Borel reducibility and symmetric models

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Borel equivalence relations

An equivalence relation E on a Polish space X is **Borel** if $E \subseteq X \times X$ is Borel.

Definition

Let E and F be Borel equivalence relations on Polish spaces X and Y respectively.

- A Borel map $f\colon X \longrightarrow Y$ is a **reduction** of E to F if for any $x, x' \in X$, $x\, E\, x' \iff f(x)\, F\, f(x')$.
- Say that E **is Borel reducible to** F, denoted $E \leq_B F$, if there is a Borel reduction.

Friedman-Stanley jumps

Definition

Let E be an equivalence relation on a set X.
A **complete classification** of E is a map $c\colon X \longrightarrow I$ such that for any $x, y \in X$, xEy iff $c(x) = c(y)$.
The elements of I are called **complete invariants** for E.

Definition

- The first Friedman-Stanley jump, $\cong_2$ (also called $=^+$) on $\mathbb{R}^\omega$ is defined such that the map

$$\langle x(i) \mid i < \omega\rangle \in \mathbb{R}^\omega \mapsto \{x(i);\, i \in \omega\} \in \mathcal{P}_2(\mathbb{N})$$

  is a complete classification.
- Similarly, $\cong_\alpha$ is classifiable by hereditarily countable elements in $\mathcal{P}_\alpha(\mathbb{N})$.

Potential complexity

Let E be a Borel equivalence relation on a Polish space X.

Definition

E is potentially $\mathbf{\Gamma}$ if there is an equivalence relation F on a Polish space Y so that $F \subseteq Y \times Y$ is $\mathbf{\Gamma}$ and E is Borel reducible to F.

Example

Consider the equality relation $=_{\mathbb{R}}$ on the reals.
Then $=_{\mathbb{R}}$ is $\mathbf{\Pi}^0_1$ but not potentially $\mathbf{\Sigma}^0_1$.

Definition

$\mathbf{\Gamma}$ is *the* potential complexity of E if it is minimal such that E is potentially $\mathbf{\Gamma}$.

The equivalence relations of Hjorth-Kechris-Louveau

Hjorth-Kechris-Louveau (1998) completely classified the possible *potential complexities* of Borel equivalence relations which are induced by closed subgroups of S_∞. (A set is in $D(\Gamma)$ if it is the difference of two sets in Γ)
For each class they found a maximal element.

Δ_1	Π^0_1	Σ^0_2	Π^0_3	$D(\Pi^0_3)$	Π^0_4	$D(\Pi^0_4)$ ...	Π^0_ω
$=_{\mathbb{N}}$	$=_{\mathbb{R}}$	E_∞	$\cong_2$ $(=^+)$		$\cong_3$ $(=^{++})$		$\cong_\omega$

$\Sigma^0_{\omega+1}$	$\Pi^0_{\omega+2}$	$D(\Pi^0_{\omega+2})$	$\Pi^0_{\omega+3}$	$D(\Pi^0_{\omega+3})$ ...
	$\cong_{\omega+1}$		$\cong_{\omega+2}$	

The equivalence relations of Hjorth-Kechris-Louveau

Definition (Hjorth-Kechris-Louveau 1998)

The relation $\cong^*_{\alpha+1,\beta}$ for $2 \leq \alpha$ and $\beta < \alpha$ is defined as follows. $\cong_4$

An invariant for $\cong^*_{3,1}$ is a set A such that $\cong^*_{4,2}$

- A is a hereditarily countable set in $\mathcal{P}_3(\mathbb{N})$ (i.e., a $\cong_3$-invariant – a set of sets of reals); $\cong^*_{4,1}$ $\cong^*_{4,0}$
- There is a trenary relation $R \subseteq A \times A \times \mathcal{P}_1(\mathbb{N})$, *definable from* A, such that; $\cong_3$
- given any $a \in A$, $R(a, -, -)$ is an injective function from A to $\mathcal{P}_1(\mathbb{N})$. $\cong^*_{3,1}$ $\cong^*_{3,0}$

Note: for $\gamma \leq \beta$, $\cong^*_{\alpha+1,\gamma} \leq_B \cong^*_{\alpha+1,\beta}$. $\cong_2$

The equivalence relations of Hjorth-Kechris-Louveau

Theorem (Hjorth-Kechris-Louveau 1998)

Let E be a Borel equivalence relation induced by a G-action where G is a closed subgroup of S_∞. Then

1. If E is potentially $\mathrm{D}(\mathbf{\Pi}^0_n)$ then $E \leq_B \cong^*_{n,n-2}$ $(n \geq 3)$;
2. If E is potentially $\mathbf{\Sigma}^0_{\lambda+1}$ then $E \leq_B \cong^*_{\lambda+1,<\lambda}$ (λ limit);
3. If E is potentially $D(\mathbf{\Pi}^0_{\lambda+n})$ then $E \leq_B \cong^*_{\lambda+n,\lambda+n-2}$ $(n \geq 2)$.

