

Generic analysis of Borel homomorphisms for the finite Friedman-Stanley jumps

Assaf Shani

Concordia University

Research partially supported by
NSF grant DMS-2246746 and NSERC grant RGPIN-2024-05827.

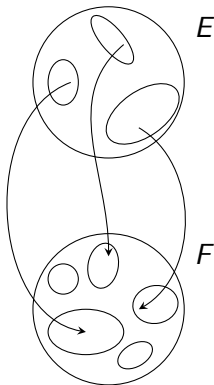
Borel homomorphisms and reductions

An equivalence relation E on a Polish space X is **analytic (Borel)** if $E \subseteq X \times X$ is analytic (Borel).

Definition

Let E and F be equivalence relations on Polish spaces X and Y respectively. $f: X \rightarrow Y$ a Borel map.

- ▶ f is a **Borel homomorphism**, $f: E \rightarrow_B F$, if $x E x' \implies f(x) F f(x')$.
- ▶ f is a **Borel reduction** of E to F if $x E x' \iff f(x) F f(x')$.
- ▶ E is **Borel reducible to F** , denoted $E \leq_B F$, if there is a Borel reduction of E to F .
- ▶ E, F are **Borel bireducible** ($E \sim_B F$) if $E \leq_B F$ & $F \leq_B E$.



Some motivations:

- “Borel definable” cardinality for definable quotient spaces.
- Possible complete invariants for classification problems.

Definition

Let E be an equivalence relation on a Polish space X .

Define E^+ on the Polish space $X^{\mathbb{N}}$ by

$$x E^+ y \iff \forall n \exists m (x(n) E y(m)) \ \& \ \forall n \exists m (y(n) E x(m)),$$

that is, $\{[x(n)]_E; n \in \mathbb{N}\} = \{[y(n)]_E; n \in \mathbb{N}\}$.

- ▶ The countable powerset operation $\mathcal{P}_{\aleph_0}(-)$, for the quotient X/E , coded on a Polish space.
- ▶ Classifiability using hereditarily countable invariants.
 - ▶ E is **concretely classifiable** if $E \leq_B =_{\mathbb{R}}$, equality ER on \mathbb{R} . (Numerical invariants.)
 - ▶ E is classifiable using countable sets of reals as invariants if $E \leq_B =_{\mathbb{R}}^+$.
 - ▶ Countable sets of countable sets of reals as invariants: $E \leq_B =_{\mathbb{R}}^{++}$.
 - ▶ ...

Classification by countable structures

Definition

E is **classifiable by countable structures** if it is Borel reducible to the isomorphism relation for some class of countable objects.

E.g.: countable graphs, countable groups ...

- ▶ Equivalently: if E is Borel reducible to an orbit equivalence relation induced by S_∞ (or a closed subgroup of S_∞ : non-Archimedean groups).

Fact

E a Borel equivalence relation. The following are equivalent.

- ▶ E is classifiable by countable structures;
- ▶ E is Borel reducible to $=_{\mathbb{R}}^{+\alpha}$ for a countable ordinal α .

Motivation

Very general goal:

Given equivalence relation E and F , is $E \leq_B F$?

Today's goal:

For $n \leq \omega$, develop methods to prove that $=_{\mathbb{R}}^{+n} \leq_B E$ for some E .

Remark:

For $=_{\mathbb{R}}^+$, the situation is well understood. Some examples:

- ▶ Foreman - Louveau 1995: $=_{\mathbb{R}}^+$ is Borel bireducible with the classification problem of ergodic discrete spectrum measure preserving transformations.
- ▶ Marker 2007: Let T be a complete first order theory whose space of types is uncountable. Then $=_{\mathbb{R}}^+ \leq_B \cong T$.

Generic dichotomy for Borel homomorphisms

Theorem (Kanovei-Sabok-Zapletal 2013)

Let E be an analytic equivalence relation. Then either

- ▶ $=_{\mathbb{R}}^+$ is Borel reducible to E , or
- ▶ any Borel homomorphism from $=_{\mathbb{R}}^+$ to E maps a comeager subset of $\mathbb{R}^{\mathbb{N}}$ into a single E -class.

Theorem (Marker 2007)

T first order theory, uncountable type space. Then $=_{\mathbb{R}}^+ \leq_B \cong T$.

- ▶ Fix a perfect set of types C , identified with \mathbb{R} .
- ▶ Naive idea: map a countable set of reals $A \subseteq C$ to a model M satisfying “precisely” A .
- ▶ Can be done if A is a Scott set: sufficiently closed under some countably many operations.
- ▶ Improved idea: $A \mapsto \text{closure}(A) \mapsto M$.
- ▶ This gives a Borel homomorphism, not trivial on comeager sets. Therefore $=_{\mathbb{R}}^+ \leq_B \simeq T$.

Some difficulties in generalizing for $n \geq 2$

Kanovei-Sabok-Zapletal 2013: E analytic ER. Then either

- ▶ $=_{\mathbb{R}}^+$ is Borel reducible to E , or
- ▶ any $f: =_{\mathbb{R}}^+ \rightarrow_B E$ maps a comeager set into a single E -class.

