Classifying invariants and Borel equivalence relations

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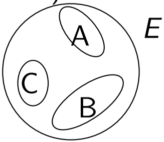
Classification problems

A **classification problem** is captured by a pair (X, E) where

- X is a collection of mathematical objects;
- E is an equivalence relation on X. (e.g. isomorphism.)

A **complete classification** is a map $c\colon X \to I$, s.t.



$$x\,E\,y \iff c(x) = c(y), \text{ for any } x, y \in X.$$

Example

- Compact orientable surface $\mapsto$ its genus $\in \mathbb{N}$.

We want the *classifying invariants* as simple as possible.

Ergodic theory examples: MPTs up to isomorphism/conjugacy.

- (Ornstein 1970) Bernoulli shift $\mapsto$ its entropy $\in [0, \infty)$.
- (Halmos-von Neumann 1942) Discrete spectrum ergodic MPT $\mapsto$ eigenvalues of its Koopman operator $\in \mathcal{P}_{\aleph_0}\mathbb{C}$.

$I =$	$\mathbb{N}$	$\mathbb{R}\,/\,\mathbb{C}$	$\mathcal{P}_{\aleph_0}(\ )$	$\mathcal{P}_{\aleph_0}\mathcal{P}_{\aleph_0}(\ )$
Invariants:		Numerical	Countable sets of	...

Borel equivalence relations, on Polish spaces

- X is a Polish space: a separable completely metrizable space.
- $E \subseteq X \times X$ is Borel, or analytic (a projection of a Borel $\subseteq X \times X \times \mathbb{R}$).

Generally: countable or separable mathematical structures can be coded as a Polish space. E.g., all the examples above.
The natural equivalence relations are analytic, sometimes Borel.

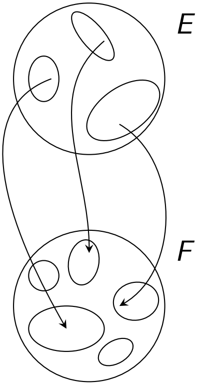
G a **Polish group**, e.g. $S_\infty =$ permutations of $\mathbb{N}$.
$a\colon G \curvearrowright X$ continuous action on a Polish space.
The induced orbit equivalence relation E_a on X:

$$x \; E_a \; y \iff g \cdot x = y \text{ for some } g \in G.$$

E_a is analytic, sometimes Borel.

Borel reducibility

E, F equivalence relations on Polish spaces X, Y.



Definition (Friedman - Stanley 1989, Harrington - Kechris - Louveau 1990)

E is **Borel reducible** to F, denoted $E \leq_B F$, if there is a Borel function $f\colon X \to Y$ s.t. for any $x_1, x_2 \in X$,

$$x_1 \; E \; x_2 \iff f(x_1) \; F \; f(x_2).$$

- Classifying invariants for F can be used for E.

Definition

E is **concretely classifiable** (numerical invariants) if there is a Borel measurable map $c\colon X \to \mathbb{R}$ so that

$$x \; E \; y \iff c(x) = c(y).$$

That is: if $E \leq_B =_{\mathbb{R}}$ (the equality relation on $\mathbb{R}$).

(Equivalently: can replace $\mathbb{R}$ by $\mathbb{C}$, $\mathbb{R}^{\mathbb{N}}$, any Polish space.)

Classifying invariants

$$E \text{ is concretely classifiable} \iff E \leq_B =_{\mathbb{R}}$$

Definition (Friedman - Stanley 1989)

Given E on X, its jump E^+ is defined on the space $X^{\mathbb{N}}$ by

$$\langle x_0, x_1, \ldots \rangle \; E^+ \; \langle y_0, y_1, \ldots \rangle \iff \forall n \exists m (x_n \; E \; y_m) \,\&\, \forall n \exists m (y_n \; E \; x_m)$$

- Define $\cong_2$ as $=^+_{\mathbb{R}}$.

$$\langle x_0, x_1, \ldots \rangle \cong_2 \langle y_0, y_1, \ldots \rangle \iff \{x_n;\; n \in \mathbb{N}\} = \{y_n;\; n \in \mathbb{N}\}$$

- E is **classifiable by countable sets of reals** if $E \leq_B \cong_2$
- Define $\cong_{\alpha+1}$ as $\cong^+_\alpha$.
- $=_{\mathbb{R}} <_B \cong_2 \;<_B\; \cong_3 \;<_B \cong_4 <_B \cdots <_B \cong_\alpha <_B \ldots$
- $\mathbb{R} \quad \mathcal{P}_{\aleph_0}\mathbb{R} \quad \mathcal{P}^2_{\aleph_0}\mathbb{R} \quad \mathcal{P}^3_{\aleph_0}\mathbb{R} \quad \ldots$

Theorem (Friedman - Stanley 1989)

For Borel E, $E <_B E^+$. (Jump operator.)

