HOMEWORK 1 (1 WEEK)

You are encouraged to discuss the homework problems with your classmates. However, your final submitted homework must reflect your personal understanding of the material. Your solution must be written by you in your own words. You are not allowed to copy an answer from another student or from any other source.

Please make sure your solutions are well organized. Make sure the structure of the proofs is clear.

Notation: Here is a quick review of some of the notation used in this assignment. *Please let me know if anything else is not familiar or not clear. Feel free to contact me or come to office hours if you have any questions.* If you have not seen a lot of formal mathematical notation before, you may also find it useful to look at pages 4-8 in [Enderton].

We use $\{\}$ to denote a set (collection) of objects. For example $\{A, B, 5\}$ is the set containing 3 members: the letter A, the letter B, and the number 5. Sometimes we use the separator $|$ to describe the elements of a set. For example, $\{X |\, X \text{ is a capital letter in the english alphabet}\}$ is the set of all letter $A, B, C, ..., Y, Z$.

$\mathbb{N} = \{0, 1, 2, ...\}$ the set of natural numbrs.

$\mathbb{Z} = \{... - 2, -1, 0, 1, 2, ...\}$ the set of integers.

$\mathbb{Q} = \left\{\frac{n}{m} |\, n, m \in \mathbb{Z}, m \neq 0\right\}$ the set of rational numbers.

We use the symbol $\in$ to denote membership. For instance, $a \in \mathbb{Q}$ says "a is a rational". $a \notin \mathbb{N}$ says "a is not a natural number".

Reading assignment. Please look over Chapter 1 parts 1.1 and 1.2 of [Enderton] for some basic definitions regarding propositional logic and connectives. I suspect most of you will find it familiar (in essence, if not in notation), and easy to understand. You can also find these in [Woodin-Slaman, 1.1 and 1.2]. (If you haven't taken a rigorous proof based class before, it is extra important to take a look at it. We will focus directly on "first order logic".) As usual, please ask if you have any questions.

Questions. Recall that for two structures (X, R^X) and (Y, R^Y), where R^X, R^Y are binary relations on X,Y, respectively, an **isomorphism** between (X, R^X) and (Y, R^Y) is a function $f \colon X \to Y$ such that

- f is one-to-one and onto;
- for any a, b in X,

$$a\ R^X\ b \iff f(a)\ R^Y\ f(b)$$

(meaning a and b "satisfy the relation" in (X, R^X) if and only if $f(a)$ and $f(b)$ "satisfy the relation" in (Y, R^Y)).

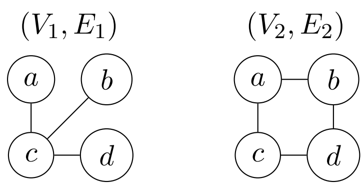
When both structures happen to be linear orders, an isomorphism precisely coincides with an order-preserving map.

When both structures are graphs, an isomorphism precisely coincides with "graph isomorphism".

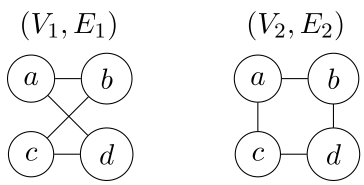
Two structures (X, R^X) and (Y, R^Y) are said to be **isomorphic** if there is *some* isomorphism between them.

Question 1. In the following cases, determine whether the two graphs (V_1, E_1) and (V_2, E_2) are isomorphic or not. That is, is there a map $f\colon V_1 \to V_2$ so that f is one-to-one and onto, and for any x, y in V_1, $x\ E_1\ y \iff f(x)\ E_2\ f(y)$. Prove your answers. (Either prove they are not isomorphic, or write an isomorphism.)

All the graphs are presented as structures with $V_1 = V_2 = \{a, b, c, d\}$. If in the graph (picture) for (V_1, E_1) there is a line, for example, between a and b, it means that $a\ E_1\ b$ and $b\ E_1\ a$ are true. If there is not line between a and b, it means that $a\ E_1\ b$ and $b\ E_1\ a$ are false in this structure.



(1)



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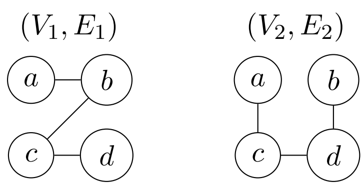


(3)

Question 2. (1) Prove that the linear orders $(\mathbb{Z}, <)$ and $(\mathbb{N}, <^*)$ are *not* isomorphic to one another. ($<^*$ here is as defined in the notes, the reverse of $<$.)
(2) Prove that the linear orders $(\mathbb{Q}, <)$ and $(\mathbb{Z}, <)$ are *not* isomorphic to one another.
(3) Let $2\mathbb{Z}$ be the set even integers, that is, all integers of the form $2k$ for some integer k. Similarly, let $3\mathbb{Z}$ be all integers of the form $3k$ for some integer k. Order both $2\mathbb{Z}$ and $3\mathbb{Z}$ with the usual ordering $<$. Prove that $(2\mathbb{Z}, <)$ and $(3\mathbb{Z}, <)$ *are* isomorphic.

The following question is a key ingredient in the isomorphism theorem for dense linear orders. [We will talk more about the proof of the theorem, and the following concepts, on Tuesday 1/27.]

Suppose $(A, <)$ and $(A', <')$ are linear orders. (As an example to keep in mind, you can think of $(\mathbb{Q}, <)$ and $(\mathbb{Q} \setminus \mathbb{Z}, <)$.)

Given sequences $\bar{a} = a_0, a_1, ..., a_{n-1}$, from A and $\bar{a}' = a'_0, a'_1, ..., a'_{n-1}$ from A', say that $\bar{a}$ and $\bar{a}'$ **have the same type** if

$$a_i < a_j \iff a'_i <' a'_j$$

for any $i, j \in \{0, ..., n-1\}$. (Equivalently: $\bar{a}$ and $\bar{a}'$ have the same type if the map sending a_i to a'_i is "order preserving".)

For a sequence $\bar{a} = a_0, a_1, ..., a_{n-1}$ from A, and some a in A, define $\bar{a}^\frown a$ as the sequence $a_0, a_1, ..., a_{n-1}, a_n$ with $a_n = a$.

Question 3. Suppose $(A, <)$ is some linear oder, and $(A', <')$ is a dense linear order. Let $\bar{a}$ and $\bar{a}'$ be sequences from A and A' accordingly, and assume that they have the same type.

Then prove that for any $a \in A$ there exists some $a' \in A'$ such that the sequences $\bar{a}^\frown a$ and $\bar{a}'^\frown a'$ also have the same type.

[Hint: it may help to start by thinking about some concrete cases, for example, when the length of the sequence n is 1 or 2, and the linear orders are some specific examples.]

Question 4. Consider the following axioms stated with the alphabet of one binary relation "$<$":

(LO 1) (Strictness) for any members x, y of the set X, if $x < y$ holds, then $y < x$ fails (we may denote this by $y \not< x$;

(LO 2) (Transitivity) for any members x, y, z of the set X, if $x < y$ and $y < z$, then $x < z$;

(LO 3) (Total (linear) ordering) for any members x, y of X, either $x < y$ or $y < x$.

(Density) for any x and y, if $x < y$ then there is some z so that $x < z < y$;

(No max) for every x there is some y with $x < y$;

(Yes min) there is some x so that for any y, either $x = y$ or $x < y$.

This is a "dense linear order with a minimum".

Suppose $(X, <^X)$ and $(Y, <^Y)$ are two linear orders satisfying the above statements, where X and Y are countable. Prove that they are isomorphic.

[Hint: you can repeat a similar argument as in the proof of Cantor's isomorphism theorem (which we will see in class 1/27). You can also avoid "redoing it", and find a way to directly quote the isomorphism theorem for dense linear orders.]