Δ_1 Π^0_1 Σ^0_2 Π^0_3 $D(\Pi^0_3)$ Π^0_4 $D(\Pi^0_4)$ … Π^0_ω

$\cong^*_{3,1}$ $\cong^*_{4,2}$

$\Sigma^0_{\omega+1}$ $\Pi^0_{\omega+2}$ $D(\Pi^0_{\omega+2})$ $\Pi^0_{\omega+3}$ $D(\Pi^0_{\omega+3})$ …

$\cong^*_{\omega+1,<\omega}$ $\cong^*_{\omega+2,\omega}$ $\cong^*_{\omega+3,\omega+1}$

Abelian group actions

Theorem (Hjorth-Kechris-Louveau 1998)

Let E be a Borel equivalence relation induced by a G-action where G is an abelian closed subgroup of S_∞. Then

1. If E is potentially $\mathrm{D}(\mathbf{\Pi}^0_n)$ then $E \leq_B \cong^*_{n,0}$ $(n \geq 3)$;
2. If E is potentially $\mathbf{\Sigma}^0_{\lambda+1}$ then $E \leq_B \cong^*_{\lambda+1,0}$ (λ limit);
3. If E is potentially $D(\mathbf{\Pi}^0_{\lambda+n})$ then $E \leq_B \cong^*_{\lambda+n,0}$ $(n \geq 2)$.

Δ_1	Π^0_1	Σ^0_2	Π^0_3	$D(\Pi^0_3)$	Π^0_4	$D(\Pi^0_4)$ …	Π^0_ω
	G is abelian			$\cong^*_{3,0}$		$\cong^*_{4,0}$	

$\Sigma^0_{\omega+1}$	$\Pi^0_{\omega+2}$	$D(\Pi^0_{\omega+2})$	$\Pi^0_{\omega+3}$	$D(\Pi^0_{\omega+3})$ …
$\cong^*_{\omega+1,0}$		$\cong^*_{\omega+2,0}$		$\cong^*_{\omega+3,0}$

Theorem (Hjorth-Kechris-Louveau 1998)

For all countable ordinals α, $\cong^*_{\alpha+3,\alpha} <_B \cong^*_{\alpha+3,\alpha+1}$.

Question (Hjorth-Kechris-Louveau 1998)

Are the reductions $\cong^*_{\omega+1,0} \leq_B \cong^*_{\omega+1,<\omega}$ and $\cong^*_{\omega+2,0} \leq_B \cong^*_{\omega+2,\omega}$ strict?

Expecting a positive answer Hjorth-Kechris-Louveau further conjectured that the entire $\cong^*_{\alpha,\beta}$ hierarchy is strict.

Theorem (S.)

$\cong^*_{\alpha+1,\beta} <_B \cong^*_{\alpha+1,\beta+1}$ for any α, β (when defined).

$$\cong^*_{\omega+1,<\omega}$$
$$\vee_B$$
$$\cong^*_{\omega+1,1}$$
$$\vee_B$$
$$\cong^*_{\omega+1,0}$$

$$\cong^*_{4,2}$$
$$\vee_B$$
$$\cong^*_{4,1}$$
$$\vee_B$$
$$\cong^*_{4,0}$$

The “Basic Cohen model”

Let $\langle x_n \mid n < \omega \rangle$ be a generic sequence of Cohen reals and $A = \{x_n;\, n \in \omega\}$ the unordered collection.
The “Basic Cohen model” where the axiom of choice fails can be expressed as

$$V(A)$$

The set-theoretic definable closure of (the transitive closure of) A.

Any set X in $V(A)$ is definable (in $V(A)$) using A, finitely many parameters $\bar{a}$ from the transitive closure of A, and a parameter v from V.
That is, X is the unique solution to $\psi(X, A, \bar{a}, v)$.

Borel reducibility and symmetric models

Theorem (S.)

Suppose E and F are Borel equivalence relations, classifiable by countable structures (and fix a collection of invariants).
Assume further that E is Borel reducible to F.
Let A be an E-invariant in some generic extension.
Then there is an F-invariant B s.t. $B \in V(A)$ and

$$V(A) = V(B).$$

Furthermore, B is definable in $V(A)$ using only A and parameters from V.

