Higher analogs of ergodicity and anti-classification

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Complete classifications

A **classification problem** is a pair (X, E):

- X is a collection of mathematical objects;
- E is an equivalence relation on X.

Definition

A **complete classification** of E is a map

$$c\colon X \longrightarrow I$$

such that for all $x, y \in X$,

$$xEy \iff c(x) = c(y).$$

We want c to be “reasonable”, and I to be as simple as possible.

Focus today: $I =$ reals, or sets of reals.

Formal definitions

Let X be a Polish space and let E be an equivalence relation on X.
• E is **concretely classifiable** if there is a Borel measurable complete classification $c\colon X \longrightarrow \mathbb{R}$. (equiv. $c\colon X \to Y$ Polish).

- Avoids arbitrary choice of representatives $c\colon X \to X$.

• E is **classifiable by countable sets of reals** if there is a Borel map $f\colon X \longrightarrow \mathbb{R}^{\mathbb{N}}$ such that

$$xEy \quad \iff \quad \{f(x)(n) \mid n \in \mathbb{N}\} = \{f(y)(n) \mid n \in \mathbb{N}\}.$$

- Avoids $c(x) = [x]_E$.

For this talk, say "reasonably definable $c\colon X \to I$"

(absoluteness for forcing extensions.)

Plan for the talk

1. Recall ergodicity as a criterion for anti-classification by real numbers.
2. Introduce an ergodicity-like criterion for anti-classification by sets of real numbers.
3. The key point:

study **virtual equivalence classes** in **choiceless models**.

(e.g. models in which countable choice for pairs fail).

(Anti-)classifiability by real numbers

Fact

Suppose a countable group Γ acts continuously on a Polish space X, and the action is generically ergodic:

$$\forall U, V \subseteq X \text{ nonempty open} \quad \exists \gamma \in \Gamma \ \ \gamma U \cap V \neq \emptyset.$$

Then the induced orbit ER E_Γ^X is not concretely classifiable.
(In fact, any invariant Borel measurable map $c \colon X \to \mathbb{R}$ sends a comeager set to a single point.)

Example

E_0 on $2^{\mathbb{N}}$: tail equivalence

$$x \, E_0 \, y \quad \Longleftrightarrow \quad \exists n \, \forall m \geq n \ \ x(m) = y(m).$$

($\bigoplus_{n \in \mathbb{N}} \mathbb{Z}_2$ acting on $2^{\mathbb{N}}$)

A forcing point of view

Theorem

E on X, $\mathbb{P}$ a poset, τ be a $\mathbb{P}$-name. Assume:

1. *τ is a name for a member of X;*
2. *τ is $(\mathbb{P}, E)$-ergodic:*

$$\forall p, q \in \mathbb{P}\ \exists \gamma \in \mathrm{Aut}(\mathbb{P}) \quad \gamma(p) \parallel q \quad \textit{and} \quad \Vdash \gamma \cdot \tau \ E\ \tau.$$

3. $\mathbb{P} \times \mathbb{P} \Vdash \tau_{left} \not{E}\ \tau_{right}$

Then there is no reasonable classification of E using invariants which are subsets of the ground model, e.g. $\mathbb{R} = \mathcal{P}(\mathbb{N})$.

Example

Let $\mathbb{C}$ be Cohen forcing (open subsets of $2^{\mathbb{N}}$).
τ name for the Cohen generic real in $2^{\mathbb{N}}$.
τ is $(\mathbb{C}, E_0)$-ergodic.

Proof sketch

E on X, $\mathbb{P} \times \mathbb{P} \Vdash \tau_{left} \not{E} \tau_{right}$, τ is $(\mathbb{P}, E)$-ergodic:

$$\forall p, q \in \mathbb{P}\ \exists \gamma \in \mathrm{Aut}(\mathbb{P}) \quad \gamma(p) \parallel q \quad \text{and} \quad \Vdash \gamma \cdot \tau\ E\ \tau.$$

Then there is no complete classification $c : X \to I$ with $I \subseteq$ subsets of the ground model (e.g. $I \subseteq \mathbb{R}$).