References

[Enderton] Herbert B. Enderton - A Mathematical Introduction to Logic

[Woodin-Slaman] Notes by Professor W. Hugh Woodin and Professor Theodore A. Slaman

Homework 2 (1 week)

Question 5. Let $(X, <^X)$ be an linear order (not assumed to be dense) so that X is countable. Prove that there is an embedding of $(X, <^X)$ into $(\mathbb{Q}, <)$. That is, there is a one-to-one (not necessarily onto) map $f\colon X \to \mathbb{Q}$ so that for any x, y in X,

$$x <^X y \iff f(x) < f(y).$$

[So, in a sense, the structure $(\mathbb{Q}, <)$ (or any countable DLO) is *maximal* among all countable linear orders.]

Question 6. The following question is specifically about the formal interpretation of terms in a structure. Follow our definitions precisely.

Let $A = \{1, 2, 3\}$. Define a function $f\colon A^3 \to A$ by $f(a, b, c) = |\{a, b, c\}|$ (the size of the unordered set $\{a, b, c\}$). For example, $f(2, 2, 2) = 1$ and $f(1, 3, 2) = 3$.
Define $h\colon A^2 \to A$ by $h(a, b) = 1$ if $a < b$, $h(a, b) = 2$ if $a = b$ and $h(a, b) = 3$ if $a > b$.

Let F be a trenary function symbol and H a binary function symbol. Let t be the term $F(H(x, y), H(y, z), H(z, x))$, where x, y, z are variables. With this list of variables $\langle x, y, z\rangle$ in mind, we view the term t as $t(x, y, z)$.

Consider the structure $\mathcal{A} = (A, f, h)$, with $f = F^{\mathcal{A}}$ and $h = H^{\mathcal{A}}$.

Find the function $t^{\mathcal{A}}(x, y, z)\colon A^3 \to A$. (Write all the possible values.)

Question 7. The following question is specifically about formal writing. Be precise and follow the definitions.

(In the following, you do not have to prove that φ "does the job", but it should be clear enough from the way you write it that the intended meaning is correct. You may use short-hand notations, but you must first define them as well.)

[We have not yet defined the interpretation of all formulas in a structure, we will finish this on Tuesday. So it may be easier to do after Tuesday's class, but you can also do it before if it is clear to you what $\exists$ or $\forall$ should mean.]

(1) Let F be a unary function symbol, $<$ a binary relation symbol, $-$ a binary operation, and $\underline{0}$ a constant symbol. We consider structures $(\mathbb{R}, -, <, 0, f)$ where $-$ is the usual subtraction operation on $\mathbb{R}$, $<$ is the usual ordering of $\mathbb{R}$, 0 is the usual 0 in $\mathbb{R}$, and f is any function from $\mathbb{R}$ to $\mathbb{R}$.

Write a formula $\varphi(x)$ "saying that" F is continuous at x. (Using the $\epsilon - \delta$ definition.)

[Formally, this means that in the structure $(\mathbb{R}, -, <, 0, f)$, the formula $\varphi(x)$ when assigned c is true if and only if f is a continuous at the point c.]

(2) Let R be a binary relation symbol. Write a sentence φ in the vocabulary $\{R\}$ "saying that" R is the graph of a function. That is, in a structure $\mathcal{A} = (A, R^{\mathcal{A}})$, the sentence φ is true if and only if there is some function $g\colon A \to A$ so that $R(a, b) \iff g(a) = b$ for any $a, b \in A$. [Recall that R is interpreted (formally) as a subset of $A \times A$. Recall the "vertical line test" for being a graph of a function.]

(3) Using the empty signature (meaning, only the relation symbol $\approx$), write a sentence φ so that a structure $\mathcal{A}$ satisfies φ if and only if the universe A of $\mathcal{A}$ has size ≥ 4. (That is, there are at least 4 distinct members.)

Not to be submitted.

Question. Consider the vocabulary for vector spaces over $\mathbb{R}$, discussed in class.

(1) Write the axioms for being a vector space over $\mathbb{R}$, in this language.
(2) ($\star$) Is there a sentence Φ in this language which captures the property "the vector space has dimension 2"? That is, a sentence Φ so that a structure $\mathcal{A}$ for this vocabulary satisfies Φ if and only if the dimension of $\mathcal{A}$ as a vector space is 2.

Homework 3 (2 weeks)

Question 8. Consider the signature $\mathcal{S} = \{F, c\}$ where F is a binary operation and c is a constant symbol. Consider the following structures

- $\mathcal{A} = (\mathbb{R}, +, 0)$ (that is, $F^{\mathcal{A}} = +$ and $c^{\mathcal{A}} = 0$);
- $\mathcal{B} = (\mathbb{R}^*, \cdot, 1)$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ are all non-zero reals;
- $\mathcal{C} = (\mathbb{R}^{>0}, \cdot, 1)$, where $\mathbb{R}^{>0}$ are all positive reals.

(1) Prove that $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic to one another. [Hint: find a sentence which is true in one but not the other.]
(2) Prove that $\mathcal{A}$ and $\mathcal{C}$ are isomorphic. [Hint: Log.] [In your proof, make it clear what is needed to be an isomorphism. Write down everything that is required to show that this is an isomorphism. You do not need to prove the properties of Log.]

Question 9. (1) Consider the vocabulary consisting of one unary function symbol F. Write a sentence φ so that for any structure $\mathcal{A} = (A, F^{\mathcal{A}})$, *if* $\mathcal{A} \models \varphi$ *then* A is infinite. [It is necessry to know the following fact: a set X is *infinite* if and only if there is *some* function $h \colon X \to X$ which is one-to-one yet is not onto. (You can see that this is impossible for a finite set.)]
(2) Let φ be the sentence you wrote above. Provide a structure $\mathcal{A} = (A, F^{\mathcal{A}})$ so that A *is* infinite, yet $\mathcal{A} \models \neg\varphi$.([1])

Given a structure $\mathcal{A}$ for a signature $\mathcal{S}$, say that a subset $C \subseteq A$ is **definable**[2] in $\mathcal{A}$ if there is a formula $\varphi(x)$ (having one free variable x) in the language so that

$$C = \left\{ a \in A \,|\, \varphi^{\mathcal{A}}(a) = 1 \right\}.$$

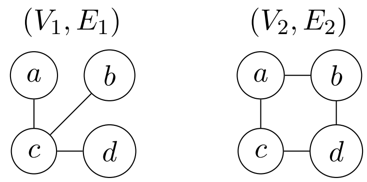
We may say that C is defined by φ (in $\mathcal{A}$). Or that C is the "set of solutions to φ (in $\mathcal{A}$)".

Question 10. (1) Let $\mathcal{A}$ be a structure, $C \subseteq A$ a definable subset, and $f \colon A \to A$ an automorphism[3] of $\mathcal{A}$. Prove that $f(c) \in C$ for any $c \in C$. Conclude that the restriction of f to C is a one-to-one and onto map from C to C. [You may use the fact that if f is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ then f^{-1} is well defined and is an isomorphism from $\mathcal{B}$ to $\mathcal{A}$. You should also use our theorem about isomorphism, which will be proven in the week Feb 14-18.]
(2) Consider the structure $\mathcal{A} = (\mathbb{N}, <)$. Prove that for every natural number $n \in \mathbb{N}$, the subset $\{n\}$ is definable in $\mathcal{A}$. [Hint, prove it first for $\{0\}$, then for $\{1\}$... Note that you need to find infinitely many formulas, one for each such set.]
(3) Consider the structure $\mathcal{A} = (\mathbb{Z}, <)$. Prove that the only definable subsets are the empty set and the entire domain $\mathbb{Z}$.
(4) Consider the following two graphs with vertices $V_1 = V_2 = \{a, b, c, d\}$

[1] The looming question here is: can you write a sentence, using any vocabulary, which precisely characterizes the infinite structures?

[2] "Without parameters". We will see more general forms of definability.

[3] That is, f is an isomorphism. We call it an automorphism when it is an isomorphism from a structure to itself



For each one, determine *all* the definable subsets of it. (Prove your answers.) [Hint: try to understand the possible automorphisms of each graph.]

