Classifying invariants for E_1
A tail of a generic real

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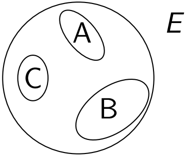
Complete classifications

Let E be an equivalence relation on X.
A **complete classification** of E is a map $c\colon X \to I$



$$x \, E \, y \iff c(x) = c(y).$$

Some "bad" examples:
- $c\colon X/E \to X$ choice function $c([x]_E) \in [x]_E$. (Not definable)
- $x \mapsto [x]_E$. (Hard to *describe* $c(x)$ from x)

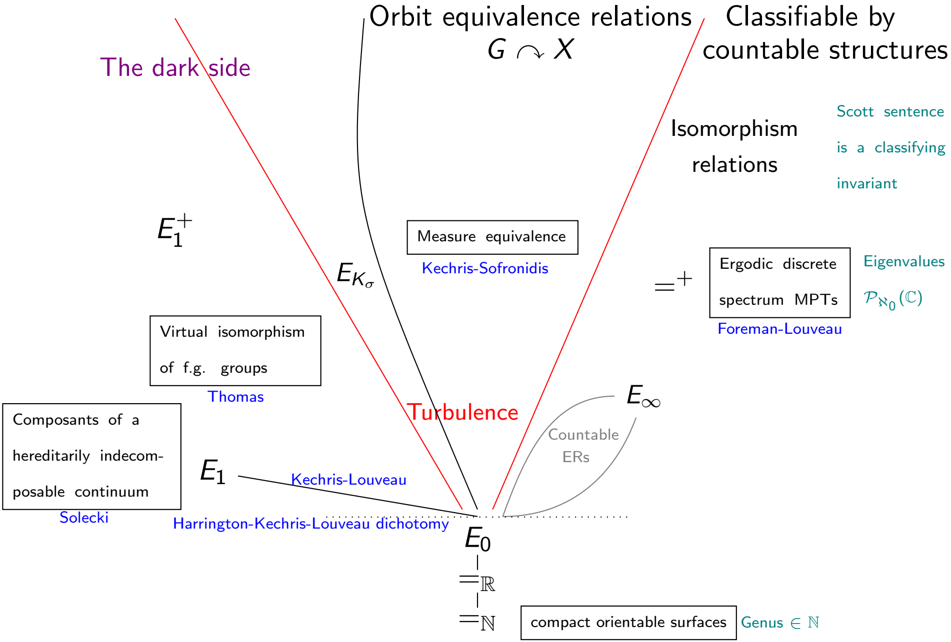
Say that c is **absolute** if:
- c is definable.
- c remains a complete classification in generic extensions.
- $c(x)^V = c(x)^{V[G]}$ for $x \in V$. ("local computation")

E,F E.R.s on Polish spaces X, Y. $f\colon X \to Y$ is a **reduction** if

$$x \, E \, y \iff f(x) \, F \, f(y).$$

E is **Borel reducible** to F, $E \leq_B F$, if there is a Borel reduction.
$\implies$ *Classifying invariants for F can be used to classify E.*

A very partial picture of Borel equivalence relations



Generically absolute classifications

Definition: $c \colon X \to I$ a definable complete classification of E.
Say that c is **generically absolute** if

- it remains a complete classification in a Cohen-real extension.
- $c(x)^V = c(x)^{V[G]}$ for $x \in V$.

Main point: allow some non-orbit relations to "be classifiable" too, while preserving the intuitions about classifications by countable structures.

Theorem

1. E_1 is generically classifiable, using $\mathfrak{b}$ many E_0-classes.
2. E_1 does not admit an absolute classification.
3. E_1 is *not* gen. class. using $< \mathbf{add}(\mathcal{B})$ many E_0-classes.

Question: is (1) optimal? (Cichon-Pawlikowsky: $\mathfrak{b}^{V[\mathrm{Cohen}]} = \mathbf{add}(\mathcal{B})^V$)

- Generic classifiability respects Borel reducibility.
- A Turbulent ER has no generically absolute classification.
- For natural CBCS ERs, same possible classifying invariants.

Conjecture E admits a generically absolute classification if and only if it does not reduce a turbulent* ER.

Classifying invariants for E_1

- E_1 on $(2^\omega)^\omega$, $x \, E_1 \, y \iff (\exists n)(\forall m > n) x(m) = y(m)$.
- Fix $x \in (2^\omega)^\omega$. Given $f \in \omega^\omega$, Let $[x \restriction f]$ be the set of all finite changes of $x \restriction f$.

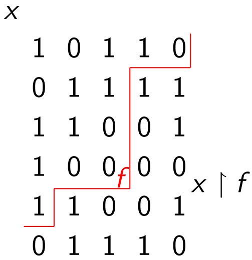
This is E_1-invariant. ($[x \restriction f]$ is an E_0-class.)

Fix $\langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$, $<^*$-unbdd, f_α increasing.

Claim: $x \mapsto \langle [x \restriction f_\alpha] \mid \alpha < \mathfrak{b} \rangle$ is a complete classification of E_1.

Moreover, this is true in any model in which $\langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$ is unbounded.

(In particular, in a Cohen-real extension.)



Proof.

- Suppose $[x \restriction f_\alpha] = [y \restriction f_\alpha]$ for all $\alpha < \mathfrak{b}$.

Fix n, $Z \subseteq \mathfrak{b}$ unbdd, so $x \restriction f_\alpha$ and $y \restriction f_\alpha$ agree past n for $\alpha \in Z$.

- Find $k \geq n$ with $\{f_\alpha(k); \alpha \in Z\}$ unbounded in ω.

(otherwise $\langle f_\alpha \mid \alpha \in Z \rangle$ is bounded).

- Now x and y agree past k, so $x \, E_1 \, y$. $\square$

An intersection model

Let $x \in \mathbb{R}^\omega$ be Cohen generic. Define the tail intersection model

$$M = \bigcap_{n<\omega} V[\langle x_n, x_{n+1}, \ldots \rangle].$$

This model was used by Kanovei-Sabok-Zapletal (2013) and Larson-Zapletal (2020), while studying E_1.
What this model looks like was left open. In particular: does it satisfy choice?

Theorem

A. Choice fails in M. (for $\mathfrak{b}$-sequences of E_0-classes)

1. E_1 is generically classifiable. (Using $\mathfrak{b}$ many of E_0-classes.)

B. $M = V(A)$ for a set (of reals) A.

2. E_1 does not admit an absolute classification.

C. Some analysis of reals in M. (Q: Does $M \models \mathrm{DC}_{<\mathbf{add}(\mathcal{B})}$?)

3. E_1 is *not* gen. class. using $< \mathbf{add}(\mathcal{B})$ many E_0-classes.

Thanks for listening!