Classification by countable sequences of countable sets of reals

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Motivation

Given an equivalence relation E on X, a **complete classification** is a map $c\colon X \to I$ such that for all $x, y \in X$,

$$x \, E \, y \iff c(x) = c(y)$$

We say that I is a set of **complete invariants** for E.

Question

Given an equivalence relation E, what is the optimal complete classification of E?

This is preserved under Borel reductions:
Let E and F be equivalence relations on Polish X and Y respectively, $g\colon X \to Y$ a Borel reduction

$$x \, E \, y \iff g(x) \, F \, g(y).$$

Then if $c\colon Y \to I$ is a complete classification of F, $c \circ g$ is a complete classification of E.

Motivation

A **complete classification** of E is a map $c\colon X \longrightarrow I$ such that for any $x, y \in X$, xEy iff $c(x) = c(y)$.

Example

$=^+$ is defined on $\mathbb{R}^\omega$ so that the map $x \mapsto \{x(i);\, i \in \omega\}$ is a complete classification. The invariants are all countable sets of reals. However, given an invariant A, it might be very hard to *verify* that A is countable (to enumerate A).

Example

Suppose Γ is a countable group acting on X. The induced orbit equivalence relation can be classified by $x \mapsto \Gamma \cdot x$.
The invariants are countable sets of reals, with the additional property that given such invariant A we can *definably* enumerate A.

Motivation

Let E be a countable Borel equivalence relation on X.
Define E^ω on X^ω by $x \; E^\omega \; y \iff (\forall n \in \omega) x(n) \; E \; y(n)$.
E^ω can be classified by $x \mapsto \langle [x(n)]_E \mid n < \omega \rangle$.
The invariants are sequences of *definably countable* sets of reals.

Vague question

What kind of equivalence relations can be classified
by countable sequences of definably countable sets of reals?

Vague answer

There is a natural candidate
for an equivalence relation, maximal with this property.

$$\begin{array}{c} ? \\ | \\ E^\omega \\ | \\ E \\ | \\ =_{\mathbb{R}} \end{array}$$

Classification by sequences of definably countable sets

Definition

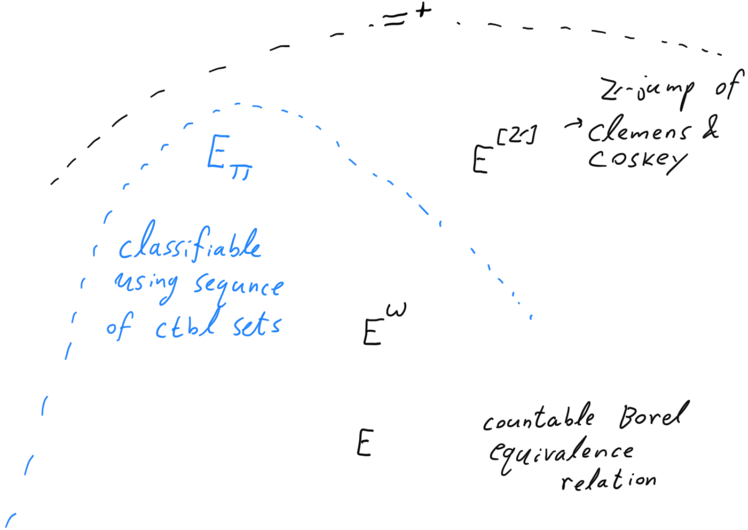
- Consider $(=^+)^\omega$ on $(\mathbb{R}^\omega)^\omega$, where $x\ (=^+)^\omega\ y$ if and only if $\{x(n)(i);\ i \in \omega\} = \{y(n)(i);\ i \in \omega\}$ for every n.
- Define $D \subseteq (\mathbb{R}^\omega)^\omega$ by $D = \{f \in (\mathbb{R}^\omega)^\omega;\ \forall n, i, j (f(n)(i) \text{ is computable from } f(n+1)(j))\}$. Define the equivalence relation $\mathbf{E}_{\mathbf{\Pi}}$ on D to be $(=^+)^\omega \restriction D$.