A formula φ is said to be **quantifier free** if the symbols $\forall, \exists$ do not appear in it at all. Alternatively, the quantifier free formulas are defined inductively:

- atomic formulas are quantifier free;
- negations, conjuctions, disjunctions, and implications between quantifier free formulas are again quantifier free formulas.

(Same as formulas, just not having the quantifier case at all.) Note that in a quantifier free formula, the free variables are *all* variables appearing in the formula.

The following question is a variation on our theorem about isomorphism. (Will be proved in week Feb 14-18). Follow a similar level of formality as in the notes.

Question 11. Suppose $h\colon \mathcal{A} \to \mathcal{B}$ is an embedding.

(1) Let φ be a quantifier free formula. Suppose $x_1, ..., x_n$ is a list containing all variables of φ. Prove that for any $a_1, ..., a_n$ in A,
$$\varphi^{\mathcal{A}}(a_1, ..., a_n) = 1 \iff \varphi^{\mathcal{B}}(h(a_1), ..., h(a_n)) = 1.$$
[Prove for the atomic formulas first, and then inductively. This time the induction does not have the quantifier stages.]

(2) Let φ be of the form $(\exists y_1)...(\exists y_k)\psi$ where ψ is a quantifier free formula. (This is "an existential formula"). Suppose $x_1, ..., x_n$ is a list containing all the free variables of φ. You may also assume that neither of the variables $x_1, ..., x_n$ appears among $y_1, .., y_k$. Prove that for any $a_1, ..., a_n$ from A,
$$\mathcal{A} \models \varphi(a_1, ..., a_n) \implies \mathcal{B} \models \varphi(h(a_1), ..., h(a_n)).$$
[Start by thinking about just one quantifier $(\exists y)\psi$. Also start by assuming that φ does not have other free variables (other than y). In this case there is "nothing to plug" and you need to show that $\mathcal{A} \models \varphi \implies \mathcal{B} \models \varphi$. Appeal to the formal definitions of interpretations in structures. Look at our proof from the isomorphism case.]

Question 12. Let us see that the result in (2) of the previous question about embeddings is essentially best possible. Meaning the $\implies$ arrow cannot be reversed. Consider the signature $\mathcal{S} = \{+, 1\}$ (binary operation and constant symbol) and the structures $(\mathbb{Z}, +, 1)$ and $(\mathbb{Q}, +, 1)$ (usual interpretations).

(1) Prove that $(\mathbb{Z}, +, 1)$ is a substructure of $(\mathbb{Q}, +, 1)$. That is, the identity map $h\colon \mathbb{Z} \to \mathbb{Q}$, $h(k) = k$, is an embedding. [There is not much to prove. Make sure you state precisely what it means, what formally needs to be true, for this to be a substructure.]

(2) Find a sentence φ which is of the form $\varphi = (\exists x)\psi$, where ψ is a quantifier free formula with one free variable x, so that $(\mathbb{Q}, +, 1) \models \varphi$ yet $(\mathbb{Z}, +, 1) \not\models \varphi$. [To

argue whether φ is true or not in the structures, you do not have to formally apply the definition of interpretations. It should be clear however why it is true or not.]

Say that a sentence φ is **logically valid** if it is true in *any* structure. That is, for any structure $\mathcal{A}$ (for the signature in which the sentence is formulated), $\mathcal{A} \models \varphi$. (We will talk more about this in the future.)

Fix some vocabulary $\mathcal{S}$. Given formulas φ, ψ, write $\varphi \leftrightarrow \psi$ to mean $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Question 13. (1) Prove that $\exists x(x \approx x)$ is logically valid.
(2) Prove that $\forall x(x \approx x)$ is logically valid.
(3) Let P be an unary relation symbol. Prove that $(\forall x)(P(x) \leftrightarrow \neg\neg P(x))$ is logically valid.
(4) Let P be an unary relation symbol. Prove that $(\exists x)P(x)$ is *not* logically valid.

[You need to appeal to the formal definition of interpretations in structures.]

Not to be submitted. Let φ and ψ be formulas whose free variables are contained in the list $x_1, ..., x_n$. Say that φ and ψ are **logically equivalent**, denoted $\varphi \equiv \psi$, if the sentence $(\forall x_1)...(\forall x_n)(\varphi \leftrightarrow \psi)$ is logically valid. [Note: this definition does not depend on the list of variables. If it is true for a shorter list, containing all free variables, it is true for a longer list, and vice versa.]

Question (Good practice for the definitions, and important to know)**.** Let φ,ψ be formulas. Prove the following logical equivalences.

(1) $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$.
(2) $\varphi \rightarrow \psi \equiv (\neg\varphi \vee \psi)$.
(3) $(\forall x)\varphi \equiv \neg(\exists x)\neg\varphi$. (Warining: x may or may not be a free variable of φ. In any case, x is not a free variable of either of the formulas.)
(4) Conclude that for any formula φ there is a formula φ' so that $\varphi \equiv \varphi'$ and φ only uses the connectives $\wedge, \neg$ and the existential quantifier $\exists$.
(5) $\neg\neg\varphi \equiv \varphi$.

Question (Requires familiarity with the basic definitions of vector spaces (linear algebra))**.** Consider the vocabulary for vector spaces over $\mathbb{R}$, discussed in class.

(1) Write the axioms for being a vector space over $\mathbb{R}$, in this language.
(2) ($\star$) Is there a sentence Φ in this language which captures the property "the vector space has dimension 2"? That is, a sentence Φ so that a structure $\mathcal{A}$ for this vocabulary satisfies Φ if and only if the dimension of $\mathcal{A}$ as a vector space is 2.
(3) Show that for structures satisfying the axioms you wrote in (1), homomorphisms between structures precisely correspond to linear maps between vector spaces.

Question (Requires familiarity with basic definitions in field theory)**.** (1) Write the axioms for being a Field, using the vocabulary $0, 1, +, \cdot$.
(2) ($\star$) How would you axiomatize being an "algebraically closed field"?
(3) How would you axiomatize that the field is of characteristic 0?
(4) Write the axioms for being an Ordered Field, using the vocabulary $0, 1, +, \cdot, <$.

Homework 4 (2 weeks)

Fix some vocabulary $\mathcal{S}$. Write $\varphi \leftrightarrow \psi$ to mean $(\varphi \to \psi) \land (\psi \to \varphi)$. Recall that two sentences are **logically equivalent**, denoted $\varphi \equiv \psi$, if the sentence $\varphi \leftrightarrow \psi$ is logically valid. That is, if for any model $\mathcal{A}$, $\mathcal{A} \models \varphi \leftrightarrow \psi$. (We also use the notation " $\models (\varphi \leftrightarrow \psi)$" for this).

Question 14. (1) Suppose $h\colon \mathcal{A} \to \mathcal{B}$ is an embedding. Let φ be of the form $(\forall y_1)...(\forall y_k)\psi$, where ψ is a quantifier free formula. (This is "a universal formula"). Suppose $x_1, ..., x_n$ is a list containing all the free variables of φ. You may also assume that neither of the variables $x_1, ..., x_n$ appear among $y_1, .., y_k$. Prove that for any $a_1, ..., a_n$ from A,

$$\mathcal{B} \models \varphi(h(a_1), ..., h(a_n)) \implies \mathcal{A} \models \varphi(a_1, ..., a_n).$$

[The same proof as Question 4 of Pset 3 would work. You can also directly use Question 4 of Pset 3 together with some logical equivalences. In particuar: $(\forall x)\varphi \equiv \neg(\exists x)\neg\varphi$. It follows that $(\forall y_1)...(\forall y_k)\psi \equiv \neg(\exists y_1)...(\exists y_k)\neg\psi$. (In other words $\neg(\forall y_1)...(\forall y_k)\psi \equiv (\exists y_1)...(\exists y_k)\neg\psi$.)]

