The Γ-jump operators of Clemens and Coskey, and actions of Polish wreath products

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Borel homomorphisms and reductions

Let $a\colon G \curvearrowright X$ be a continuous action of a Polish group G on a Polish space X.

The action induces the *orbit equivalence relation* E_a on X defined by $x \; E_a \; y \iff \exists g \in G(g \cdot x = y)$.

How much of G does E_a remember?

Definition: Let E and F be equivalence relations on Polish spaces X and Y respectively.

- A Borel map $f\colon X \to Y$ is a **homomorphism** from E to F, ($f\colon E \to_B F$), if for $x, x' \in X$, $x \, E \, x' \implies f(x) \; F \; f(x')$.
- A Borel map $f\colon X \to Y$ is a **reduction** of E to F if for any $x, x' \in X$, $x \, E \, x' \iff f(x) \; F \; f(x')$.
- E **is Borel reducible to** F, denoted $E \leq_B F$, if there is a Borel reduction of E to F.
- E and F are **Borel bireducible**, $(E \sim_B F)$, if $E \leq_B F$ and $F \leq_B E$.

The Γ-jumps of Clemens and Coskey

Let E be an equivalence relation on X and Γ a countable group.
E^Γ is defined on X^Γ by $x \mathrel{E^\Gamma} y \iff (\forall\gamma\in\Gamma) x(\gamma) \mathrel{E} y(\gamma)$.
Γ acts on X^Γ by shifts: $\gamma \cdot x(\alpha) = x(\gamma^{-1}\alpha)$.

Definition (Clemens-Coskey)

The Γ-**jump of** E, $E^{[\Gamma]}$, is defined on X^Γ by

$$x \mathrel{E^{[\Gamma]}} y \iff (\exists\gamma\in\Gamma)\gamma\cdot x \mathrel{E^\Gamma} y.$$

$J_0^{[\Gamma]} = \Delta(2)$; $J_{\alpha+1}^{[\Gamma]} = (J_\alpha^{[\Gamma]})^{[\Gamma]}$; $J_\alpha^{[\Gamma]} = \left(\bigoplus_{\beta<\alpha} J_\beta^{[\Gamma]}\right)^{[\Gamma]}$ for limit α.
$J_1^{[\Gamma]}$: $\Gamma \curvearrowright 2^\Gamma$. $J_2^{[\Gamma]}$: $\Gamma \wr \Gamma = \Gamma \rtimes \Gamma^\Gamma \curvearrowright (2^\Gamma)^\Gamma$.

Theorem (Clemens-Coskey)

$J_\alpha^{[\Gamma]}$, $\alpha < \omega_1$, are cofinal for Borel ERs induced by $\Gamma \wr \Gamma = \Gamma \rtimes \Gamma^\Gamma$.
$E \mapsto E^{[\Gamma]}$ is a jump operator on Borel ERs, for many Γ ($\mathbb{Z}$, $\mathbb{Z}_2^{<\omega}$).
Iso. on ctbl scattered linear orders of rank $1+\alpha$ is $\sim_B$ with $J_\alpha^{[\mathbb{Z}]}$.

Main result

Question (Clemens-Coskey)

When is $J_\alpha^{[\Gamma]}$ Borel reducible to $J_\beta^{[\Delta]}$ for some $\beta < \omega_1$?

Theorem (S.)

Assume that for any group homomorphism ϕ from Γ to a quotient of a subgroup of Δ,

- the image of ϕ is finite;
- the Kernel of ϕ is isomorphic to Γ.

(For example, $\mathbb{Z}$ and $\mathbb{Z}_2^{<\omega}$.)

Then any Borel homomorphism $f\colon J_2^{[\Gamma]} \to_B J_\beta^{[\Delta]}$ sends a comeager set into a single orbit. ("$J_2^{[\Gamma]}$ is generically $J_\beta^{[\Delta]}$-ergodic".)

In particular, there is an action of $\Gamma \wr \Gamma$ which is not Borel reducible to any action of $\Delta \wr \Delta$.