Classification by countable structures

$\langle x_0, x_1, \ldots \rangle \; E^+ \; \langle y_0, y_1, \ldots \rangle \iff \forall n \exists m (x_n \; E \; y_m) \,\&\, \forall n \exists m (y_n \; E \; x_m)$
e.g.: $\langle x_0, x_1, \ldots \rangle \cong_2 \langle y_0, y_1, \ldots \rangle \iff \{x_n;\, n \in \mathbb{N}\} = \{y_n;\, n \in \mathbb{N}\}$

- E is **classifiable by countable sets of reals** if $E \leq_B \cong_2$
- $=_{\mathbb{R}} <_B \cong_2 \; <_B \; \cong_3 \; <_B \cong_4 <_B \cdots <_B \cong_\alpha <_B \cdots$

Definition

E is **classifiable by countable structures** if it is Borel reducible to the isomorphism relation for some class of countable objects.
E.g.: countable graphs, countable groups ...

- Equivalently: if E is Borel reducible to an orbit equivalence relation induced by S_∞ (or a closed subgroup of S_∞).

Fact

E a Borel equivalence relation. The following are equivalent.

- *E is classifiable by countable structures;*
- *E is Borel reducible to $\cong_\alpha$ for a countable ordinal α.*

Classification using countable structures

Borel equivalence relations
Classifiable by
countable structures

(S_∞ actions)

$\vdots$

Friedman-Stanley 1989

$\cong_\alpha \ \sim_B$ Isomorphism for well founded trees of rank $\leq \alpha + 2$

$\cong_4$

$\cong_3$

Foreman-Louveau 1996

$\cong_2 \ \sim_B$ Ergodic discrete spectrum MPTs — Classification using countable sets of reals

Halmos-von Neumann 1942

$=_{\mathbb{R}}$ Bernoulli shifts — Ornstein 1970

$=_{\mathbb{N}}$ compact orientable surfaces

Concretely classifiable

A finer hierarchy of classifying invariants

Borel equivalence relations
Classifiable by
countable structures

(S_∞ actions)

Hjorth-Kechris-Louveau 1998
$\cong^*_{n,k}$, $n \geq 3$, $0 \leq k \leq n-2$

$\vdots$

$\cong_4$

$\cong^*_{4,2}$

$\cong^*_{4,1}$

$\cong^*_{4,0}$

$\cong_3$

$\cong^*_{3,1}$

$\cong^*_{3,0}$

$\cong_2$

Classification using countable sets of reals

$=_\mathbb{R}$

$=_\mathbb{N}$

Concretely classifiable

Potential complexity

Let E be a Borel equivalence relation on a Polish space X.

Definition

E is **potentially $\mathbf{\Gamma}$** if there is an equivalence relation F on a Polish space Y so that $F \subseteq Y \times Y$ is $\mathbf{\Gamma}$ and E is Borel reducible to F.

Example

Consider the equality relation $=_{\mathbb{R}}$ on the reals.
Then $=_{\mathbb{R}}$ is $\mathbf{\Pi}^0_1$ but not potentially $\mathbf{\Sigma}^0_1$.

Definition

$\mathbf{\Gamma}$ is *the* potential complexity of E if it is minimal such that E is potentially $\mathbf{\Gamma}$.

The equivalence relations of Hjorth-Kechris-Louveau

Hjorth-Kechris-Louveau (1998) completely classified the possible *potential complexities* of Borel equivalence relations which are induced by closed subgroups of S_∞.

Δ_1	Π^0_1	Σ^0_2	Π^0_3	$D(\Pi^0_3)$	Π^0_4	$D(\Pi^0_4)$	Π^0_5 ...
$=_\mathbb{N}$	$=_\mathbb{R}$	E_∞	$\cong_2$	$\cong^*_{3,1}$	$\cong_3$	$\cong^*_{4,2}$	$\cong_4$

(A set is in $D(\Gamma)$ if it is the difference of two sets in Γ)
For each class they found a maximal element.