(2) Consider the signature $\mathcal{S} = \{+, 1\}$ as in Question 5 of Pset 3. Let φ be the sentence from that question. Prove that the sentence φ is *not* logically equivalent to any universal formula. That is, if $\theta = (\forall y_1)...(\forall y_k)\zeta$ where ζ is a quantifier free formula, then it is *not* true that $\models (\varphi \leftrightarrow \theta)$.

(3) The above example (and Question 5 of Pset 3) should give you some idea of what quantifier free, existential, or universal formulas can or cannot express in natural mathematical examples. Here is a more abstract version of this same idea, which you can prove using the same outline.

Consider the vocabulary consisting of just one unary relation symbol P. Let φ be the sentence $(\exists x)P(x)$. Prove that φ is *not* logically equivalent to any universal formula in this language with signature $\{P\}$.

Question 15. Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, ...$ be structures (for the same signature) so that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ ($\mathcal{A}_n$ is a substructure of $\mathcal{A}_{n+1}$). Let $A = \bigcup_{n=0,1,2,...} A_n$. Define a structure $\mathcal{A}$ with universe A as follows. If R is an n-ary relation in the vocabulary and $a_1, ..., a_n$ are in A, take some M so that $a_1, ..., a_n$ are in A_M and define $R^{\mathcal{A}}(a_1, ..., a_n)$ to hold if and only if $R^{\mathcal{A}_M}(a_1, ..., a_m)$ holds. If F is an n-ary function symbol in the language and $a_1, ..., a_n$ are in A, take some M so that $a_1, ..., a_n$ are in A_M and define $F^{\mathcal{A}}(a_1, ..., a_n) = F^{\mathcal{A}_M}(a_1, ..., a_m)$.

(1) Show that $\mathcal{A}$ "is well defined". [Perhaps you have not heard that term before. Start by thinking critically at the above given definition. Does it really make sense? Is $R(a_1, ..., a_n)$ given a clear and unique truth value by that definition? Note that there may be different values of M for which $a_1, ..., a_n$ is in A_M.]
(2) Show that each $\mathcal{A}_n$ is a substructure of $\mathcal{A}$. [There is not much to do here.]
(3) Assume that for each n, $\mathcal{A}_n \prec \mathcal{A}_{n+1}$ ($\mathcal{A}_n$ is an elementary substructure of $\mathcal{A}_{n+1}$). Prove that each $\mathcal{A}_n$ is an elementary substructure of $\mathcal{A}$.
(4) Here is an example of why being an elementary substructure is important above. Unlike it, "elementary equivalence", and even "being isomorphic" is not "preserved under unions".
Let $\mathcal{A}_n = (\mathbb{Q} \cap [\frac{1}{n}, 1), <)$ with the usual order.
(a) Show that $\mathcal{A}_n$ is a substructure of $\mathcal{A}_{n+1}$.

(b) Show that $\mathcal{A}_n$ is isomorphic to $\mathcal{A}_{n+1}$ (so in particular elementary equivalent).
(c) Let $\mathcal{A}$ be the union model as above. Show that $\mathcal{A}_n$ is not elementary equivalent to $\mathcal{A}$ (in particular not isomorphic).

Given a model $\mathcal{A}$ and a natural number n, say that a set $C \subseteq A^m$ is **definable** in $\mathcal{A}$ (*with parameters*) if there is some formula φ, a list of variables $x_1, ..., x_m, y_1, ..., y_n$ including all free variables of φ, and there are fixed members $b_1, ..., b_n$ of A so that

$$C = \{(a_1, ..., a_m) \in A^m | \mathcal{A} \models \varphi(a_1, ..., a_m, b_1, ..., b_n)\}.$$

Note that such φ and $b_1, ..., b_n$ are not necessarily unique. There could be many ways to define the same set.

If $n = 0$ (there are no b's at all), we say that C is definable *without parameters*. Generally, one may care about definability with a particularly chosen parameter.

Question 16. Fix a model $\mathcal{A}$ in a signature $\mathcal{S}$.

(1) Prove that any finite subset of A is definable in $\mathcal{A}$. That is, given $a_1, ..., a_k \in A$ the set $C = \{a_1, ..., a_k\} \subseteq A = A^1$ is definable. [You do not need to use the signature. This is true for any signature, also the empty signature.]
(2) ([4]) Assume that A and S are countable. Let $\mathcal{D}$ be the collection of all C such that for some m, $C \subseteq A^m$ is definable in $\mathcal{A}$ with parameters. Prove that $\mathcal{D}$ is countable. [Hint: if you used that "countable unions of countable sets is countable" many times, you probably need to use it even more.]

Recall that in the last Pset we considered sets $C \subseteq A = A^1$ which are definable *without parameters*.

Question 17. Consider the structure $\mathcal{A} = (\mathbb{R}, +^{\mathcal{A}}, \cdot^{\mathcal{A}}, 0^{\mathcal{A}}, 1^{\mathcal{A}})$ (standard interpretations).

(1) Prove that the set $C \subseteq \mathbb{R}$, $C = \{a \in \mathbb{R} | a \geq 0\}$ is definable in $\mathcal{A}$ without parameters.
(2) Prove that the set of one member $\{\sqrt{2}\} \subseteq \mathbb{R}$ is definable in $\mathcal{A}$ without parameters.
(3) Is $\{\sqrt{2}\}$ definable without parameters in the structure $(\mathbb{R}, +, \cdot)$?
(4) Is $\{\sqrt{2}\}$ definable without parameters in the structure $(\mathbb{R}, +)$?

Question (Not to be submitted)**.** Let $\mathcal{S}$ be some vocabulary and $\mathcal{S}' \subseteq \mathcal{S}$. Given a structure $\mathcal{A}$ for $\mathcal{S}$, it's $\mathcal{S}'$-**reduct** is the structure $\mathcal{A}'$ defined by $A' = A$, $F^{\mathcal{A}'} = F^{\mathcal{A}}$ and $R^{\mathcal{A}'} = R^{\mathcal{A}}$ for any F, R in $\mathcal{S}'$. If we write $\mathcal{A}$ as $(A, R^{\mathcal{A}}, F^{\mathcal{A}})_{R,F \in \mathcal{S}}$ then we may write $\mathcal{A}'$ as $(A, R^{\mathcal{A}}, F^{\mathcal{A}})_{R,F \in \mathcal{S}'}$. Note that any formula in the language for $\mathcal{S}'$ is also a formula for the language $\mathcal{S}$. Prove that for $\mathcal{S}$-structures $\mathcal{A}, \mathcal{B}$,

- If $\varphi(x_1, ..., x_n)$ is an $\mathcal{S}'$ formula and $a_1, ..., a_n$ in A, then $\mathcal{A} \models \varphi(a_1, ..., a_n)$ if and only if $\mathcal{A}' \models \varphi(a_1, ..., a_n)$.
- If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as $\mathcal{S}$ structures, then they are isomorphic as $\mathcal{S}'$-structures.
- If $\mathcal{A}, \mathcal{B}$ are elementary equivalent as $\mathcal{S}$-structures, then they are elementary equivalent as $\mathcal{S}'$-structures.
- If $\mathcal{B}$ is an elementary substructure of $\mathcal{A}$ as $\mathcal{S}$-structures, then it is also the case as $\mathcal{S}'$-structures.

[4] Recall: if A is a countable infinite set, like $\mathbb{N}$, then the "powerset" of A, the collection of *all* subsets of A, is not countable.

(There is not much to prove here. It is very useful though. For example in the following question.)

Question 18. Let $\mathcal{S}$ be a vocabulary, which has a constant symbol c. In the following questions, we write $\bar{x}$ for the list of the n (distinct) variables $x_1, ..., x_n$ and $\bar{x}'$ for the list of $n-1$ variables $x_2, ..., x_n$.

(1) Let t be a term so that the variables appearing in t are contained in the list $\bar{x}$. Define the term $t[c]$ by replacing the symbol x_1 with c everywhere in the string of symbols t. $t[c]$ may be viewed as a term $t[c](\bar{x}')$, since x_1 does not appear in $t[c]$. Let $\mathcal{A}$ be a structure in the vocabulary $\mathcal{S}$. Prove that for any $a_2, ..., a_n$ in A

$$t[c]^{\mathcal{A}}(\bar{x}')(a_2, ..., a_n) = t^{\mathcal{A}}(\bar{x})(c^{\mathcal{A}}, a_2, ..., a_n).$$

[Hint: induction.]