Main technical point

- Assume that all group homomorphisms from Γ to a quotient of a subgroup of Δ are trivial. (e.g., $\mathbb{Z}_2^{<\omega}$, $\mathbb{Z}_3^{<\omega}$.)
- Let E be a generically ergodic countable Borel equivalence relation, (such as $J_1^{[\Gamma]} = \Delta(2)^{[\Gamma]}$,) and F an analytic equivalence relation.
- We show that a homomorphism $E^{[\Gamma]} \to_B F^{[\Delta]}$ is in fact a homomorphism $E^{[\Gamma]} \to_B F^{\Delta}$, on a comeager set.
- Assuming $E^{[\Gamma]}$ is generically F-ergodic we conclude that it is also generically $F^{[\Delta]}$-ergodic.
- Inductively, $E^{[\Gamma]}$ is generically $J_\beta^{[\Delta]}$-ergodic for all $\beta < \omega_1$.

A symmetric model for $E^{[\Gamma]}$

Let E be a generically ergodic countable Borel equivalence relation on X. Fix $x \in X^\Gamma$, Cohen generic over V, and let

$$\vec{A} = \langle [x(\alpha)]_E \mid \alpha \in \Gamma \rangle, \text{ and } A = \left\{ \gamma \cdot \vec{A};\, \gamma \in \Gamma \right\}.$$

A is a classifying invariant for $E^{[\Gamma]}$.
We study the model $V(A)$.
Its key property is the following.

Lemma

In $V(A)$, the members of A are indiscernible over A and parameters in V. That is, given a formula ψ and $v \in V$,

$$\psi^{V(A)}(A, \vec{A}, v) \iff \psi^{V(A)}(A, \gamma \cdot \vec{A}, v).$$

(In particular, $\vec{A}$ is not definable from A.)

Borel homomorphisms and pins in $V(A)$

Let $\mathbb{P}$ be the poset of finite conditions p approximating a choice function through some $\vec{B}$ in A. Let τ be the name for $\bigcup \dot{G}$. Then

$$\mathbb{P} \times \mathbb{P} \Vdash \tau_{left} \; E^{[\Gamma]} \; \tau_{right}.$$

That is, $(\mathbb{P}, \tau)$ **is an** $E^{[\Gamma]}$**-pin**.
Fix a Borel homomorphism $f \colon E^{[\Gamma]} \to_B F^{[\Delta]}$. Then

$$\mathbb{P} \times \mathbb{P} \Vdash f(\tau_{left}) \; F^{[\Delta]} \; f(\tau_{right}).$$

Goal:

Show that in fact $\mathbb{P} \times \mathbb{P} \Vdash f(\tau_{left}) \; F^{\Delta} \; f(\tau_{right})$.

Then f is essentially a homomorphism to F^{Δ} (on a comeager set).

Proof that in fact $\mathbb{P} \times \mathbb{P} \Vdash f(\tau_{left})\ F^{\Delta}\ f(\tau_{right})$

Fact (Larson-Zapletal "Geometric Set Theory")

There is a condition $p \in \mathbb{P}$ such that
$(\mathbb{P} \restriction p) \times (\mathbb{P} \restriction p) \Vdash f(\tau_{left})\ F^{\Delta}\ f(\tau_{right})$.

w.l.o.g. p "chose" $\vec{A}$.

$\star(\vec{A})$ there is some $p \in \mathbb{P}$ that "chose" $\vec{A}$, such that
$(\mathbb{P} \restriction p) \times (\mathbb{P} \restriction p) \Vdash f(\tau_{left})\ F^{\Delta}\ f(\tau_{right})$.

By indiscernibility, any shift of $\vec{A}$ in A also satisfy this statement.
For $\gamma \in \Gamma$, there is $\delta \in \Delta$ s.t. if p, q satisfy $\star(\vec{A}), \star(\gamma \cdot \vec{A})$, then

$$(\mathbb{P} \restriction p) \times (\mathbb{P} \restriction q) \Vdash f(\tau_{left})\ F^{\Delta}\ \delta \cdot f(\tau_{right}).$$

By indiscernibility: the map $\gamma \mapsto \{\delta \in \Delta \text{ as above}\}$ is a group homomorphism from Γ to a quotient of a subgroup of Δ.
<u>If</u> this group homomorphism is trivial, then
$\mathbb{P} \times \mathbb{P} \Vdash f(\tau_{left})\ F^{\Delta}\ f(\tau_{right})$, as required.

Thanks for listening!