Theorem (Hjorth-Kechris-Louveau 1998)

E induced by a closed subgroup of S_∞. Then

1. E is potentially $\mathbf{\Pi}^0_{n+1}$ iff $E \leq_B \cong_n$ $(n \geq 2)$;
2. E is potentially $\mathbf{\Sigma}^0_{n+1}$ iff
   E is potentially $\mathrm{D}(\mathbf{\Pi}^0_n)$ iff $E \leq_B \cong^*_{n,n-2}$ $(n \geq 3)$.

Question: What about potential complexities of other ERs?

The equivalence relations of Hjorth-Kechris-Louveau

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Δ_1	Π^0_1	Σ^0_2	Π^0_3	$D(\Pi^0_3)$	Π^0_4	$D(\Pi^0_4)$	Π^0_5 ...
$=_{\mathbb{N}}$	$=_{\mathbb{R}}$	E_∞	$\cong_2$	$\cong^*_{3,0}$	$\cong_3$	$\cong^*_{4,0}$	$\cong_4$

(A set is in $D(\Gamma)$ if it is the difference of two sets in Γ)
For each class they found a maximal element.

Theorem (Hjorth-Kechris-Louveau 1998)

E induced by an abelian closed subgroup of S_∞. Then

1. E is potentially $\mathbf{\Pi}^0_{n+1}$ iff $E \leq_B \cong_n$ $(n \geq 2)$;
2. E is potentially $\mathbf{\Sigma}^0_{n+1}$ iff
   E is potentially $\mathrm{D}(\mathbf{\Pi}^0_n)$ iff $E \leq_B \cong^*_{n,0}$ $(n \geq 3)$.

The equivalence relations of Hjorth-Kechris-Louveau

$\cong^*_{5,3}$

$\cong^*_{5,2}$

$\cong^*_{5,1}$

$\cong^*_{5,0}$

$\cong_4$

$\cong^*_{4,2}$

$\cong^*_{4,1}$

$\cong^*_{4,0}$

$\cong_3$

$\cong^*_{3,1}$

$\cong^*_{3,0}$

$\cong_2$

Definition (Hjorth-Kechris-Louveau 1998)

A classifying invariant for $\cong^*_{3,1}$ is a pair (A, R) such that

- $A \in \mathcal{P}^2_{\aleph_0}(\mathbb{R})$ (i.e., a $\cong_3$-invariant – a set of sets of reals);
- $R \subseteq A \times A \times \mathbb{R}$, given X, Y in A, there is r s.t. $R(X, Y, r)$,

$$R(X, Y_1, r) \wedge R(X, Y_2, r) \implies Y_1 = Y_2.$$

For $\cong^*_{3,0}$: replace $\mathbb{R}$ with $\mathbb{N}$.

Theorem (Hjorth-Kechris-Louveau 1998)

$\cong_{n-1} <_B \cong^*_{n,0} \leq_B \cong^*_{n,n-3} <_B \cong^*_{n,n-2} <_B \cong_n$.

Theorem (S. 2021)

For any $3 \leq n$, $k < n-2$, $\cong^*_{n,k} <_B \cong^*_{n,k+1}$.

Classification of countable Archimedean ordered groups

In joint work with F. Calderoni, D. Marker, and L. Motto Ros, we studied countable Archimedean ordered groups, up to order-isomorphism, denoted $\cong_{\mathrm{ArGp}}$.

Archimedean property for an ordered group $(G, +, <)$:
For any positive group elements x, y there is $n \in \mathbb{N}$ s.t. $x < n \cdot y$

Theorem (Calderoni, Marker, Motto Ros, and S.)

- $\cong_{\mathrm{ArGp}}$ is Borel reducible to $\cong^*_{3,1}$.
- $\cong_{\mathrm{ArGp}}$ is not Borel reducible to $\cong^*_{3,0}$.

In particular
$\cong_{\mathrm{ArGp}}$ is classifiable using countable sets of countable sets of reals,
$\cong_{\mathrm{ArGp}}$ is not classifiable using countable sets of reals,
and the potential complexity of $\cong_{\mathrm{ArGp}}$ is $D(\mathbf{\Pi}^0_3)$.

Archimedean ordered groups

Archimedean property for an ordered group $(G, +, <)$:
For any positive group elements x, y there is $n \in \mathbb{N}$ s.t. $x < n \cdot y$

Theorem (Hölder 1901)

Any Archimedean ordered group is isomorphism to a subgroup of $(\mathbb{R}, +)$.

