Borel equivalence relations and weak choice principles

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Motivation

Suppose E is a Borel equivalence relation on X and $c\colon X \longrightarrow I$ is a (*natural*) complete classification.
Roughly speaking, we will try to measure the complexity of E by the following question:
Given an invariant $A \in I$, how hard is it to find a representative of it, i.e. $x \in X$ such that $c(x) = A$.

Weak choice principles

Consider particularly equivalence relations of the form E^ω, where E is a countable Borel equivalence relation.
These have a complete classification by sequences of countable sets

$$\langle x_i \mid i < \omega \rangle \mapsto \langle [x_i]_E \mid i < \omega \rangle \,.$$

Definition

Let E be a countable equivalence relation on a Polish space X.
Then **choice for countable sequences of E classes**, abbreviated $\mathrm{CC}[E^\omega]$, stands for the following statement:
Suppose $A = \langle A_n \mid n < \omega \rangle$ is a countable sequence of sets $A_n \subseteq X$ such that each A_n is an E-class. Then $\prod_n A_n$ is not empty.

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Theorem (S.)

Suppose E and F are countable Borel equivalence relations on Polish spaces X and Y respectively, and μ is a Borel probability measure on X. If E is F-ergodic with respect to μ, then there is a model in which $\mathrm{CC}[F^\omega]$ holds yet $\mathrm{CC}[E^\omega]$ fails.

Strong ergodicitiy

Let (X, μ) be a standard probability space, E a Borel equivalence relation on X and F a Borel equivalence relation on a Polish space Y.

Say that **E is F-ergodic** (with respect to μ) if for any homomorphism $f \colon X \longrightarrow Y$ of E to F maps a measure 1 set into a single F-class.

i.e., there is a measure 1 set $C \subseteq X$ such that for any $x, y \in C$, $f(x)$ and $f(y)$ are F-related.

Fact

There are many pairs of countable Borel equivalence relations E and F s.t. E is F-ergodic and F is E-ergodic.

Separation

Theorem (S.)

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Corollary

There are many pairs of countable Borel equivalence relations E, F such that the choice principles $\mathrm{CC}[E^\omega]$ and $\mathrm{CC}[F^\omega]$ are independent over ZF.

Proof sketch

The proof goes through the following two lemmas.

Lemma

Suppose E^ω is F^ω-ergodic, w.r.t. μ^ω. Let $\langle x_n \mid n < \omega \rangle$ be μ^ω-Random generic over V. Let $A_n = [x_n]_E$. Then $V(\langle A_n \mid n < \omega \rangle) \models \mathrm{CC}[F^\omega] \wedge \neg \mathrm{CC}[E^\omega]$.

Remark

To make this connection we use tools from Zapletal "Idealized Forcing" and Kanovei-Sabok-Zapletal "Canonical Ramsey theory on Polish Space".

Product ergodicity lemma

Lemma

Suppose E and F are countable Borel equivalence relations on X and Y respectively. Let μ be an E-quasi-invariant Borel probability measure on X and assume that E is F-ergodic with respect to μ. Then E^ω is F-ergodic with respect to μ^ω.

Remark

- This is known when F is $=_{\mathbb{R}}$ (i.e., this classical notion of ergodicity).
- For finite products, a direct measure theoretic argument works.
- The proof of the lemma uses symmetric models techniques.

Proof sketch

Lemma

Suppose E and F are countable Borel equivalence relations on X and Y respectively. Let μ be an E-quasi-invariant Borel probability measure on X and assume that E is F-ergodic with respect to μ. Then E^ω is F-ergodic with respect to μ^ω.

- Take a *random*-generic E^ω-invariant, $\langle A_n \mid n < \omega \rangle$, and consider the model $V(\langle A_n \mid n < \omega \rangle)$.
- A homomorphism between E^ω and F gives an F-invariant in this model.
- (Main point) Every real in $V(\langle A_n \mid n < \omega \rangle)$ belongs to $V[x_1, \ldots, x_m]$ for some $m < \omega$.
- Reduce to the case of a homomorphism between E^m and F.

Further results

Let $\mathrm{CC}[\mathbb{R}^\omega]$ be the axiom of choice for countable sequences of countable sets of reals.
E.g., it is known to hold in the "Basic Cohen Model".
Note that for any countable Borel equivalence relation E, $\mathrm{CC}[\mathbb{R}^\omega]$ implies $\mathrm{CC}[E^\omega]$.

Theorem (S.)

There is a model in which $\mathrm{CC}[\mathbb{R}^\omega]$ fails, yet $\mathrm{CC}[E^\omega]$ holds for every countable Borel equivalence relation E.
Moreover: This model "corresponds" to a *natural* Borel equivalence, which is strictly above E_∞^ω, strictly below $=^+$ and is pinned.