Actions of tame abelian product groups

Assaf Shani

Joint work with Shaun Allison

CMU

UCLA Logic Seminar,
May 2020

Tame Polish groups

Let $a\colon G \curvearrowright X$ be a continuous action of a Polish group G on a Polish space X.
The action induces an *orbit equivalence relation* E_a on X defined by $x \mathrel{E_a} y \iff \exists g \in G(g \cdot x = y)$.

Definition

A Polish group G is *tame* if for any continuous action $a\colon G \curvearrowright X$, $E_a \subseteq X \times X$ is Borel.

Theorem (Becker-Kechris)

G is tame $\iff \exists \alpha < \omega_1(E_a$ is Π^0_α for any continuous $a\colon G \curvearrowright X)$

Example

- If G is compact, all continuous actions are Π^0_1.
- If G is locally compact, all continuous actions are Σ^0_2.

Abelian product groups

G is *tame* if for any continuous action a, E_a is Borel.

Theorem (Solecki '95)

Not tame	Tame
$\mathbb{Z}^\omega$, $(\mathbb{Z}_p^{<\omega})^\omega$	$(\mathbb{Z}(p^\infty))^\omega$, $\prod_p \mathbb{Z}(p^\infty)$, $(\oplus_p \mathbb{Z}_p)^\omega$, $\prod_p \mathbb{Z}_p^{<\omega}$

Where $\mathbb{Z}(p^\infty) \simeq \left\{z \in \mathbb{C};\ \exists n(z^{p^n} = 1)\right\}$.
($\mathbb{Z}(p^\infty)$ has no infinite descending chain of subgroups.)

More generally:
Let $H_0, H_1, \ldots$ be ctbl abelian groups. Then $\prod_n H_n$ is tame iff:

- For all but finitely many n, H_n is torsion.
- For any prime p, for all but finitely many n, the p-component of H_n is of the form $F \times \mathbb{Z}(p^\infty)$, for some finite F.

Ding-Gao '17 characterized all tame abelian non-archimedean Polish groups.

Potential complexity

Definition

Given equivalence relations E and F on Polish space X, Y, say that *E is Borel reducible to F*, $E \leq_B F$, if there is a Borel map $f\colon X \to Y$ satisfying $x \, E \, x' \iff f(x) \, F \, f(x')$.

Definition

For a potential class Γ, say that E is potentially Γ if $E \leq_B F$ for some F in Γ.

Example

$=_{\mathbb{R}}$, the equality relation on $\mathbb{R}$, is Π^0_1 and not potentially Σ^0_1.

Theorem (Ding-Gao '17, extending Solecki '95)

Suppose that G is a tame abelian non-archimedean Polish group. Then all actions of G are potentially Π^0_6.

Ding and Gao noted that the existing examples were potentially Π^0_3, and conjectured that Π^0_3 is the optimal bound.

Main result

Theorem (Ding-Gao '17, extending Solecki '95)

Suppose that G is a tame abelian non-archimedean Polish group. Then all actions of G are potentially Π^0_6.

Ding and Gao noted that the existing examples were potentially Π^0_3, and conjectured that Π^0_3 is the optimal bound.

Theorem (Allison - S.)

The optimal bound is $D(\Pi^0_5)$.

$X \in D(\Gamma) \iff \exists A, B \in \Gamma (X = A \setminus B)$.

We will prove below: there is an action of a tame abelian product group which is not potentially Π^0_3, and sketch the arguments for a non Π^0_5 action.
(The latter action is by $\mathbb{Z}^{<\omega} \times \prod_p \mathbb{Z}_p^{<\omega} \times (\bigoplus_p \mathbb{Z}_p)^\omega \times (\mathbb{Z}(q^\infty))^\omega$)

Classification of potential complexities

Suppose E is induced by an action of a closed subgroup of S_∞.

Theorem (Hjorth-Kechris-Louveau '98)

The potential complexity of E is either

$$\Delta^0_1,\ \Pi^0_1,\ \Sigma^0_2,\ \Pi^0_3,\ D(\Pi^0_3),\ \Pi^0_4,\ D(\Pi^0_4),\ \Pi^0_5,\ D(\Pi^0_5),\ \Pi^0_6,\ldots$$

For example, if E is Σ^0_5 then E is potentially $D(\Pi^0_4)$.

Let $=_\mathbb{R}$ be the equality relation on $\mathbb{R}$.
Define $=^+$ (the first Friedman-Stanley jump) on $\mathbb{R}^\omega$ by
$x =^+ y \iff A^1_x := \{x(n);\, n\in\omega\} = \{y(n);\, n\in\omega\} =: A^1_y$.
Define $=^{++}$ (the second Friedman-Stanley jump) on $(\mathbb{R}^\omega)^\omega$ by
$x =^{++} y \iff A^2_x := \left\{A^1_{x(n)};\, n\in\omega\right\} = \left\{A^1_{y(n)};\, n\in\omega\right\} =: A^2_y$.

