Bi-weekly homework assignments.

Homework 1

Question 1. Show that the following expressions can be written as Δ_0 formulas.

(1) $x = \{a, b\}$;
(2) $x = \bigcup y$;
(3) $x = (a, b)$;
(4) $X = A \times B$;
(5) R is a relation;
(6) f is a function.

(Recall that (a, b) is defined as $\{\{a\}, \{a, b\}\}$ and $A \times B$ is defined as $\{(a, b) : a \in A, b \in B\}$.

Question 2. Show that the following properties have the corresponding complexity:

(1) "κ is a cardinal" is Π_1;
(2) "κ is a regular cardinal" is Π_1.
(3) "$|X| = |Y|$" is Σ_1;
(4) "$|X| < |Y|$" is Δ_2 assuming ZF, and is Π_1, assuming ZFC.

Question 3. Assume ZF.

(1) Show that $(V_\omega, \in)$ satisfies all axioms of ZFC but the axiom of infinity.
(2) Suppose $\alpha > \omega$ is a limit ordinal. Show that V_α satisfies all axioms of ZF apart from the axiom of replacement. Assuming AC, show that AC holds in V_α as well.
(3) Show that the axiom of replacement fails in $V_{\omega+\omega}$ and in V_{ω_1}.
(4) Show that the powerset axiom fails in $H(\aleph_1)$.

Question 4. A cardinal κ is called **(strongly) inaccessible** if κ is regular and for any $\alpha < \kappa$, $|2^\alpha| < \kappa$. Assuming ZFC, show that if κ is strongly inaccessible then $H(\kappa) = V_\kappa$. Conclude that then V_κ satisfies all the axioms of ZFC.

Question 5. Exercise 2.20 in the notes.

Question 6 (Recursive definition)**.** Suppose $\theta(x, y)$ is such that given a function f with domain an ordinal α, there is a unique y such that $\theta(f, y)$ holds. We may use θ to define recursively functions f_α by

- $f_0 = \emptyset$;
- given f_α, $f_{\alpha+1}$ has domain $\alpha + 1$ such that $f_{\alpha+1} \restriction \alpha = f_\alpha$ and $f_{\alpha+1}(\alpha)$ is the unique y for which $\theta(f_\alpha, y)$ holds;
- $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$ if α is a limit ordinal.

Write a formula $\phi(w, y)$ which is true if and only if w is an ordinal and $f_{w+1}(w) = y$ (equivalently, $f_\alpha(w) = y$ for some $\alpha > w$). That is, ϕ defines a class function, defined on all the ordinals, recursively using θ. Show that if θ is Σ_n then ϕ is Δ_n.

Homework 2

Recall the following definition involved in Godel's L.

Given a set M, we view M as a model $(M, \in)$. Say that X is definable (with parameters) over M if $X \subseteq M$ and there is a formula $\phi(x, x_1, ..., x_k)$ and parameters $a_1, ..., a_k$ in M such that

$$x \in X \iff M \models \phi(x, a_1, ..., a_k).$$

Let $\mathrm{Def}(M)$ be the set of all $X \subseteq M$ which are definable over M.

Recall that the hierarchy L_α is defined recursively on the ordinals by

- $L_0 = \emptyset$;
- $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$;
- $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ for limit α.

Question 7. (1) Show that the property "$X = \mathrm{Def}(M)$" (of the pair X, M) is Δ_1.
(2) Show that the property "$X = L_\alpha$" (of the pair X, α) is Δ_1.
(3) Show that L_α is hereditarily ordinal definable, for every ordinal α. In other words, $L \subseteq \mathrm{HOD}$.

It follows that L is absolute between transitive models of ZF. In particular, if we follow the "construction of L" inside L, we get L again. In other words, $L \models \forall x \exists \alpha (x \in L_\alpha)$, where "$x \in L_\alpha$" is the Δ_1 expression you found above.

Question 8. Show that the property "X is in HOD" is Σ_2.

Question 9. (1) Show that $H(\aleph_1) \not\prec_{\Sigma_2} V$.
(2) Show that $H(\aleph_\omega) \not\prec_{\Sigma_2} V$. (Recall that $\aleph_{n+1} = \aleph_n^+$ and $\aleph_\omega = \sup_n \aleph_n = \bigcup_n \aleph_n$.)
(3) Show that $V_{\omega_1} \not\prec_{\Sigma_1} V$.
(4) Show that $H(\aleph_1) \prec_{\Sigma_1} V_{\omega_1}$.