Already for $=_{\mathbb{R}}^{++}$:

- ▶ On a comeager subset $C \subseteq (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$, $(=_{\mathbb{R}}^{++} \upharpoonright C) \leq_B =_{\mathbb{R}}^+$.
 $C =$ all $x \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ s.t. $(n, m) \neq (l, k) \implies x(n, m) \neq x(l, k)$.
- ▶ There **is** an “interesting” Borel homomorphism $=_{\mathbb{R}}^{++} \rightarrow_B =_{\mathbb{R}}^+$:
 $(x_{i,j} \mid i, j \in \mathbb{N}) \mapsto (x_{\langle i,j \rangle} \mid i, j \in \mathbb{N})$.

More generally:

- ▶ For $n \geq 2$, need a different presentation / topology.
- ▶ Need to consider the homomorphisms $=_{\mathbb{R}}^{+n} \rightarrow_B =_{\mathbb{R}}^{+k}$, $k < n$, essentially taking a hereditarily countable set of rank n to the set of its rank k elements.

Main result

Theorem (S.)

There are equivalence relations F_n on Polish spaces X_n , s.t.

1. $F_n \sim_B =_{\mathbb{R}}^{+n}$, $n = 1, 2, 3, \dots, \omega$, and

there are Borel homomorphism $u_k^n: F_n \rightarrow_B F_k$, $k < n \leq \omega$, s.t.

2. Let E be classifiable by countable structures. Then either
 - ▶ F_n is Borel reducible to E , or
 - ▶ every Borel homomorphism $f: F_n \rightarrow_B E$ factors through u_k^n on a comeager set, for $k < n$. (That is, there is a homomorphism $h: F_k \rightarrow_B E$ so that $(h \circ u) \upharpoonright E f$ on a comeager set.)

**To prove $=_{\mathbb{R}}^{+n} \leq_B E$,
enough to find a
“non-trivial” homo-
morphism.**

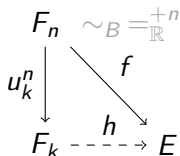


Figure: $(\forall f: F_n \rightarrow_B E)(\exists k < n \exists h: F_k \rightarrow_B E)$

Definition of F_n and u_m^n

- ▶ $X_n \subseteq ((2^{\mathbb{N}})^{\mathbb{N}})^n$, for $n = 1, 2, 3, \dots, \omega$. Fix $x \in X_n$.
- ▶ $A_1^x = \{x(0)(k); k \in \mathbb{N}\} \subseteq 2^{\mathbb{N}}$.

$$a_1^{x,l} = \{x(0)(k); x(1)(l)(k) = 1\} \subseteq A_1^x$$

\vdots	\vdots	\vdots	\vdots	\dots	\vdots	\vdots	\vdots	\dots
*	1	0	1	\dots	*	—	*	\dots
*	1	1	0	\dots	*	*	—	\dots
*	0	1	1	\dots	—	*	*	\dots
*	0	1	0	\dots	—	*	—	\dots
$x(0)$	$x(1)(0)$	$x(1)(1)$	$x(1)(2)$	\dots	$a_1^{x,0}$	$a_1^{x,1}$	$a_1^{x,2}$	\dots
$(2^{\mathbb{N}})^{\mathbb{N}}$	$2^{\mathbb{N}}$	$2^{\mathbb{N}}$	$2^{\mathbb{N}}$	\dots	$2^{\mathbb{N}}$	$2^{\mathbb{N}}$	$2^{\mathbb{N}}$	\dots

- ▶ $A_2^x = \{a_1^{x,l}; l \in \mathbb{N}\}$; $a_2^{x,l} = \{a_1^{x,k}; x(2)(l)(k) = 1\} \subseteq A_2^x \dots$

$$x \mathbf{F}_n y \iff \mathbf{A}_n^x = \mathbf{A}_n^y$$

- ▶ $u_m^n: X_n \rightarrow X_m$, for $m < n$, projection.

An application to a question of Clemens

The following answers positively a question of Clemens.

Corollary (S.)

Suppose $E <_B =^{\omega}_{\mathbb{R}}$. Then for any Borel homomorphism $f: =^{\omega}_{\mathbb{R}} \rightarrow_B E$, $=^{\omega}_{\mathbb{R}}$ retains its complexity on a fiber, that is, there is y in the domain of E so that $=^{\omega}_{\mathbb{R}} \sim_B =^{\omega}_{\mathbb{R}} \upharpoonright \{x; f(x) E y\}$.
(That is, $=^{\omega}_{\mathbb{R}}$ is **regular**.)

- ▶ Can replace $=^{\omega}_{\mathbb{R}}$ with F_{ω} .
- ▶ By the main theorem, any $f: F_{\omega} \rightarrow_B E$ factors through u_k^{ω} for some k , on a comeager set.
- ▶ From the definitions, F_{ω} is equivalent to its restriction to any fiber of u_k^{ω} .
- ▶ It remains to show that F_{ω} retains its complexity on comeager sets: $F_{\omega} \leq_B F_{\omega} \upharpoonright C$ for any comeager C .