Theorem (Hjorth-Kechris-Louveau '98)

For $n\geq 2$, E is potentially Π^0_n if and only if $E \leq_B =^{+(n-2)}$.

We need to find an action which is not Borel reducible to $=^{+++}$

Classification by countable structures

Let F be an equivalence relation on Y. A **complete classification of** F is a map $c\colon Y \longrightarrow I$ such that for any $x, y \in Y$,

$$x \, F \, y \iff c(x) = c(y).$$

Complete classifications: (using hereditarily countable structures)

- $=_{\mathbb{R}}$ on $\mathbb{R}$: $x \mapsto x$;
- E a countable Borel equivalence relation: $x \mapsto [x]_E$;
- E^ω for countable E: $x \mapsto \langle [x(n)]_E \mid n < \omega \rangle$;
- $=^+$ on $\mathbb{R}^\omega$: $x \mapsto A^1_x = \{x(n);\, n \in \omega\}$;
- $=^{++}$ on $(\mathbb{R}^\omega)^\omega$: $x \mapsto A^2_x = \left\{A^1_{x(n)};\, n \in \omega\right\}$;
- $=^{+++}$ on $((\mathbb{R}^\omega)^\omega)^\omega$: $x \mapsto A^3_x = \left\{A^2_{x(n)};\, n \in \omega\right\}$.

Borel reducibility and symmetric models

Theorem (S. '19)

Suppose E and F are Borel equivalence relations, classifiable by countable structures (and fix a collection of invariants).
Assume further that E is Borel reducible to F.
Let A be an E-invariant in some generic extension of V.
Then there is an F-invariant B s.t. $B \in V(A)$ and

$$V(A) = V(B).$$

Furthermore, B is definable in $V(A)$ using only A and parameters from V.

- For example, to prove E is not Borel reducible to $=_{\mathbb{R}}$, need to show $V(A) \neq V(B)$ when B is a (definable) real.
- To show $E \not\leq =^{+}$, need to show $V(A) \neq V(B)$ when B is a (definable) set of reals.

A simple example

Example

Let $\mathbb{P}$ be Cohen forcing to add a real in 2^ω by finite approximations. Let $x \in 2^\omega$ be generic and $A = [x]_{E_0}$ its E_0-invariant.

Claim (Lévy)

If r is a real in $V(A)$ (or $r \subseteq V$) which is definable from A and parameters in V alone then $r \in V$.

It follows that $V(A) \neq V(r)$, and so $E_0 \not\leq =_{\mathbb{R}}$.

Proof of claim: For any $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$ such that π preserves A and πq and p are compatible ($\mathbb{Z}_2^{<\omega}$ acts ergodically on $\mathbb{P}$!). So if r is definable from A, any statement $v \in r$ for $v \in V$ is decided by the empty condition. $\square$

In other words: the type of $[x]_{E_0}$ over V does not depend on x.

This is just rephrasing of "$\mathbb{Z}_2^{<\omega} \curvearrowright 2^\omega$ is generically ergodic".

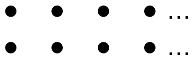
The same works for any gen. ergodic action, e.g. $\Gamma \curvearrowright 2^\Gamma$.

How to get some $\not\leq_B =^+$ proof

Proposition

Γ acts ergodically on $\mathbb{P}$ $(\forall p, q \in \mathbb{P} \exists \gamma \in \Gamma(\gamma \cdot q \parallel p))$, x is $\mathbb{P}$-generic over V. Then

1. if $B \subseteq V$ is definable from $A = \Gamma \cdot x$ then $B \in V$.
   (so if B is a definable real, $V(A) \neq V(B)$.)
2. if $\mathbb{P}$ adds no reals, then for any $B \subseteq \mathbb{R}$, if B is definable from A then $V(A) \neq V(B)$.

- $\langle A_n \mid n < \omega \rangle$ a sequence of pairs, $|A_n| = 2$. 
- $\mathbb{P} =$ finite choice functions in $\prod_n A_n$. $\mathbb{Z}_2^{<\omega} \curvearrowright \mathbb{P}$ ergodically.
- For $\mathbb{P}$ to add no reals, we must have $\prod_n A_n = \emptyset$ in V!