(2) Let φ be a formula whose free variables are contained in $\bar{x}$. Define $\varphi[c]$ by replacing every <u>free</u> occurrence of x_1 by c. Let $\mathcal{A}$ be a structure in the vocabulary $\mathcal{S}$. Prove that for any $a_2, ..., a_n$ in A

$$(a_2, ..., a_n) \in \varphi[c]^{\mathcal{A}}(\bar{x}') \iff (c^{\mathcal{A}}, a_2..., a_n) \in \varphi^{\mathcal{A}}(\bar{x}).$$

Instructions: you can assume that x_1 does not have any bounded appearances. In this case we replace every symbol x_1 by c. In other words, when you prove by induction on the construction of a formula, and you are at the stage where $\varphi = (\exists y)\psi$, you may assume that y is not x_1.[5]

(3) Let $\mathcal{S} = \mathcal{S}_0 \cup \{c\}$. Suppose Γ is a set of sentences for the vocabulary $\mathcal{S}_0$ (so not using c). Let φ be a sentence for the vocabulary $\mathcal{S}_0$, which is of the form $\varphi = (\forall x)\psi$. We may view $\psi(x)$ as a formula for the vocabulary $\mathcal{S}$ as well, and consider $\psi[c]$ as above. Note that $\psi[c]$ is a sentence [why?]. Prove that the following are equivalent:
   - $\Gamma \models \varphi$ (this statement is entirely stated for the vocabluary $\mathcal{S}_0$).
   - $\Gamma \models \psi[c]$ (this statement is stated for the vocabulary $\mathcal{S}$).

   [Hint: Let $\mathcal{A}$ be a structure for $\mathcal{S}_0$ which is a model for Γ. You can *expand* it to a structure for $\mathcal{S}$. More precisely: you can expand it to a structure $\mathcal{A}^+$ for $\mathcal{S}$ so that $\mathcal{A}$ is the reduct of $\mathcal{A}^+$ to the signature $\mathcal{S}_0$. To expand $\mathcal{A}$, all that is necessary is to determine how would this expansion interpret the new symbol c. You can choose how to interpret it, and *any* such choice will be a structure for $\mathcal{S}$, which is still a model of Γ.]

Recall the definition of an equivalence relation. Recall that an equivalence relation E on a set X partitions the set X to disjoint equivalence classes. I will post a short review of equivalence relations.

Question 19. Let $\mathcal{A}$ be a structure for a signature $\mathcal{S}$. Define a relation E on A as follows.[6] For a, b in A, say that $a\ E\ b$ if there exists *some* automorphism $f\colon A \to A$ from $\mathcal{A}$ to $\mathcal{A}$ so that $f(a) = b$.

(1) Prove that E is an equivalence relation.

[5] It is important however to note that if there are bounded occurrences, we do not want to change those, as this would not give us the right meaning.

[6] This is *not* a "definition" in a formal language or in the structure. This is an *external definition*. That is, the usual way we define things in mathematics.

(2) Let $\mathcal{S} = \{<, c_1, c_2, c_3, ...\}$ where $<$ is a binary relation and $c_1, c_2, ...$ are constant symbols. Consider the structure $\mathcal{A}$ for $\mathcal{S}$ with $A = \mathbb{Q}$, $<^{\mathcal{A}} = <$ is the usual ordering of $\mathbb{Q}$, and $c_n^{\mathcal{A}} = \frac{1}{n}$. We may write $\mathcal{A}$ as $(\mathbb{Q}, <, 1, \frac{1}{2}, \frac{1}{3}, ...)$. Let E be the equivalence relation defined above. Find the equivalence classes of E. In other words, write how E partitions $\mathbb{Q}$ into equivalence classes.

[Hint: (1) For a constant symbol c an automorphism f must send $f(c^{\mathcal{A}}) = c^{\mathcal{A}}$. (2) To prove that there is *no* automorphism with certain properties, use the same ideas as when you proved in the past that certain isomorphisms do not exists. (3) To prove that *there is* an automorphism with certain properties, it may be useful to build it "by pieces". That is, you may write $\mathbb{Q}$ as the unions of disjoint sets A_n and define the automorphism of each one individually (you did something similar in Pset 1). Note that you *do* need to say why the final map you get is an automorphism once defined of the entire structure.]

Homework 5 (2 weeks)

Question 20. Let $\mathcal{S} = \{<, c_1, c_2, c_3, ...\}$ where $<$ is a binary relation and $c_1, c_2, ...$ are constant symbols. We interpret below $<$ as the usual ordering on $\mathbb{Q}$. (Technically it should be $<^{\mathcal{A}}$. We are abusing notation by using the same $<$ as the symbol in the vocabulary and the interpretation in $\mathbb{Q}$.)

(1) Define $\mathcal{A} = (\mathbb{Q}, <, 1, \frac{1}{2}, ...)$. That is, $A = \mathbb{Q}$ and $c_n^{\mathcal{A}} = \frac{1}{n}$.
(2) Define $\mathcal{B} = (\mathbb{Q} \setminus \{0\}, <, 1, \frac{1}{2}, ...)$.
(3) Define $\mathcal{C} = (\mathbb{Q}^+, <, 1, \frac{1}{2}, ...)$ where $\mathbb{Q}^+$ are all (strictly) positive rational numbers.

Note that $\mathcal{C}$ is a substructure of $\mathcal{B}$ which is a substructure of $\mathcal{A}$.

Given $\bar{a} = a_1, ..., a_n$ and $\bar{b} = b_1, ..., b_n$ from $\mathbb{Q}$, say that $\bar{a}$ and $\bar{b}$ have the same type if for any $i, j = 1, ..., n$:

- $a_i = a_j \iff b_i = b_j$;
- $a_i < a_j \iff b_i < b_j$;
- $a_i = \frac{1}{m} \iff b_i = \frac{1}{m}$, for any $m = 1, 2, ...$;
- $a_i < \frac{1}{m} \iff b_i < \frac{1}{m}$ for any $m = 1, 2, ...$

(1) Prove that if $\bar{a}$ and $\bar{b}$ have the same type then there is an automorphism of $\mathcal{B}$, $f : \mathbb{Q} \to \mathbb{Q}$, so that $f(a_i) = b_i$ for $i = 1, ..., n$.[7]
(2) Let $\mathcal{C}' = (\mathbb{Q}, <, -1, -2, -3, ...)$. That is $c_n^{\mathcal{C}'} = -n$. Prove that $\mathcal{C}'$ and $\mathcal{C}$ are isomorphic.
(3) Prove that neither two of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are isomorphic to one another.
(4) Prove that $\mathcal{B} \preceq \mathcal{A}$, $\mathcal{C} \preceq \mathcal{A}$, and $\mathcal{C} \preceq \mathcal{B}$. Conclude that $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C}$ (equivalently, $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B}) = \mathrm{Th}(\mathcal{C})$). [Hint: Suppose $\mathcal{D}$ is either $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$. Let $\varphi(x_1, ..., x_k)$ be a formula in the vocabulary $\{<, c_1, c_2, ...\}$ and fix $d_1, ..., d_k$ in D. Then there is some n and a formula $\psi(x_1, ..., x_k, y_1, ..., y_n)$ in the vocabulary $\{<\}$ so that:
   - $\mathcal{A} \models \varphi(d_1, ..., d_k) \iff (\mathbb{Q}, <) \models \psi(d_1, ..., d_k, 1, \frac{1}{2}, ..., \frac{1}{n})$;
   - $\mathcal{B} \models \varphi(d_1, ..., d_k) \iff (\mathbb{Q} \setminus \{0\}, <) \models \psi(d_1, ..., d_k, 1, \frac{1}{2}, ..., \frac{1}{n})$;
   - $\mathcal{C} \models \varphi(d_1, ..., d_k) \iff (\mathbb{Q}^+, <) \models \psi(d_1, ..., d_k, 1, \frac{1}{2}, ..., \frac{1}{n})$.