Lemma (Hion's Lemma 1954)

$G, H \leq \mathbb{R}$, ϕ: $G \to H$ ordered preserving isomorphism. Then

$$\text{there is } \lambda > 0 \text{ with } \phi(x) = \lambda \cdot x, \text{ for all } x \in G.$$

Archimedean ordered groups: classifying invariants

$G, H \leq \mathbb{R}$, $\phi\colon G \to H$ ordered preserving homomorphism.
Then $\phi(x) = \lambda \cdot x$, for all $x \in G$, for a fixed $\lambda > 0$.

Definition

Given $G \leq \mathbb{R}$, define $G/a = \{g/a;\, g \in G\}$,

$$A_G = \{G/a;\, 0 < a \in G\} \in \mathcal{P}^2_{\aleph_0}(\mathbb{R}).$$

Proposition

The map $G \mapsto A_G$ is a complete classification of $\cong_{\mathrm{ArGp}}$.

Proof.

Suppose $x \mapsto \lambda \cdot x$ isomorphism $G \to H$.
For any $a \in G \setminus \{0\}$, $G/a = \lambda \cdot G/\lambda \cdot a = H/\lambda \cdot a \in A_H$.
So $A_G \subseteq A_H$. Therefore $A_G = A_H$. □

Moreover: Given $X = G/a$, $Y = G/b$ in A_G $(a, b \in G)$,

$$\text{let } r = b/a\,(\in G/a), \quad \text{then } Y = X/r.(\iff R(X, Y, r))$$

This is a $\cong^*_{3,1}$-classifying invariant.

Classification of countable Archimedean ordered groups

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Archimedean property for an ordered group $(G, +, <)$:
For any positive group elements x, y there is $n \in \mathbb{N}$ s.t. $x < n \cdot y$

Theorem (Calderoni, Marker, Motto Ros, and S.)

- $\cong_{\mathrm{ArGp}}$ is Borel reducible to $\cong^*_{3,1}$.
- $\cong_{\mathrm{ArGp}}$ is not Borel reducible to $\cong^*_{3,0}$.

Question

Is $\cong^*_{3,1} \leq_B \cong_{\mathrm{ArGp}}$? ($\cong^*_{3,0} \leq_B \cong_{\mathrm{ArGp}}$?) (We proved $\cong_2 \leq_B \cong_{\mathrm{ArGp}}$)
More generally, how to construct a Borel reduction from $\cong^*_{n,k}$?
Find a simpler combinatorial presentation of $\cong^*_{n,k}$.

The Γ-jumps of Clemens and Coskey

Let E be an equivalence relation on X and Γ a countable group.

Definition (Clemens - Coskey 2022)

The Γ-**jump of** E, $E^{[\Gamma]}$, is defined on X^Γ by

$$x \; E^{[\Gamma]} \; y \iff (\exists \gamma \in \Gamma)(\forall \alpha \in \Gamma) x(\gamma^{-1}\alpha) \; E \; y(\alpha).$$

Iterated Γ-jumps:
$J_1^{[\Gamma]} = (=_{\{0,1\}})^{[\Gamma]}, \; J_2^{[\Gamma]} = (J_1^{[\Gamma]})^{[\Gamma]}, \; \ldots \; J_\alpha^{[\Gamma]} \; \ldots$

Theorem (Clemens - Coskey 2022)

- Iso. on ctbl scattered linear orders of rank $1+\alpha$ is $\sim_B J_\alpha^{[\mathbb{Z}]}$.
- E Borel, induced by $\Gamma \wr \Gamma = \Gamma \ltimes \Gamma^\Gamma$, then $E \leq_B J_\alpha^{[\Gamma]}$ for some α.
- $E \mapsto E^{[\Gamma]}$
  - Is a (proper) jump operator for, e.g., $\Gamma = \mathbb{Z}$, or $\mathbb{Z}_p^{<\mathbb{N}}$;
  - Not a (proper) jump operator for $\mathbb{Z}(p^\infty)$ (quasi-cyclic p-group)

Question (Clemens-Coskey)

For which Γ is $E \mapsto E^{[\Gamma]}$ a jump operator? For $\Gamma = \bigoplus_{p \text{ prime}} \mathbb{Z}_p$?