Relative definability. Let A be a set. Say that a set X is **ordinal definable over** A (X is $\mathrm{OD}(A)$) if there is a formula $\phi(x, y_1, ..., y_n, z, w_1, ..., w_k)$, ordinals $\alpha_1, ..., \alpha_n$ and $a_1, ..., a_k \in A$ such that

$$x \in X \iff \phi(x, \alpha_1, ..., \alpha_n, A, a_1, ..., a_k).$$

As before, this is equivalence to saying that there exists an ordinal α such that $A \in V_\alpha$, $\alpha_1, ..., \alpha_n < \alpha$ and

$$x \in X \iff \phi^{V_\alpha}(x, \alpha_1, ..., \alpha_n, A, a_1, ..., a_k)$$

Say that X is **hereditarily ordinal definable over** A if every member of $\mathrm{t.c.}(X)$, the transitive closure of X, is ordinal definable over A.

Let $\mathrm{HOD}(A)$ be the collection of all sets which are hereditarily ordinal definable over A.

Question 10 (Assume ZF)**.** (1) For any set A prove that $\mathrm{HOD}(A)$ satisfies ZF.
(2) Assume that A is a set of ordinals. Prove that $\mathrm{HOD}(A)$ satisfies ZFC, and $A \in \mathrm{HOD}(A)$.
(3) Assume that A is a set of sets of ordinals, that is, there is some ordinal η such that $A \subseteq \mathcal{P}(\eta)$. Prove that $A \in \mathrm{HOD}(A)$.
(4) Assume that A is a transitive set. Prove that $A \in \mathrm{HOD}(A)$.

Relative constructability. The following provides a quick introduction to versions of the constructible universe L, when relativized to a set A. These are not to be submitted. Also they assume you are already familiar with the basics of L.

Definition 1 (Hajnal's relativized L-construction)**.** Given a set A, define

- $L_0(A) = \mathrm{t.c.}(A)$, the transitive closure of A;
- $L_{\alpha+1}(A) = \mathrm{Def}(L_\alpha(A))$;

- $L_\alpha(A) = \bigcup_{\beta<\alpha} L_\beta(A)$ for limit α.

Let $L(A)$ be the union of all $L_\alpha(A)$ for ordinals α. Then $L(A)$ is a definable transitive class.

For a set M a relation R on M and $X \subseteq M$, say that X is definable (with parameters) over $(M, \in, R)$ if there is a formula ϕ in the language of set theory *expanded by the relation* R (that is in the atomic case we also consider formulas of the form $R(x_1, ..., x_m)$, if R is an m-ary relation), and parameters $a_1, ..., a_k$ in M such that

$$x \in X \iff M \models \phi(x, a_1, ..., a_k).$$

Let $\mathrm{Def}^A(M)$ be the set of all $X \subseteq M$ which are definable over $(M, \in, R)$.

Definition 2 (Levy's relativized L-construction)**.** Given a set, or a definable class, A, define

- $L_0[A] = \emptyset$;
- $L_{\alpha+1}[A] = \mathrm{Def}^{A\cap L_\alpha[A]}(L_\alpha[A])$;
- $L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A]$.

Let $L[A]$ be the union of all $L_\alpha[A]$ for ordinals α. Then $L[A]$ is a definable class.

Question (Assuming ZF)**.**
(1) For any set A, $L(A)$ is a model of ZF and it is the minimal transitive class model of ZF which contains A and all the ordinals.
(2) For any set or definable class A, $L[A]$ is a model of ZFC. It is the smallest transitive model of ZF which contains all the ordinals and satisfies the following:

$$\text{for any } X \in L[A],\ X \cap A \in L[A].$$

Question. If A is a set of ordinals, then $L[A] = L(A)$.

Remark. In general the two models could be different. For example if A is not a set of ordinals, it might be that $A \notin L[A]$. Also, if A is not a set of ordinals it might be that $L(A)$ fails to satisfy the axiom of choice.

Homework 3

Fix a countable transitive model M of ZFC.

Question 11. Let $(\mathbb{P}, \leq)$ be a poset and τ a $\mathbb{P}$-name in M. Find a name $\tau^\star$ such that τ and $\tau^\star$ have the same rank and for any filter $G \subseteq \mathbb{P}$, $\tau^\star[G]$ is the transitive closure of $\tau[G]$. (And prove this.)

Question 12. Exercises 3.22 and 3.23 in the notes.

Question 13.
(1) Prove clauses (6) and (7) of Theorem 3.32 in the notes.

Let $\mathbb{P}$ be the poset of all finite approximations for a subset of ω.

(2) Find two distinct names σ and τ such that $\sigma[G] = \tau[G]$ for any generic filter $G \subseteq \mathbb{P}$.
(3) Find a statement $\phi(x_1, ..., x_n)$, names $\tau_1, ..., \tau_n$ and a condition $p \in \mathbb{P}$ such that $p \Vdash \phi(\tau_1, ..., \tau_n)$ yet $p \not\Vdash^* \phi(\tau_1, ..., \tau_n)$.