   In each instance we assume that $d_1, ..., d_k$ are in A, B, C respectively.]

So $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are all different models of the same theory. Question: [Not to be submitted, to be discussed later.] Among the structures (for the same theory) $\mathcal{A}, \mathcal{B}, \mathcal{C}$. Which is the smallest? Which is the largest?

The following question shows that the complete theory of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ can be seen as the logical consequences of very simple axioms.

Question 21. Let $\mathcal{S} = \{<, c_1, c_2, c_3, ...\}$ where $<$ is a binary relation and $c_1, c_2, ...$ are constant symbols. Let T be the theory consisting of the following axioms:

- The 6 DLO axioms saying that $<$ is a linear order, it is dense and has no minimum and no maximum.
- For each $n = 1, 2, ...$, the sentence $\psi_n = c_n > c_{n+1}$ is in T.

(1) Let $\mathcal{D} = (D, <^{\mathcal{D}}, c_1^{\mathcal{D}}, c_2^{\mathcal{D}}, ...)$ be *an arbitrary model* of T, and assume that D is countable. Prove that $\mathcal{D}$ is isomorphic to either $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$.

[7] This is not true in $\mathcal{A}$. You saw in the last Pset that if $f : \mathbb{Q} \to \mathbb{Q}$ is an automorphism of $\mathcal{A}$, then $f(0) = 0$.

(2) Conclude that Con(T), the logical consequences of T, is a complete theory, which is equal to $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B}) = \mathrm{Th}(\mathcal{C})$.

Question 22. Let φ, ψ_1, ψ_2 be sentences. Write a formal deduction (that is, fully write the entire tree of deduction) to prove the following.

(1) $\{\psi_1 \wedge \psi_2\} \vdash \psi_2 \wedge \psi_1$.
(2) $\{\psi_1 \wedge \psi_2, \varphi\} \vdash \varphi \wedge \psi_2$
(3) $\{\varphi\} \vdash \neg\neg\varphi$.
(4) Suppose $\varphi(x)$ is a formula and d a constant symbol. $\{\varphi[d]\} \vdash (\exists x)\varphi$.

Question 23. Prove that the following are equivalent.

(1) $T \vdash \varphi$ (by our definition: there is a formal deduction tree for $T \cup \{\neg\varphi\}$ all of whose branches contain a contradiction.)
(2) There is a formal deduction tree for T (only) so that every branch either contains a contradiction or contains a node with the formula φ.

Homework 6 (2 weeks)

Question 24. Exercise 7.31 in the notes.

Question 25. Exercise 7.36 in the notes. Write a full proof for each of the remaining cases.

Let $\mathcal{S}$ be a signature and $\mathcal{A}$ an $\mathcal{S}$-structure. Say that $\mathcal{A}$ is **named** if for every $a \in A$ there is a constant term t so that $t^{\mathcal{A}} = a$.

Remark:
(1) The models we constructed in Henkin's theorem were all named.
(2) For the model $\mathcal{A}$ to be named, we must have some constant symbols in $\mathcal{S}$. (Otherwise there are no constant terms.)

Question 26. (1) Fix a signature $\mathcal{S}$ and two $\mathcal{S}$-structures $\mathcal{A}, \mathcal{B}$. Assume that $\mathcal{A}$ is named and $\mathcal{A} \simeq \mathcal{B}$ (are isomorphic). Prove that $\mathcal{B}$ is named.

(2) Let $\mathcal{S}$ be a countable signature and $\mathcal{A}$ an infinite countable $\mathcal{S}$-structure. Prove that there is a countable $\mathcal{S}$ structure $\mathcal{B}$ so that $\mathcal{A} \equiv \mathcal{B}$ (elementary equivalent) and $\mathcal{B}$ is *not* named.

[Hint: first consider an expansion by a new constant.]

(3) Let $\mathcal{N} = (\mathbb{N}, 0, 1, +)$. Prove that $\mathcal{N}$ is named. Conclude that there is a countable structure $\mathcal{B}$ so that $\mathcal{B} \equiv (\mathbb{N}, 0, 1, +)$ yet $\mathcal{B} \not\simeq (\mathbb{N}, 0, 1, +)$.

All graphs below will be symmetric and with no loops (a vertex has no edge with itself).

Let (V, E) be a graph. A function $c \colon V \to C$ is called a **coloring** of the graph, if for any $a, b \in V$, *if* $a\ E\ b$, *then* $c(a) \neq c(b)$. Namely: when we "move along an edge" the color must change.
For a number k, a k-coloring is a coloring with k many colors: $C = \{1, ..., k\}$.
A graph (V, E) is k-colorable if it has some k-coloring.

Remark: A more familiar notion of coloring is that of a map (of say, countries). In this case, you want to color each country so that no two adjacent countries have the same color. This is the same problem, when you take the vertices of the graph to be the countries, and you connect to vertices by an edge if the countries share a border.

Question 27. Let (V, E) be a countable graph. Prove that (V, E) is k-colorable if and only every finite subgraph of it is k-colorable.

(Here a subgraph is exactly what it sounds like: we take some finite subset of vertices $V_0 \subseteq V$, and we have the same edges between them as in the big graph. This precisely means that it is a substructure.)

Let $\mathcal{A}$ be a structure for a signature $\mathcal{S}$. Define an expanded signature $S_{\mathcal{A}} = S \cup \{c_a |\, a \in A\}$, where c_a is a new constant symbol.
We consider a collection of sentences in $S_{\mathcal{A}}$, which we will call the (atmoic) **diagram** of $\mathcal{A}$, denoted $D_{\mathcal{A}}$, which are simply supposed to code the structure $\mathcal{A}$.
Fix $\bar{x} = x_1, ..., x_n$. Fix an atomic formula $\varphi(\bar{x}) = R(t_1, ..., t_k)$ where $t_1(\bar{x}), ..., t_k(\bar{x})$ are terms and R is a k-ary relation symbol in $\mathcal{S}$. Fix $a_1, ..., a_n \in A$.
If $(t_1^{\mathcal{A}}(\bar{a}), ..., t_k^{\mathcal{A}}(\bar{a})) \in R^{\mathcal{A}}$, then add the sentence $R(t_1[\bar{c}], ..., t_k[\bar{c}])$ to $D_{\mathcal{A}}$.
Here $\bar{c} = c_{a_1}, ..., c_{a_n}$ and $t_i[\bar{c}]$ is the constant term we get by substituting each variable x_i by the constant symbol c_{a_i}.
(It is best to simply think of a relational language $\mathcal{S}$, and the case where simply $n = k$

and $t_i = x_i$, $\varphi(\bar{x}) = R(x_1, ..., x_k)$ and $\varphi[\bar{c}] = R(c_{a_1}, ..., c_{a_k})$.)
Similarly, if $(t_1^{\mathcal{A}}(\bar{a}), ..., t_k^{\mathcal{A}}(\bar{a})) \notin R^{\mathcal{A}}$, then add the sentence $\neg R(t_1[\bar{c}], ..., t_k[\bar{c}])$ to $D_{\mathcal{A}}$.
(R is also allowed to be $\approx$ here.)

More generally, we consider a collection of sentences in $S_{\mathcal{A}}$, which we will call the **elementary diagram** of $\mathcal{A}$, $D^e_{\mathcal{A}}$, which are supposed to code *all* truths in $\mathcal{A}$.
Fix *any* formula $\varphi(\bar{x})$. Fix $\bar{a} = a_1, ..., a_n \in A$. Let $\bar{c} = c_{a_1}, ..., c_{a_n}$, and $\varphi[\bar{c}]$ the result of substituting every x_i by c_i.[8]
If $\varphi^{\mathcal{A}}(\bar{a}) = 1$, then put the sentence $\varphi[\bar{c}]$ in $D^e_{\mathcal{A}}$.
If $\varphi^{\mathcal{A}}(\bar{a}) = 0$, then put the sentence $\neg\varphi[\bar{c}]$ in $D^e_{\mathcal{A}}$.