In fact, for E-invariant map $c\colon X \longrightarrow I \subseteq \mathcal{P}(V)$ there is $A \in I$ s.t. $\Vdash c(\tau) = A$. (So not a complete classification).
If not: find $p, q \in \mathbb{P}$ and $m \in V$ so that

$$p \Vdash m \in c(\tau), \qquad q \Vdash m \notin c(\tau).$$

By ergodicity: there is $\gamma \in \mathrm{Aut}(\mathbb{P})$ with $\gamma \cdot p \parallel q$ and

$$\gamma \cdot p \Vdash m \in c(\gamma \cdot \tau) = c(\tau), \qquad q \Vdash m \notin c(\tau).$$

A contradiction.

(Anti-)classifiability by sets of reals (?)

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E on X, $\mathbb{P}$ a poset, τ be a $\mathbb{P}$-name. Assume:

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3. $\mathbb{P} \times \mathbb{P} \Vdash \tau_{left} \not{E}\ \tau_{right}$

Then there is no reasonable classification of E using invariants which are subsets of the ground model.

Assume further that $\mathbb{P}$ does not add any reals. *Then there is no reasonable classification of E using sets of reals!*

(Anti-)classifiability by sets of reals (?)

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(Anti-)classifiability by sets of reals: the theorem

Theorem (ZF)

E on X, $\mathbb{P}$ a poset, τ be a $\mathbb{P}$-name. Assume:

1. *τ is a name for a virtual E-class;*
2. *τ is $(\mathbb{P}, E)$-ergodic:*

$$\forall p, q \in \mathbb{P}\ \exists \gamma \in \mathrm{Aut}(\mathbb{P}) \quad \gamma(p) \parallel q \quad \textit{and} \quad \Vdash \gamma \cdot \tau\ E\ \tau.$$

3. $\mathbb{P} \times \mathbb{P} \Vdash \tau_{left} \not{E}\ \tau_{right}$

Then there is no reasonable classification of E using invariants which are subsets of the ground model.

Assume further that $\mathbb{P}$ does not add any reals. Then there is no reasonable classification of E using sets of reals!

What is a virtual class?

Example

Consider $E_0^{\mathbb{N}}$ on $(2^{\mathbb{N}})^{\mathbb{N}}$: $x\, E_0^{\mathbb{N}}\, y \iff \forall n \;\; x(n)\, E_0\, y(n)$.
The $E_0^{\mathbb{N}}$-class of x is determined by the sequence

$$\langle [x(n)]_{E_0} \mid n \in \mathbb{N} \rangle \,.$$

Any sequence $\langle A_n \mid n \in \mathbb{N} \rangle$ of E_0-classes is a *virtual* $E_0^{\mathbb{N}}$-class.
(Note: may be $\prod_n A_n = \emptyset$)

Actual definition of a **virtual** E-**class**: a pair $(\mathbb{Q}, \sigma)$ such that:

1. $\mathbb{Q}$ is a forcing notion;
2. σ is a $\mathbb{Q}$-name for an element of X;
3. $\mathbb{Q} \times \mathbb{Q} \Vdash \;\; \sigma_{\mathrm{left}} \; E \; \sigma_{\mathrm{right}}$.

Example

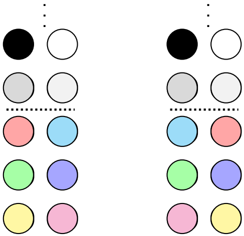
$\mathbb{Q}$ adds by finite approximations a choice function $\sigma \in \prod_n A_n$.

Applying the theorem, an example (definition)

Define E on $\left((2^{\mathbb{N}})^2\right)^{\mathbb{N}}$ by

$$xEy \iff x(n)(i)\ E_0\ y(n)(i)$$

up to finitely many swaps.



Virtual E-classes: $(A_n, B_n)_{n\in\mathbb{N}}$, where A_n, B_n are E_0-classes.

Equivalence of virtual classes: $(A_n, B_n)_{n\in\mathbb{N}}\ \ E\ \ (C_n, D_n)_{n\in\mathbb{N}}$ if they are equal after finitely many swaps.

We apply the theorem to show:

Proposition

E is not classifiable by sets of reals.

First we describe the model in which the theorem is applied.

Applying the theorem, an example (ground model setup)

Force Cohen generics

$$x \in \left((2^{\mathbb{N}})^2\right)^{\mathbb{N}}.$$

For each n, let

$$Z_n = \left\{[x(n)(0)]_{E_0}, [x(n)(1)]_{E_0}\right\}.$$

$$V \subseteq V\big(\langle Z_n \mid n \in \mathbb{N}\rangle\big) \subseteq V[x].$$

$V(A) =$ minimal transitive ZF extension of V containing A.