A non Π^0_3 action of $\mathbb{Z}_p^{<\omega} \times \Gamma^\omega$

Fix a ctbl group Γ, $X = (2^\Gamma)^2$. (e.g. $\Gamma = \mathbb{Z}(q^\infty)$ for some fixed q).
Γ acts diagonally on X (generically ergodic).
$\mathbb{Z}_2$ acts by flipping (generically free). (The actions commute.)
Let $x \in X^\omega$ be Cohen generic, $a_n = \Gamma \cdot x(n)$, $A_n = \mathbb{Z}_2 \cdot a_n$,
$\bar{A} = \langle A_n \mid n < \omega \rangle$. Each A_n is a set of two generic Γ-orbits.

$$\begin{matrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a'_0 & a'_1 & a'_2 & a'_3 & \dots \end{matrix} \quad (\text{let } x'(n) = \text{flip of } x(n),\ a'_n = \Gamma \cdot x'(n))$$

Proposition (In $V(\bar{A})$)

If $p, q \in \prod_{k<N} A_k$ then p, q are indiscernibles over $\bar{A}$ and V.

Corollary

Let $\mathbb{P}$ be the poset of all finite choice functions in $\prod_{k<\omega} A_k$.
Forcing with $\mathbb{P}$ over $V(\bar{A})$ adds no reals (nor subsets of V).

So if $a \in \prod_{k<\omega} A_k$ is $\mathbb{P}$-generic over $V(\bar{A})$, $A = \mathbb{Z}_2^{<\omega} \cdot a =$ all finite alterations of a, $B \subseteq \mathbb{R}$ is definable from A, then $V(A) \neq V(B)$.

A non Π^0_3 action of $\mathbb{Z}_p^{<\omega} \times \Gamma^\omega$ - continued

If $a \in \prod_{k<\omega} A_k$ is $\mathbb{P}$-generic over $V(\bar{A})$, $A = \mathbb{Z}_2^{<\omega} \cdot a =$ all finite alterations of a, $B \subseteq \mathbb{R}$ is definable from A, then $V(A) \neq V(B)$.
The commuting actions of Γ and $\mathbb{Z}_2$ on X give us an action $a \colon \mathbb{Z}_2^{<\omega} \times \Gamma^\omega \curvearrowright X^\omega$.
If $\Gamma = \mathbb{Z}(p^\infty)$ for some prime p, then $\mathbb{Z}_2^{<\omega} \times \Gamma^\omega$ is tame.
Note that the map sending $x \in X^\omega$ to $\mathbb{Z}_2^{<\omega} \cdot \langle a_n \mid n < \omega \rangle$ where $a_n = \Gamma \cdot x(n)$, is a complete classification of E_a.

The set A above is a generic E_a-invariant, and so $E_a \not\leq_B =^+$.

E_a is "E_0 with $\{0,1\}$ replaced by Γ-orbits".

Complex actions of the wild group $(\mathbb{Z}_p^{<\omega})^\omega$

Step 0 $\alpha^0 \colon \Gamma \curvearrowright X$ gen. ergodic and $\beta^0 \colon \mathbb{Z}_2 \curvearrowright X$ gen. free

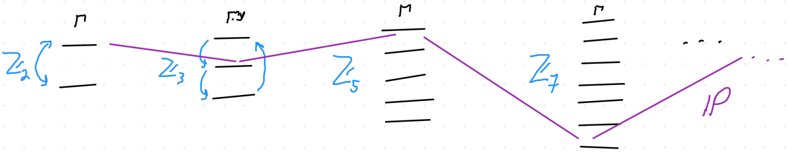
Step 1 $\alpha^1 = (\beta^0)^{<\omega} \times (\alpha^0)^\omega \colon \mathbb{Z}_2^{<\omega} \times \Gamma^\omega \curvearrowright X^\omega$.
and we have $\beta^1 \colon \mathbb{Z}_2 \curvearrowright X^\omega$, diagonal action of β^0

Step 2 $\alpha^2 = (\beta^1)^{<\omega} \times (\alpha^1)^\omega$ acting on $(X^\omega)^\omega$.
β^2 is the diagonal action of β^1. ... and repeat

For each n there is a model V_n with $\bar{A} = \langle A_n \mid n < \omega \rangle$ such that A_n is a $\beta^n \times \alpha^n$-orbit, that is, a set of two α^n-orbits, such that the poset $\mathbb{P}$ for adding a choice function through $\bar{A}$ adds no sets in $\mathcal{P}^n(\mathbb{R})$ to V_n.

- $\mathbb{Z}_2^{<\omega}$ acts ergodically on $\mathbb{P}$, so if $a \in \prod_n A_n$ is $\mathbb{P}$-generic then $A = \mathbb{Z}_2^{<\omega} \cdot a$ is an α^{n+1}-invariant and as before $V_n(A) \neq V_n(B)$ for any $=^{+n}$-invariant B. So $E_{\alpha^{n+1}} \not\leq_B =^{+n}$
- The models V_n are constructed inductively such that $V_n \subseteq V_{n+1}$ have same subsets of V_{n-1} and so the reals, sets of reals, etc., are eventually stabilized.