The following question is not to be submitted, the results are almost the same as the definition.

Question. Let $\mathcal{B}$ be an $\mathcal{S}_{\mathcal{A}}$-structure. Define a function $f\colon A \to B$ by $f(a) = c_a^{\mathcal{B}}$.

(1) Assume that $\mathcal{B} \models D_{\mathcal{A}}$. Prove that f is an embedding.
(2) Assume that $\mathcal{B} \models D^e_{\mathcal{A}}$, prove that f is an **elementary embedding**: that is, for any formula $\varphi(x_1, ..., x_n)$, for any $a_1, ..., a_n$ from A,
$$\mathcal{A} \models \varphi(a_1, ..., a_n) \iff \mathcal{B} \models \varphi(f(a_1), ..., f(a_n)).$$[9]
(3) Another point of view is this: define $A' = \left\{c_a^{\mathcal{B}} \,|\, a \in A\right\}$, and define the structure $\mathcal{A}'$ with universe A' in the natural way. Then the map f is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$.
Furthermore, if $\mathcal{B} \models D_{\mathcal{A}}$ then $\mathcal{A}'$ is a substructure of $\mathcal{B}$.
If $\mathcal{B} \models D^e_{\mathcal{A}}$, then $\mathcal{A}'$ is an elementary substructure of $\mathcal{B}$.
So, by identifying $\mathcal{A}$ with $\mathcal{A}'$, we may view $\mathcal{B}$ as an *extension* of $\mathcal{A}$.

Not to be submitted.

Question (Requires some familiarity with real analysis)**.** Let $\mathcal{A} = (\mathbb{R}, 0, 1, \cdot, +, <)$ and assume that $\mathcal{A} \preceq \mathcal{B}$. Assume further that there is some $b \in B \setminus \mathbb{R}$.

- Prove that there is some $\epsilon \in B$ so that $\epsilon >^{\mathcal{B}} 0$ and $\epsilon <^{\mathcal{B}} \frac{1}{n}$ for every $n = 1, 2, ...$
- Prove that there is some $b \in B$ so that $n <^{\mathcal{B}} b$ for all $n = 1, 2, ...$

Question (Requires familiarity with basic definitions in field theory)**.** Recall the axioms of a field, a field of characteristic 0, and field of characteristic p. (In the vocabulary $0, 1, +, \cdot$.) Let φ be a sentence in this vocabulary. Assume that φ is provable from the axioms of "a field of characteristic 0". Show that φ is provable from the axioms of "a field of characteristic p" for all large enough p.

Question (Requires familiarity with field theory and uncountable cardinals)**.** Suppose F_1, F_2 are algebraically closed fields of characteristic 0. Assume further that F_1 and F_2 have the same *uncountable* cardinality. Prove that F_1 and F_2 are isomorphic.

(Note that not all countable algebraically closed fields are isomorphic.)

Conclude that all algebraically closed fields (of any size) are elementary equivalent to one another (and therefore to $(\mathbb{C}, +, \cdot, 0, 1)$. Here you will need to use the "upwards Loweinheim-Skolem" theorem.

[8] Recall there are some subtleties with substitution when quantifiers are involved. However, we may simply assume that the variables $\bar{x}$ do not have any "quantified appearences", and then substitution is very natural.

[9] This is a mild generalization of elementary substructure.

Conclude that $(\mathbb{C}, +, \cdot, 0, 1)$ is precisely the logical consequence of the axioms "algebraically closed field of characteristic zero".
In particular, these logical consequences form a complete theory.

Homework 7 (1 week)

Question 28. Work with $\mathcal{A} = (\mathbb{N}, +, \cdot, 1, 0)$ and $T = \text{Th}(\mathcal{A})$. To abuse notation less than usual, let us consider different symbols for the signature: let $\mathcal{S} = \{\tilde{+}, \tilde{\cdot}, \tilde{1}$. So $\mathcal{A}$ is the $\mathcal{S}$-structure with domain $\mathbb{N}$ and $\tilde{+}^{\mathcal{A}} = +$, $\tilde{\cdot}^{\mathcal{A}} = \cdot$ (the standard operations), and $\tilde{1}^{\mathcal{A}} = 1 \in \mathbb{N}$, $\tilde{0}^{\mathcal{A}} = 0 \in \mathbb{N}$.
For each natural number m we have an $\mathcal{S}$-term t_m so that $t_m^{\mathcal{A}} = m$. [These can be defined inductively: $t_1 = \tilde{1}$, $t_{k+1} = t_k \tilde{+} \tilde{1}$. We usually write it less formally as $t_m = 1 + ... + 1$ m many times.]

(1) Write an $\mathcal{S}$-formula $\delta_m(x)$ so that $\mathcal{A} \models \delta_m(n)$ if and only if the number n is divisible by the number m.[10]

Let $p_0, p_1, p_2, p_3, ...$ be a list of all prime numbers (2,3,5,7,...)
Fix a non-empty set $X \subseteq \mathbb{N} = \{0, 1, 2, ...\}$. (e.g. $X = \{1, 3\}$, $X = \mathbb{N}$, $X = \{0, 2, 4, 6, ...\}$, $X = \{2, 4, 8, 16, ...\}$.)
Define a set of formulas (with the variable x) as follows.
$p = \{\delta_{p_k}(x) | \, k \in X\} \cup \{\neg\delta_{p_k}(x) | \, k \in \mathbb{N} \setminus X\}$.

(2) Prove that p is a type. (Working in $T = \text{Th}(A)$). [This should not be long. Make sure your solution shows an understanding of what being a type means.]

What does this type mean? For example, if $X = \{0, 2, 4, 6, ...\}$, then a realization of this type is a "non-standard" natural number, which is divisible by $p_0, p_2, p_4, ...$ yet is not divisible by $p_1, p_3, p_5, ...$

Recall that $S_n(T)$ are all the *complete* n-types. While generally there are at least as many types as there are formulas (and sets of formulas), the *complete* types may be more restricted.

Question 29. Work with $\mathcal{A} = (\mathbb{Q}, <)$ and $T = \text{Th}(\mathcal{A})$.

(1) Prove that $S_1(T)$ has exactly one complete type. That is, $S_1(T) = \{p\}$ for a complete type p.
(2) Prove that $S_2(T)$ has exactly 2 complete types in it.
- [Not for submission] Describe $S_n(T)$ for every n. Prove that they are all isolated.

[Hint: automorphisms! Suppose $\varphi(x)$ is a formula in some type $p \in S_1(T)$. Can there be a type $q \in S_1(T)$ so that $\neg\varphi(x) \in q$?]

Not to be submitted.

Question. Let $T = \text{Th}(\mathbb{N}, +, \cdot, 1, 0)$. Conclude from Question 1 that $S_1(T)$ is uncountable.

Question. Recall the theory $T = \text{Th}(\mathbb{Q}, <, 1, \frac{1}{2}, \frac{1}{3}, ...)$ in the signature $\{<, c_1, c_2, ...\}$, which we saw is precisely the logical consequences of DLO + $\{c_{n+1} < c_n | \, n = 1, 2, ...\}$. Prove that T is not "finitely axiomatizable". That is, there do not exist finitely many sentences $\theta_1, ..., \theta_k$ so that $\text{Con}(\{\theta_1, ..., \theta_k\}) = T$.

[Hint: first, show that no finite subset of DLO+$\{c_{n+1} < c_n | \, n = 1, 2, ...\}$ can work. Note however that we do not assume that $\theta_1, ..., \theta_k$ are of this form.]

Question. [Requires familiarity with basic definitions in field theory]

[10]Note that m here is "external" while n is "internal".

(1) Prove that the theory T of "a field with characteristic 0" is not finitely axiomatizable.
(2) Prove that the theory T of "an algebraically closed field" is not finitely axiomatizable.