Fact (In $V\big(\langle Z_n \mid n \in \mathbb{N}\rangle\big)$)

1. $\prod_n Z_n = \emptyset$; in fact,
2. the forcing $\mathbb{P}$ to add a member $z \in \prod_n Z_n$ by finite approximations adds no reals.

Let τ be a $\mathbb{P}$ name for a sequence of pairs, so that $\tau(n)(0) = z(0)$ and $\tau(n)(1)$ is the other member of Z_n.
Note: $\mathbb{P}$ forces that τ is a virtual E-class.

Applying the theorem, an example (proof)

Work in $V\big(\langle Z_n \mid n \in \mathbb{N}\rangle\big)$.
$\mathbb{P}$ adds $z \in \prod_n Z_n$ by finite approximations.
τ a $\mathbb{P}$ name for a virtual E-class
$\tau(n)(0) = z(0)$ and $\tau(n)(1)$ is the other member of Z_n.

1. $\mathbb{P}$ adds no reals.
2. $\mathbb{P} \times \mathbb{P} \Vdash \tau_{left} \not\mathrel{E} \tau_{right}$.
3. τ is a $(\mathbb{P}, E)$-ergodic:

For $p, q \in \mathbb{P}$, consider $\gamma \in \mathrm{Aut}(\mathbb{P})$ swapping finitely many choices so that $\gamma \cdot p \parallel q$. Note:

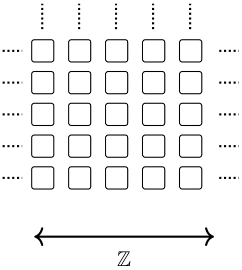
$$\gamma \cdot \tau \; E \; \tau.$$

By the theorem, E is not classifiable by sets of reals.

Another example (definition)

Define E on $\left((2^{\mathbb{N}})^{\mathbb{N}}\right)^{\mathbb{Z}}$ by

$$xEy \iff \exists \gamma \in \mathbb{Z} \quad \gamma \cdot x \; E_0^{\mathbb{N}} \; y.$$



Virtual class:

$\mathbb{Z} \times \mathbb{N}$-array of E_0-classes.

Equivalence of virtual classes:

$\mathbb{Z}$-shift.

Proposition (Allison–S., 2023)

E is not classifiable using sets of reals.

Another example (ground model setup)

Force Cohen generics $x \in \prod_{p \text{ prime}} (2^{\mathbb{N}})^p$.
For each prime p, define A_p to be the set of cyclic permutations of

$$\langle [x(p)(0)]_{E_0}, \ldots, [x(p)(p-1)]_{E_0} \rangle \,.$$

Fact (In $V(\langle A_p \mid p \text{ prime} \rangle)$)

1. $\prod_{p \text{ prime}} A_p = \emptyset$; in fact,
2. the forcing $\mathbb{P}$ to add a member $y \in \prod_p A_p$ by finite approximations adds no reals.

Let τ be the $\mathbb{P}$-name for a $\mathbb{Z} \times \mathbb{N}$-array of E_0-classes:
n'th row of τ is $y(n\text{'th prime})$ repeated along the $\mathbb{Z}$-axis.

$$\ldots y(3)(2), y(3)(0), y(3)(1), y(3)(2), y(3)(0) \ldots$$

$$\ldots y(2)(1), y(2)(0), y(2)(1), y(2)(0), y(2)(1) \ldots$$

τ is a $\mathbb{P}$-name for a virtual E-class.

Another example (proof of anti-classification)

Work in $V\big(\langle A_p \mid p \text{ prime}\rangle\big)$.
$\mathbb{P}$ adds $y \in \prod_p A_p$ by finite approximations.
τ a $\mathbb{P}$ name for a virtual E-class, $\tau(n) =$ repeat $y(n\text{th prime})$.

1. $\mathbb{P}$ adds no reals.
2. $\mathbb{P} \times \mathbb{P} \Vdash \tau_{left} \not\!E\ \tau_{right}$.
3. τ is a $(\mathbb{P}, E)$-ergodic:

For $n \in \mathbb{Z}$, let $\gamma_n \in \mathrm{Aut}(\mathbb{P})$ shift rows by n (all at once).
Note: $\gamma_n \cdot \tau\ E\ \tau$. Finally,

$$\forall p, q \in \mathbb{P} \quad \exists \gamma_n(\gamma_n \cdot p \parallel q)$$

by the Chinese Remainder Theorem.
By the theorem, E is not classifiable by sets of reals.