Non Π^0_3 actions of $\bigoplus_p \mathbb{Z}_p \times \Gamma^\omega$ and $\mathbb{Z} \times \Gamma^\omega$



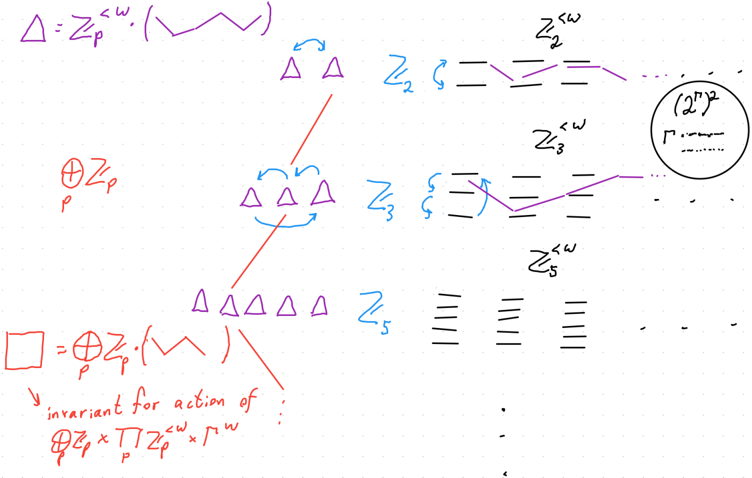
$\bigoplus_p \mathbb{Z}_p$ acts ergodically (on $\mathbb{P}$) $\rightsquigarrow$ non pot. Π^0_3 action of $\bigoplus_p \mathbb{Z}_p \times \Gamma^\omega$

$\mathbb{Z}$ acts diagonally.

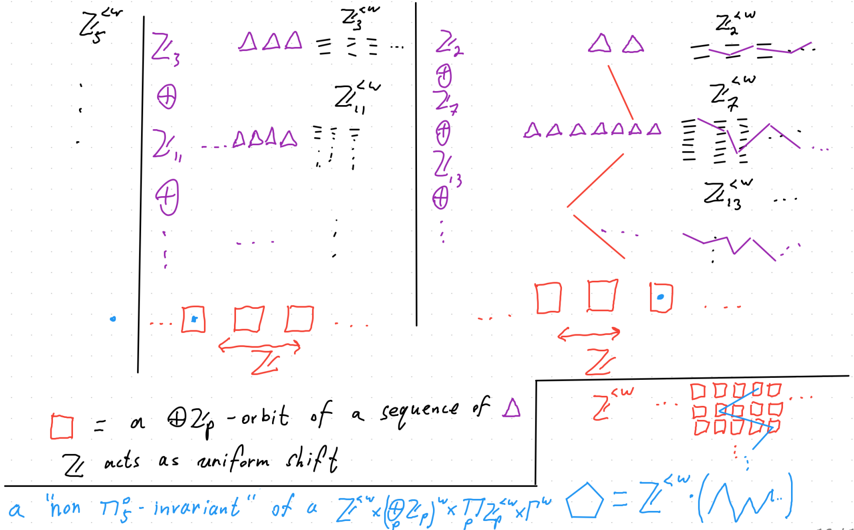
Also ergodic, by Chinese remainder theorem.

$\rightsquigarrow$ non pot. Π^0_3 action of $\mathbb{Z} \times \Gamma^\omega$

Non Π^0_4 action of $\bigoplus_p \mathbb{Z}_p \times \prod_p \mathbb{Z}_p^{<\omega} \times \Gamma^\omega$



Non Π^0_5 action of $\mathbb{Z}^{<\omega} \times (\bigoplus_p \mathbb{Z}_p)^\omega \times \prod_p \mathbb{Z}_p^{<\omega} \times \Gamma^\omega$



Optimal bounds

Theorem (Allison - S.)

Optimal bounds for tame abelian product groups $\prod_n H_n$

Property	Main example	Optimal bound
$\forall n(H_n \text{ is finite})$		Π^0_1
$\forall^\infty n$	ctbl Γ	Σ^0_2
$\forall n\ \nexists(H_n > \Gamma_0 > \Gamma_1 > \dots)$	$\mathbb{Z}(p^\infty)^\omega$	Π^0_3
$\forall^\infty n$	$\Gamma \times \dots \uparrow$	$D(\Pi^0_3)$
$\forall p, n(\{\gamma \in H_n;\ \lvert\gamma\rvert = p\} \text{ is finite})$	$(\oplus_p \mathbb{Z}_p)^\omega \times \dots \uparrow$	Π^0_4
$\forall p \forall^\infty n$	$\Gamma \times \dots \uparrow$	$D(\Pi^0_4)$
$\forall n(H_n \text{ is torsion})$	$\prod_p \mathbb{Z}_p^{<\omega} \times \dots \uparrow$	Π^0_5
$\forall^\infty n$	$\Gamma \times \dots \uparrow$	$D(\Pi^0_5)$

Thanks!