Remark

The proof uses tools from Zapletal "Idealized Forcing" (2008) and Kanovei-Sabok-Zapletal "Canonical Ramsey theory on Polish Spaces" (2013).

A simple example

Assume E is Borel reducible to F and A is a generic E-invariant. Then $V(A) = V(B)$ for some F-invariant B which is definable in $V(A)$ using only A and parameters from V.

Example

The "Basic Cohen Model" is $V(A)$ for a generic $=^+$-invariant A. $V(A)$ is not of the form $V(r)$ for any real r (an $=_{\mathbb{R}}$-invariant). (Recall that for any real r, $V(r)$ satisfies choice.)

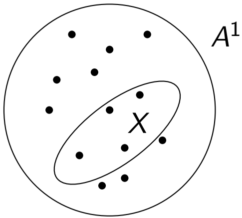
$$\text{It follows that } =^+ \text{ is not Borel reducible to } =_{\mathbb{R}}$$

To prove the main theorem,
we need to find "good" invariants for $\cong^*_{\alpha,\beta}$.

$\cong^*_{3,1}$ is not Borel reducible to $\cong^*_{3,0}$

Let $V(A^1)$ be the Basic Cohen model as before.
Let $X \subseteq A^1$ be generic over $V(A^1)$.

$$A = \left\{X \Delta \bar{a};\ \bar{a} \subseteq A^1 \text{ is finite}\right\} \in \mathcal{P}_3(\mathbb{N}).$$



For any $Y \in A$ the map $Z \mapsto Z \Delta Y$ is injective from A to the reals.

Thus A is a $\cong^*_{3,1}$-invariant. Note that $V(A) = V(A^1)[X]$.

To prove $\cong^*_{3,1} \not\leq_B \cong^*_{3,0}$ it suffices to show

Proposition

$V(A) \neq V(B)$ whenever $B \in V(A)$ is a set of sets of reals and B is countable <u>and B is definable from A</u>.

Proof of the proposition

Assume for contradiction that B is a countable set of sets of reals B, definable from A alone, such that $V(A) = V(B)$.
Then $X \in V(B)$. Assume that for some $U \in B$

$$X \text{ is defined by } \psi(X, B, U).$$

Applying finite permutations to the poset adding X, we get that for any $a \in A^1$ there is $U_a \in B$ such that

$$X\Delta\{a\} \text{ is defined by } \psi(X\Delta\{a\}, B, U_a).$$

A is preserved under finite changes of X and therefore so is B since B is definable from A alone.
This gives an injective map from the Cohen set A^1 to B.
Since B is countable, so is A^1. This is a contradiction since:
Fact: $V(A^1)$ and $V(A^1)[X]$ have the same reals.

Dealing with $\cong^*_{\omega+1,<\omega}$ and $\cong^*_{\omega+2,\omega}$

- The trick above produces "good" invariants for the $\cong^*$ equivalence relations starting from "good" invariants for the Friedman-Stanley jumps.
- Monro (1973) produced models $V(A^n)$, $A^n \in \mathcal{P}_{n+1}(\mathbb{N})$, in which the generalized Kinna-Wagner principles KWP^{n-1} fail. It can be shown that $V(A^n) \neq V(B)$ for any $B \in \mathcal{P}_n(\mathbb{N})$.
- Karagila (2019) constructed a model $M_\omega = V(A^\omega)$ in which $\mathrm{KWP}^{\mathrm{n}}$ fails for all n. He asked whether Monro's constructed can be continued past ω.
- The only previously known failure of KWP^ω is in the *Bristol model*. (The construction uses L-like conbinatorial principles.)
- It is open which large cardinals are consistent with high failure of Kinna-Wagner principles (Woodin's *Axiom of Choice Conjecture* implies that extendible cardinals are not.)

Invariants for the Friedman-Stanley jumps

Theorem (S.)

For any $\alpha < \omega_1$ there is a Monro-style model $V(A^\alpha)$.

- A^α is a generic $\cong_\alpha$-invariant;
- $V(A^\alpha)$ is not of the form $V(B)$ for any set B in $\mathcal{P}_{<\alpha}(\mathbb{N})$;
- KWP^α fails in $V(A^{\alpha+1})$;
- Works over any V.

Corollary

- (Friedman-Stanley) $\cong_{\alpha+1}$ is not Borel reducible to $\cong_\alpha$.
- Together with a few more tricks, the main theorem follows. That is, the $\cong^*_{\alpha,\beta}$ hierarchy is strict.