An invariant is a sequence of sets of reals $\langle A_n \mid n < \omega \rangle$ such that, using a member of A_{n+1} as a parameter, we can definably enumerate A_n.

Theorem (S.)

1. *For any countable E, E_Π is strictly above E^ω in the Borel reducibility hierarchy. (And is incomparable with the $\mathbb{Z}$-jumps $E^{[\mathbb{Z}]}$ defined by Clemens and Coskey.)*
2. *E_Π^ω is Borel bi-reducible to E_Π (and is "maximal").*

Pinned equivalence relations below $=^+$



Pinned equivalence relations

"Definition"

Suppose E is classifiable by countable structures via $x \mapsto A_x$. E is **pinned** if, for any set A, if there is some x in *some* generic extension, such that $A = A_x$, then $A = A_x$ for some x (in the ground model).

Example

1. $=^+$ is not pinned. The set of reals $\mathbb{R}$ is not A_x (not countable), but it *is* A_x where x is an enumeration of $\mathbb{R}$ in a collapse generic extension.
2. If E is a countable Borel equivalence relation, then E, E^ω, and $E^{[\mathbb{Z}]}$ are all pinned. Also E_Π is pinned.
   (Given an invariant $A = \langle A_n \mid n < \omega \rangle$, take $x = \langle x_n \mid n < \omega \rangle$ with $x_n \in A_n$, then $A = A_x$.)

Pinned equivalence relations

Theorem (Hjorth '99 - Thompson '06)

Let G be a Polish group, the following are equivalent:

- *G admits a complete left-invariant metric (CLI);*
- *all orbit equivalence relations induced by G-actions are pinned.*

For example, if E is a countable Borel equivalence relation, then E^ω and $E^{[\mathbb{Z}]}$ are induced by CLI group actions.

Question

Is E_Π Borel reducible to a CLI action?

Remark

Panagiotopoulos and Lupini ('18) introduced a different obstruction to being reducible to a CLI action. It is not known whether or not it is equivalent to being pinned.

Proof that E_Π is not Borel reducible to E^ω

Recall: E_Π is defined to have natural invariants of the form $\langle A_n \mid n < \omega \rangle$ such that for each n there is a member of A_{n+1} which is an enumeration of A_n.

The irreducibility proof relies on finding model of ZF separating the following very weak choice principles:

1. There is a countable sequence of countable sets of reals with not choice function, yet
2. for any CBER E, any countable sequence of E-classes admits a choice function.

Moreover the sequence in (1) looks like an invariant for E_Π.
In this model E_Π is not pinned, yet E^ω is pinned for any CBER E (also $E^{[\mathbb{Z}]}$ are all pinned).

Recall: the "basic Cohen model".
a_0, a_1, a_2... generic sequence of Cohen reals. $A = \{a_n;\ n \in \omega\}$.
Let $V(A)$ be the minimal transitive ZF extension of V which contains the set A.

Separation of fragments of choice

The "basic Cohen model": $a_0, a_1, a_2...$ generic Cohen reals, $A = \{a_n;\ n \in \omega\}$, $V(A)$ = def. closure of A over V.

- Choice fails in $V(A)$ (A cannot be enumerated);
- $V(A)$ does satisfy countable choice for countable sets of reals.

Choice for countable sets of reals vs choice for E-classes.

- Let $a^0 = a^0_0, a^0_1, a^0_2, ...$ generic Cohen reals, $A_0 = \left\langle a^0_n \mid n \in \omega \right\rangle$.
- Let $\pi_0, \pi_1, \pi_2, ...$ be Cohen generic permutations of ω. Define $a^1_n = a^0 \circ \pi_i$. $A_1 = \left\{a^1_n;\ n \in \omega\right\}$.
- ... A_{n+1} is a set of mutually generic enumerations of A_n.

Theorem

In $V(\langle A_n \mid n < \omega\rangle)$: $\prod_n A_n = \emptyset$ (so E_Π is not pinned), yet for any CBER E, and any sequence $\langle B_n \mid n < \omega\rangle$ of E classes, $\prod_n B_n \neq \emptyset$ (so E^ω is pinned).

Thanks for listening!