The next questions shows another natural mathematical concept which can be captured using types, regarding connected components of graphs.

Let $G = (V, E)$ be a (symmetric) graph. Fix $a, b \in V$ vertices.
Say that $x_0, x_1 ..., x_n$ is a **path** between a and b (of length n) if $x_0 = a$, $x_n = b$ and for each $i = 0, ..., n-1$ there is an edge between x_i and x_{i+1} ($x_i \; E \; x_{i+1}$).
For example, for any a the path x_0, where $x_0 = a = x_n$, is a path of length 0 between a to itself.
If $a \; E \; b$, then $x_0 = a, x_1 = b$ is a path of length 1 between a and b.

Say that a and b are in **the same connected component** of the graph if *there is* a (finite) path from a to b.
Note that "there is a path between a and b" is an equivalent relation on V whose equivalence classes are the connected components.

Question. Work in the language for graphs $\mathcal{S} = \{E\}$ and let $\mathcal{A} = (A, E^{\mathcal{A}})$ be a graph. Assume that $\mathcal{A}$ has more than 1 connected components.

(1) Write a 2-type $p = p(x, y)$ so that for any $a, b \in A$, the pair (a, b) realizes the type p if and only if a and b are in different connected components.
Find a graph $\mathcal{B}$ with only 1 connected component (so p is not realized in $\mathcal{B}$) so that p is still a type ("finitely realizable").
(2) Is there a type $q = q(x, y)$ so that for any $a, b \in A$, the pair (a, b) realizes the type q if and only if a and b are in the same connected component?

Homework 8

Question 30. Suppose $\mathcal{A} = (\mathbb{N}, 0, 1, \cdot, +, <)$, $\mathcal{A} \preceq \mathcal{B}$ (so $\mathcal{B} = (B, 0^{\mathcal{B}}, 1^{\mathcal{B}}, \cdot^{\mathcal{B}}, +^{\mathcal{B}}, <^{\mathcal{B}})$ and in particular $\mathbb{N} \subseteq B$, $0^{\mathcal{B}} = 0$, $1^{\mathcal{B}} = 1$). Prove that if $b \in B \setminus \mathbb{N}$ then $n <^{\mathcal{B}} b$ for all $n \in \mathbb{N}$.

Question 31. In this question you are asked to prove that certain countable structures are saturated or atomic.
This can be done "directly", with enough effort. But that is not the point here.
The point is to see how our big theorems here, regarding the existence of such models (in certain situations), can be used to conclude facts about a given model. [For convenience, some relevant theorems are attached below.]
You can give a quick proof using these big theorems, as well as our understanding the models of these theories. Fully justify your arguments! In particular, you need to argue why such saturated or atomic model exists.

(1) Prove that the structure $(\mathbb{Q}, <)$ is both saturated and atomic.
(2) Prove that $\mathcal{B} = (\mathbb{Q} \setminus \{0\}, <, 1, \frac{1}{2}, ...)$ is saturated.
(3) Prove that $\mathcal{C} = (\mathbb{Q}^+, <, 1, \frac{1}{2}, ...)$ is atomic.

We saw that elementary equivalent structures may be non-isomorphic. In other words, two structures could have the same theory, yet be non-isomorphic. This can happen even if the structure are countable.

We will see now that for finite structures (structures with a finite domain) the theory completely determines the structure. In other words, elementary equivalence implies isomorphism.

Question 32. (1) [Warm up] Consider the vocabulary $\{R, F\}$ where R is a binary relation and F is a binary function. Let $\mathcal{A} = (A, R^{\mathcal{A}}, F^{\mathcal{A}})$ be a *finite* structure (meaning A is finite). Prove that there is a sentence φ in the language so that $\mathcal{A} \models \varphi$ and *for any* structure $\mathcal{B}$, if $\mathcal{B} \models \varphi$ then $\mathcal{B} \simeq \mathcal{A}$. [Hint: let n be the size of A. Write a sentence saying that there are precisely n things and which describes how R and F behave.]

(2) [Finite signature] Let $\mathcal{S}$ be any *finite* vocabulary. Let $\mathcal{A}$ be a *finite* structure in this vocabulary. Prove that there is a sentence φ in the language so that $\mathcal{A} \models \varphi$ and *for any* structure $\mathcal{B}$, if $\mathcal{B} \models \varphi$ then $\mathcal{B} \simeq \mathcal{A}$.
This could be a little messy. You don't have to be too formal here in writing a sentence.
Referring to your solution for (1), explain the main ideas of what this sentence should say, and how this can be done in our language.

This should already be quite convincing. Nevertheless, it *does* make senese to talk about finite structures even if the signature is infinite!
The result is still true, with an additional trick, as explained below.
This part is not for submission.

(•) Let $\mathcal{S}$ be *any* vocabulary (not assumed to be finite, or even countable). Let $\mathcal{A}$ be a *finite* structure in this vocabulary. Prove that for any structure $\mathcal{B}$, if $\mathcal{B} \equiv \mathcal{A}$ (they are elementary equivalent) then $\mathcal{B} \simeq \mathcal{A}$.
[Hint: (a) Show that if $|A| = n$ (A has exactly n members) then $|B| = n$ as well. (b) There are only finitely many functions from A to B (n^n many). You just want to show that one of these must be an isomorphism.

(c) Approach by contradiction, that no function works. If a function $f\colon A \to B$ is *not* an isomorphism, there is some symbol in the language whose interpretation f fails to respect.
(d) Conclude that there are two finite structures for a finite signature, which are elementary equivalent yet not isomorphic. This would contradict part (2). (The facts about reducts from Pset 4 are useful.)]

Not to be submitted.

Question. Let $\mathcal{A} = (\mathbb{N}, +, \cdot, 0, 1)$. Prove that $\mathcal{A}$ is atomic.

Recall: we saw in class that if T has only countably many types, then T has an atomic model.
On the other hand, we saw in Pset 7 that for $T = \mathrm{Th}(\mathcal{A})$, $S_1(T)$ is uncountable.
So the existence of an atomic model is not equivalent to having "not too many types".
(Unlike the existence of a saturated model, which is equivalent to all the S_n's being countable.

Question (Requires familiarity with field theory)**.** Let $\mathbb{Q}^{\mathrm{alg}}$ be the algebraic closure of the rationals. Show that the (countable) structure $(\mathbb{Q}^{\mathrm{alg}}, +, \cdot, 0, 1)$ (an algebraically closed field with characteristic 0) is atomic.
Let $\mathbb{Q}^{\mathrm{alg}}(x_1, x_2, ...)$ be the algebraic closure of the (countably) infinite degree transcendental extension of $\mathbb{Q}$. Show that this model is saturated.

[This can be done by "direct computations" using the concept of quantifier elimination (see Marker's book).
However, you can also give a quick argument using our results, together with some structure theory about algebraically closed fields. (Specifically, the fact that we understand all algebraically closed fields of characteristic 0, up to isomorphism, in terms of their transcendence degree.]

Relevant theorems for Question 2. Recall that a countable structure $\mathcal{A}$ is *saturated* for any parameters $\bar{d} = d_1, ..., d_k$ from A and for any n-type p with parameter $\bar{d}$, p is realized in $\mathcal{A}$. [$\mathcal{A}$ is "very large".]

Recall that a countable structure $\mathcal{A}$ is *atomic* if for any $\bar{a} = a_1, ..., a_n$ in A, the type $p = \mathrm{tp}^{\mathcal{A}}(\bar{a})$ is isolated. [$\mathcal{A}$ is "very small".]

Recall that an isolated type p (w.r.t. a theory T) is realized in *any model* of T. If a type p is omitted in some model, it must be not isolated.

Theorem 1. Let T be a complete theory. If T does not have a countable saturated model, then $I(T)$ is infinite (and in fact uncountable!)

Theorem 2. Let T be a complete theory. If T has a countable saturated model, then it has a countable atomic model.