Spectrum of the meager ideal

Corollary (S.)

For any $n \leq \omega$, F_n retains its complexity on comeager sets:
 $F_n \leq_B F_n \upharpoonright C$ for any comeager set C .

In particular, $=_{\mathbb{R}}^{+n}$ is in the **spectrum of the meager ideal**.
This was proved by Kanovei, Sabok, and Zapletal for $n = 1$.
For $n > 1$, the different presentation F_n is necessary.

- ▶ Fix a comeager set C (assume it is F_n -invariant). Fix $f: F_n \rightarrow_B F_n \upharpoonright C$ which is the identity on C .
- ▶ From the definitions, u_k^n is not a reduction on any comeager set, for $k < n$.
- ▶ So f does not factor through u_k^n , for $k < n$.
- ▶ By the main theorem, $F_n \leq_B F_n \upharpoonright C$.

What else is good about F_n ? Group action

$S_\infty = \text{Sym}(\mathbb{N})$, $S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}} \rightsquigarrow =_{\mathbb{R}}^+$ (on a large set).

Consider the action $S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}}$.

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ * & 1 & 0 & 1 & \dots \\ * & 1 & 1 & 0 & \dots \\ * & 0 & 1 & 1 & \dots \\ * & 0 & 1 & 0 & \dots \end{array}$$

$S_\infty \quad \curvearrowright \quad \curvearrowright \quad \curvearrowright \quad \curvearrowright$

F_2 is induced (on a large set) by the action

$$S_\infty \times S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$$

Similarly: F_n is induced by a natural action of $(S_\infty)^n$ on $((2^{\mathbb{N}})^{\mathbb{N}})^n$.

In contrast, $=_{\mathbb{R}}^{++}$ is naturally induced by an action of

$$S_\infty \times (S_\infty)^{\mathbb{N}} \text{ on } (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$$

What else is good about F_n ? Borel complexity

Note: $=^+$ is \mathfrak{N}_3^0 ; $=^{++}$ is \mathfrak{N}_5^0 ; $=^{+++}$ is \mathfrak{N}_7^0 .

Theorem (Hjorth-Kechris-Louveau 1998)

$=^{+n}$ is *potentially* \mathfrak{N}_{2+n}^0 : it is Borel reducible to a \mathfrak{N}_{2+n}^0 ER.
In fact it is maximal potentially \mathfrak{N}_{2+n}^0 for S_∞ -actions.

Note:

F_n is \mathfrak{N}_{2+n}^0 .

e.g., F_2 is \mathfrak{N}_4^0 . Main point: given x, y , we want

$$\forall n \exists m (\forall i, j [x(0)(i) = y(0)(j) \rightarrow x(1)(n)(i) = y(1)(m)(j)])$$

*	1	1	0	*	0	1	1
*	0	1	1	*	0	0	0
*	0	1	0	*	1	1	0
$x(0)$	$x(1)(0)$	$x(1)(1)$	$x(1)(2)$	$y(0)$	$y(1)(0)$	$y(1)(1)$	$y(1)(2)$

New ideas going into the proof

1. To build a reduction from F_n to some equivalence relation, need a construction which is invariant under the action

$$S_\infty \times S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$$

2. For this action, need to understand Vaught transforms:

$$\{x; \text{ for a comeager set of } g \in S_\infty \times S_\infty, g \cdot x \in C\},$$

for comeager sets C .

3. Lemma: given a Borel homomorphism f from F_n to E , E classifiable by countable structures, there are:

$$\begin{array}{ccccccc} F_1 & \leftarrow & F_2 & \leftarrow & F_3 & \leftarrow & \dots & \leftarrow & F_n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ E_1 & \leftarrow & E_2 & \leftarrow & E_3 & \leftarrow & \dots & \leftarrow & E \end{array}$$

which commute on comeager sets.

Some questions

- ▶ Find a model-theoretic condition on a first order theory T so that $=_{\mathbb{R}}^{+n} \leq_B \simeq_T$. (Done by Marker 2007 for $=_{\mathbb{R}}^+$.)
- ▶ Extend to transfinite jumps, $=_{\mathbb{R}}^{+\alpha}$ for a countable ordinal α .
- ▶ Find an equivalent jump operation which also gives a good topology.
- ▶ Prove generic dichotomies for homomorphisms for other equivalence relations.
- ▶ Prove a measure theoretic version of the dichotomy for $=_{\mathbb{R}}^{+n}$. (Done by Kanovei-Sabok-Zapletal 2013 for $=_{\mathbb{R}}^+$.)
- ▶ Find a set theoretic condition for $=_{\mathbb{R}}^{+n} \leq_B E$, e.g. based on properties of E in $L(\mathbb{R})$ or the Solovay model. (Done by Larson-Zapletal 2020 for $=_{\mathbb{R}}^+$.)