Above countable products of countable equivalence relations

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The Γ-jumps of Clemens and Coskey

Definition (Clemens-Coskey)

Let E be an equivalence relation on X and Γ a countable group. The Γ-**jump of** E, $E^{[\Gamma]}$, is defined on X^Γ by

$$x \; E^{[\Gamma]} \; y \iff (\exists \gamma \in \Gamma)(\forall \alpha \in \Gamma) x(\gamma^{-1}\alpha) \; E \; y(\alpha).$$

E^ω is defined on X^ω by $x \; E^\omega \; y \iff (\forall n \in \omega) x(n) \; E \; y(n)$.

Example

$E_0 \sim_B (=_{\{0,1\}})^{[\mathbb{Z}]}$ and $E_\infty \sim_B (=_{\{0,1\}})^{[\mathbb{F}_2]}$.

Theorem (Clemens-Coskey)

$E \mapsto E^{[\mathbb{Z}]}$ is a jump operator on Borel equivalence relations.

The Γ-jumps of Clemens and Coskey

$x \; E^{\omega} \; y \iff (\forall n \in \omega) x(n) \; E \; y(n)$
$x \; E^{[\Gamma]} \; y \iff (\exists \gamma \in \Gamma)(\forall \alpha \in \Gamma) x(\gamma^{-1}\alpha) \; E \; y(\alpha).$

Theorem (Clemens-Coskey)

Suppose E is a generically ergodic countable Borel equivalence relation and Γ a countable infinite group. Then $E^{\omega} <_B E^{[\Gamma]}$.

Question (Clemens-Coskey)

Is $E_{\infty}^{[\mathbb{Z}]} <_B E_{\infty}^{[\mathbb{F}_2]}$?

Theorem (S.)

Suppose E is a generically ergodic countable Borel equivalence relation.

$$E^{[\mathbb{Z}]} <_B E^{[\mathbb{Z}^2]} <_B E^{[\mathbb{Z}^3]} <_B \ldots <_B E^{[\mathbb{F}_2]}.$$

Complete classifications

Let F be an equivalence relation on Y. A **complete classification of** F is a map $c\colon Y \longrightarrow I$ such that for any $x, y \in Y$,

$$x\, F\, y \iff c(x) = c(y).$$

Complete classifications: (using hereditarily countable structures)

- $=_{[0,1]}$ on $[0,1]$: $x \mapsto x$;
- E a countable Borel equivalence relation: $x \mapsto [x]_E$;
- E^ω: $x \mapsto \langle [x(n)]_E \mid n < \omega \rangle$
- $E^{[\Gamma]}$: Given $x \in X^\Gamma$, for $\gamma \in \Gamma$ let $A_\gamma = [x(\gamma)]_E$.

$$x \mapsto \left\{ (\gamma, A_\alpha, A_{\gamma^{-1}\alpha});\, \gamma, \alpha \in \Gamma \right\}.$$

"A set of E-classes and an action of Γ on it"

Borel reducibility and symmetric models

Theorem (S.)

Suppose E and F are Borel equivalence relations, classifiable by countable structures (and fix a collection of invariants).
Assume further that E is Borel reducible to F.
Let A be an E-invariant in some generic extension.
Then there is an F-invariant B s.t. $B \in V(A)$ and

$$V(A) = V(B).$$

Furthermore, B is definable in $V(A)$ using only A and parameters from V.

Remark

The proof uses tools from Zapletal "Idealized Forcing" (2008) and Kanovei-Sabok-Zapletal "Canonical Ramsey theory on Polish Spaces" (2013).

A simple example

Assume E is Borel reducible to F and A is a generic E-invariant. Then $V(A) = V(B)$ for some F-invariant B which is definable in $V(A)$ using only A and parameters from V.

Example

Let x be a Cohen generic and $A = [x]_{E_0}$ its E_0-invariant. If r is a real in $V(A)$ which is definable from A and parameters in V alone then $r \in V$, so $V(r) \neq V(A)$.

It follows that E_0 is not Borel reducible to $=_{[0,1]}$

To prove the main theorem,
we need to study models generated by invariants for $E^{[\Gamma]}$.

$E^{[\mathbb{Z}^2]}$ is not Borel reducible to $E^{[\mathbb{Z}]}$

Assume towards a contradiction that $E^{[\mathbb{Z}^2]} \leq_B E^{[\mathbb{Z}]}$.
Let $x \in X^{\mathbb{Z}^2}$ be Cohen-generic and A its $E^{[\mathbb{Z}^2]}$-invariant.

$$\begin{array}{ccc} \dots & & \dots \\ A_{-1,1} & A_{0,1} & A_{1,1} \\ A_{-1,0} & A_{0,0} & A_{1,0} \\ A_{-1,-1} & A_{0,-1} & A_{1,-1} \\ \dots & & \dots \end{array} \qquad B_{-3} \quad B_{-2} \quad B_{-1} \quad B_0 \quad B_1 \quad B_2 \quad B_3$$

Assume that B_0 and $A_{0,0}$ are bi-definable over A and $v \in V$.

$E^{[\mathbb{Z}^2]}$ is not Borel reducible to $E^{[\mathbb{Z}]}$

$$\begin{array}{ccc} \cdots & & \cdots \\ A_{-1,1} & A_{0,1} & A_{1,1} \\ A_{-1,0} & A_{0,0} & A_{1,0} \\ A_{-1,-1} & A_{0,-1} & A_{1,-1} \\ \cdots & & \cdots \end{array} \qquad B_{-3} \quad B_{-2} \quad B_{-1} \quad B_0 \quad B_1 \quad B_2 \quad B_3$$

Proposition (Strong failure of Marker Lemma)

In $V(A)$, the elements of $\{A_\gamma;\, \gamma \in \Gamma\}$ are indiscernibles over A and parameters in V.

$A_{0,0} \longleftrightarrow B_0$ bi-definable (over A and $v \in V$).
Then for some $5 \in \mathbb{Z}$, $A_{1,0} \longleftrightarrow B_5$.
Then $A_{m,0} \longleftrightarrow B_{5 \cdot m}$ for all $m \in \mathbb{Z}$.
($\{A_{m,0};\, m \in \mathbb{Z}\} \longleftrightarrow$ an arithmetic sequence with difference 5)
Now for each n, $\{A_{m,n};\, m \in \mathbb{Z}\}$ "corresponds" to an arithmetic sequence in B with common difference 5. Furthermore, these are disjoint for distinct values of n, a contradiction.

More general results

Theorem (S.)

Let Γ and Δ be countable groups and E a generically ergodic countable Borel equivalence relation. The following are equivalent:

1. $E^{[\Gamma]}$ is not generically $E_\infty^{[\Delta]}$-ergodic.
2. There is a subgroup $\tilde{\Delta}$ of Δ, a normal subgroup H of $\tilde{\Delta}$ and a group homomorphism from Γ to $\tilde{\Delta}/H$ with finite kernel;

Using similar arguments as before, plus:

Theorem (S.)

Let E and F be Borel equivalence relations classifiable by countable structures. The following are equivalent:

1. E is generically F-ergodic;
2. If A is the E-invariant of a generic Cohen-real, then for any F-invariant $B \in V(A)$, definable from A and parameters in V, B is in V.