$ and $q_2 \Vdash \check{x} \notin \dot{G}$. Since S_p is countable, we may find some $x \in Y \setminus S_p$. By the algebraic independence of the members of Y, it follows that $x \notin D(S_p)$. Define

$$S_{q_1} = S_{q_2} = S_p \cup \{x\}, \;\; G_{q_1} = G_p \cup \{x\}, \;\; G_{q_2} = G_p.$$

Then $q_1 = (S_{q_1}, G_{q_1})$ and $q_2 = (S_{q_2}, G_{q_2})$ are as required. □

Claim 9.20. $\mathbb{P}$ is countably closed. In particular, no reals are added when forcing with $\mathbb{P}$.

Claim 9.21. $\dot{A}_G$ is $\mathbb{P}$-ergodic.

Proof. Fix $a \in Y$. Let $\mathbb{P}_a = \{p \in \mathbb{P} : a \in S_p\}$, a dense open subset of $\mathbb{P}$. Define $\pi_a(p) = (S_p, a \cdot G_p)$. Then π_a is an automorphism of $\mathbb{P}_a$, which preserves $\dot{A}_G$.

Given any $p, q \in \mathbb{P}$, fix some $a \in Y \setminus (S_p \cup S_q)$. By the algebraic independence of the members of Y, it follows that $a \cdot D(S_p) \cap D(S_q) = \emptyset$, and therefore $\pi_a(p)$ and q are compatible. □

We conclude from Theorem 9.11 that $\sim_{\mathcal{A}}$ is not Borel reducible to $=^+$.

9.2. **Forcing over choiceless models.** Let $\mathcal{P}(On)$ denote all subsets of ordinals, $\mathcal{P}^2(On)$ denote all sets of sets of ordinals, and so on. As in Proposition 9.2, often times an irreducibility to $=^+$ can be strengthened to 'unclassifiability using sets of sets of ordinals as invariants'. We show this for $=^{++}$ in Corollary 9.25 below. First, note that in Theorem 9.11, we may replace $\mathbb{N}$ by some ordinal, and the same proof works. Furthermore, the proof did not use the axiom of choice.

Theorem 9.22 (ZF)**.** Let $\mathbb{P}$ be a forcing poset, $\dot{A}$ a non-trivial $\mathbb{P}$-name for a potential E-invariant. Assume that $\dot{A}$ is $\mathbb{P}$-ergodic. Assume further that forcing with $\mathbb{P}$ adds no new subsets of $\mathcal{P}^{\alpha}(On)$. Then there is no absolute complete classification of E using subsets of $\mathcal{P}^{\alpha}(On)$ as invariants.

When $\alpha = 0$ we recover Proposition 9.2. When $\alpha = 1$ we conclude a generalization to 'irreducibility to $=^+$': unclassifiability by sets of ordinals as invariants.

For $\alpha \geq 1$, for the hypothesis in Theorem 9.22 to hold, the axiom of choice must fail. We present below some instances of this scenario, and how it naturally leads to an irreducibility result.

9.2.1. *Forcing over the basic Cohen model.* Let $x \in \mathbb{R}^{\mathbb{N}}$ be Cohen generic over V, $A = \{x(n) : n \in \mathbb{N}\}$. We consider $V(A)$ as 'the basic Cohen model' (see [Kan08a]). This model has been extensively studied in the literature. A basic fact is that in $V(A)$, A is a Dedekind-finite set, that is, there is no sequence of distinct members of A. In particular, there is no $y \in \mathbb{R}^{\mathbb{N}}$ enumerating A. It follows that A is a non-trivial potential invariant for $=^+$, in $V(A)$.

Working in $V(A)$, let $\mathbb{P}$ be the poset of all finite partial functions $p\colon A \to 2$, ordered by extension. We identify a generic filter with a 'generic function' $A \to 2$, and in turn with a 'generic subset' $G \subseteq A$. Let $\dot{B}$ be a name for the set $B = \{H \subseteq A : H \triangle G \text{ is finite}\}$, all subsets of A which have finite symmetric difference with G. Note that, in the extension $V(A)[G]$, B is a potential invariant for $=^{++}$, as it is a set of sets of reals.

Claim 9.23. $\dot{B}$ is $\mathbb{P}$-ergodic.

We conclude from Theorem 9.9 that, in $V(A)$, there is no absolute complete classification of $=^{++}$ using invariants which are subsets of $V(A)$ in any $\mathbb{P}$-generic extension. The key point here is the following, due to Monro [Mon73].

Lemma 9.24. Forcing with $\mathbb{P}$ adds no subsets of V to $V(A)$.