The Γ-jump for different Γ

$x \; E^{[\Gamma]} \; y \iff (\exists \gamma \in \Gamma)(\forall \alpha \in \Gamma) x(\gamma^{-1}\alpha) \; E \; y(\alpha)$

$J_1^{[\Gamma]} = (=_{0,1})^{[\Gamma]}, \; J_2^{[\Gamma]} = (J_1^{[\Gamma]})^{[\Gamma]}, \; \dots \; J_\alpha^{[\Gamma]} \; \dots$

- Iso. on ctbl scattered linear orders of rank $1 + \alpha$ is $\sim_B J_\alpha^{[\mathbb{Z}]}$.
- E Borel, induced by $\Gamma \wr \Gamma = \Gamma \rtimes \Gamma^\Gamma$, then $E \leq_B J_\alpha^{[\Gamma]}$ for some α.
- Proper jump: YES for $\mathbb{Z}, \mathbb{Z}_p^{<\mathbb{N}}$; NO for $\mathbb{Z}(p^\infty)$; ??? for $\bigoplus_{p \text{ prime}} \mathbb{Z}_p$

Remark

$J_1^{[\Gamma]} = (=_{\{0,1\}})^{[\Gamma]}$ is the orbit ER induced by the shift $\Gamma \curvearrowright \{0,1\}^\Gamma$.

By a theorem of Gao and Jackson (2015), $J_1^{[\Gamma]} \sim_B J_1^{[\Delta]}$ for any *abelian* Γ, Δ.

Question (Clemens - Coskey)

Is this also true for $J_2^{[\Gamma]}, J_2^{[\Delta]}$?

To what extent does $E \mapsto E^{[\Gamma]}$ depend on Γ?

Theorem (S.)

E.g.: if Γ, Δ are two of $\mathbb{Z}$, $\bigoplus_{p \text{ prime}} \mathbb{Z}_p$, or $\mathbb{Z}_p^{<\mathbb{N}}$ for a prime p,

Then $J_2^{[\Gamma]}$ is not Borel reducible to $J_\alpha^{[\Delta]}$, for any α.

$\cong^*_{\alpha,\mathbf{0}}$ - classifying invariants for Γ-jumps

$x\ E^{[\Gamma]}\ y \iff (\exists \gamma \in \Gamma)(\forall \alpha \in \Gamma) x(\gamma^{-1}\alpha)\ E\ y(\alpha)$
- Propoer jump operation for $\mathbb{Z}, \mathbb{Z}_p^{<\mathbb{N}}$.

Remark

If E is $\mathbf{\Pi}^0_n$ then $E^{[\Gamma]}$ is $\mathbf{\Sigma}^0_{n+1}$.

Example ($\mathbb{Z}$-jump of $\cong_2$)

We can think of $\cong_2^{[\mathbb{Z}]}$ as follows:
A space of $\mathbb{Z}$-sequences of countable sets of reals
$\vec{A} = \langle \ldots, A_{-1}, A_0, A_1, \ldots \rangle$. $\mathbb{Z}$ acts by shifting.
Note: a sequence of sets of reals can be coded as a set of reals.
Classifying invariants (of "type" $\cong^*_{3,0}$):
a set of sets of reals $\left\{ k \cdot \vec{A};\ k \in \mathbb{Z} \right\}$,
with a relation $R(\vec{A}, \vec{B}, k) \iff k \cdot \vec{A} = \vec{B}$.

Corollary

This also shows $\cong_n <_B \cong_{n+1,0}$, given that $\cong_n <_B \cong_n^{[\mathbb{Z}]}$.

Potential complexity of Γ-jumps

In fact, there is a relationship between the equivalence relations $J_\alpha^{[\mathbb{Z}_2^{<\mathbb{N}}]}$ and the ERs constructed by Hjorth-Kechris-Louveau to realize $D(\boldsymbol{\Pi}_n^0)$ as a potential complexity.

Theorem (Clemens - Coskey 2022)

Potential complexities for $\mathbb{Z}_2^{<\mathbb{N}}$:

	$J_2^{[\mathbb{Z}_2^{<\mathbb{N}}]}$		$J_3^{[\mathbb{Z}_2^{<\mathbb{N}}]}$		$J_4^{[\mathbb{Z}_2^{<\mathbb{N}}]}$
Π_3^0	$D(\Pi_3^0)$	Π_4^0	$D(\Pi_4^0)$	Π_5^0	$D(\Pi_5^0)$ ...

Question (Clemens - Coskey)

What are the precise potential complexities of $J_\alpha^{[\mathbb{Z}]}$?

Remark

- (Clemens - Coskey) potential complexity of $J_2^{[\mathbb{Z}]}$ is $\boldsymbol{\Pi}_3^0$;
- (S.) potential complexity of $J_2^{[\mathbb{Z}^2]}$ is $D(\boldsymbol{\Pi}_3^0)$;