In particular, no sets of ordinals are added. It follows that any set of sets of ordinals, in a $\mathbb{P}$-generic extension of $V(A)$, is a subset of $V(A)$.

Corollary 9.25. There is no absolute complete classification of $=^{++}$ using invariants which are sets of sets of ordinals. In particular, $=^{++} \not\leq_B =^{+}$.

This proof naturally applies to another equivalence relation which is strictly Borel reducible to $=^{++}$.

Consider the space $X = (2^{\mathbb{N}})^{\mathbb{N}} \times 2^{\mathbb{N}}$. For $(x,t) \in X$, define $A_x = \{x(n) : n \in \mathbb{N}\}$, $G_{(x,t)} = \{x(n) : t(n) = 1\}$, and

$$B_{(x,t)} = \left\{G \subseteq A_x : G \triangle G_{(x,t)} \text{ is finite}\right\}.$$

Define an equivalence relation E on X so that the map $x \mapsto B_x$ is a complete classification of E.

It is readily seen that $=^{+} \leq_B E \leq_B =^{++}$. E is an example of an equivalence relation with potential complexity $D(\mathbf{\Pi}^0_3)$.[6] It follows from the results in [HKL98] that $E <_B =^{++}$. We show here that the reduction $=^{+} <_B E$ is strict as well.

Working again in the basic Cohen model $V(A)$, we see that $\dot{B}$ is a name for a potential invariant for E. The same argument above shows

Corollary 9.26. There is no absolute complete classification of E using invariants which are sets of sets of ordinals. In particular, $E \not\leq_B =^{+}$.

As discussed right before Section 9.2.1, the key point here is Lemma 9.24, witnessing the failure of choice in $V(A)$ in a strong way. We will use the following Continuity Lemma. See [Fel71, p.133], also [CMRS20, p.19] for a proof in this presentation of the basic Cohen model.

Lemma 9.27 (Continuity Lemma)**.** Let ϕ be a formula, $\bar{a} = a_0, ..., a_{n-1}$ a finite sequence of distinct members of A, and $v \in V$. Suppose $\phi(A, \bar{a}, v)$ holds in $V(A)$. Then there are open sets $U_0, ..., U_{n-1}$ such that $a_i \in U_i$ and for any $\bar{b} = b_0, ..., b_{n-1}$ consisting of distinct elements from A, if $b_i \in U_i$ for all $i \leq n-1$, then $\phi(A, \bar{b}, v)$ holds in $V(A)$.

Proof of Lemma 9.24. Suppose τ is a $\mathbb{P}$-name for a subset of V, $\tau \in V(A)$. As in Fact 7.1, let $\bar{a} \subseteq A$ be finite, $w \in V$ a parameter, and ψ a formula so that in $V(A)$, τ is defined as the unique solution to $\psi(\tau, A, \bar{a}, w)$.

We will show that for any condition $p \in \mathbb{P}$ and any $v \in V$, if p forces $\check{v} \in \tau$ then $p \restriction \bar{a}$ forces the same. It then follows that for any generic filter $G \subseteq P$ over $V(A)$,

[6] See [HKL98] for a treatment of potential complexity. The equivalence relation E here is Borel reducible to the equivalence relation $\cong^{*}_{3,1}$ there. See also [Sha21, Sha].

the values of $\tau[G]$ are determined by the single condition $q = G \restriction \bar{a}$. That is, $\tau[G]$ may be defined in $V(A)$ as the set of all v for which $q \Vdash \check{v} \in \tau$.

Fix a condition $p \in \mathbb{P}$ and $v \in V$ so that $p \Vdash \check{v} \in \tau$. Assume that the domain of p is of the form $\bar{a}, \bar{b}$, with $\bar{b}$ disjoint from $\bar{a}$. By the Continuity Lemma, there are infinitely many distinct tuples $\bar{b}'$, disjoint from $\bar{a}$, such that $p[\bar{b}'] \Vdash v \in \tau$ as well, where $p[\bar{b}']$ is the condition with domain $\bar{a}, \bar{b}'$ defined on $\bar{b}'$ as p is defined on $\bar{b}$. Note that $p[\bar{b}']$ extends $p \restriction \bar{a}$.

Now for any r extending $p \restriction \bar{a}$, there is some $\bar{b}'$ such that $p[\bar{b}'] \Vdash \check{v} \in \tau$ and $p[\bar{b}']$ is compatible with r (take $\bar{b}'$ disjoint from the domain of r). It follows that $p \restriction \bar{a}$ forces $\check{v} \in \tau$. □

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