Question 14 (Forcing with a dense set)**.** Let $(\mathbb{P}, \leq)$ be a poset and $D \subseteq \mathbb{P}$ a dense set. (All in M.) We can view $(D, \leq)$ as a poset (assume also that $1_{\mathbb{P}} \in D$).

- Given a filter G for $(\mathbb{P}, \leq)$, generic over M, show that there is a filter H for $(D, \leq)$, generic over M, such that $M[G] = M[H]$.
- Conversely, given a filter H for $(D, \leq)$, generic over M, show that there is a filter G for $(\mathbb{P}, \leq)$, generic over M, such that $M[G] = M[H]$.

(In this case we consider $(\mathbb{P}, \leq)$ and $(D, \leq)$ as *equivalent* in the sense that they produce the same extensions.)

Question 15. Let κ be an uncountable cardinal and suppose N is a countable elementary submodel of $H(\kappa)$. (N is not assumed to be transitive.)

(1) Suppose $X \in N$ and X is a countable set. Show that $N \models$ "X is countable" and $X \subseteq N$.
(2) Show that $N \cap \omega_1$ is an ordinal (that is, it is transitive).
(3) Suppose $\kappa > \omega_1$. Show that $N \cap \omega_2$ is not an ordinal.

Homework 4

Question 16. (1) Let $\mathbb{P}$ be the poset of all finite approximations for a subset of ω. Find an infinite antichain $A \subseteq \mathbb{P}$. (Without using the axiom of choice.)
(2) Let $\mathbb{P}$ be the poset of all intervals $p = (x, y)$ for $x, y \in \mathbb{R}$, $x < y$. $p \leq q \iff p \subseteq q$. Find a countable subset of $\mathbb{P}$ which is dense in $\mathbb{P}$.

Question 17 (Product forcing)**.** Let $(\mathbb{P}, \leq^{\mathbb{P}})$ and $(\mathbb{Q}, \leq^{\mathbb{Q}})$ be posets (in M). Define $\leq$ on $\mathbb{P} \times \mathbb{Q}$ by

$$(p, q) \leq (p', q') \iff p \leq^{\mathbb{P}} p' \wedge q \leq^{\mathbb{Q}} q'.$$

Note that $(\mathbb{P} \times \mathbb{Q}, \leq)$ is a poset with maximal element $(1_{\mathbb{P}}, 1_{\mathbb{Q}})$.

We show that forcing with $\mathbb{P} \times \mathbb{Q}$ is equivalent to forcing first with $\mathbb{P}$, and then forcing with $\mathbb{Q}$ over the generic extension.

(1) Suppose $K \subseteq \mathbb{P} \times \mathbb{Q}$ is a generic filter over M. Let G and H be the left and right projections of K respectively. Show that $G \subseteq \mathbb{P}$ is a generic filter over M and that $H \subseteq \mathbb{Q}$ is a generic filter over $M[G]$. Furthermore, show that $M[K] = M[G][H]$, that is, the generic extension of $M[G]$ using the poset $\mathbb{Q}$.
(2) Suppose $G \subseteq \mathbb{P}$ is generic over M and $H \subseteq \mathbb{Q}$ is generic over $M[G]$. Show that $K = G \times H \subseteq \mathbb{P} \times \mathbb{Q}$ is generic over M and that $M[G][H] = M[K]$.
(3) Show that $\mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{P}_2$ are isomorphic. (Recall that $\mathbb{P}_\alpha$ is the poset of all finite binary functions with domain a subset of $\alpha \times \omega$.)

Question 18 (Defining a name by cases)**.** Suppose $(\mathbb{P}, \leq)$ is a poset and $A \subseteq \mathbb{P}$ is an antichain. (Both $(\mathbb{P}, \leq)$ and A are in M.) Suppose that, in M, there is a map assigning for each $p \in A$ a name τ_p. Define a name τ in M such that for any filter $G \subseteq \mathbb{P}$, generic over M, if $G \cap A = \{p\}$ then $\tau[G] = \tau_p[G]$ and if $G \cap A = \emptyset$ then $\tau[G] = \emptyset$. (And prove that this is the case.)

Question 19 (Canonical witness (using AC))**.** Let $\phi(x, \tau_1, ..., \tau_n)$ be a formula in the forcing language. Find a name τ so that for any condition $p \in \mathbb{P}$, if $p \Vdash \exists x \phi(x, \tau_1, ..., \tau_n)$, then $p \Vdash \phi(\tau, \tau_1, ..., \tau_n)$.

Homework 5

For a partial function $p\colon \mathrm{dom}(p) \to \{0,1\}$, $\mathrm{dom}(p) \subseteq \omega$, and a subset $a \subseteq \omega$, say that a agrees with p if the characteristic function of a extends p. That is, if $p(n) = 1 \iff n \in a$, for all $n \in \mathrm{dom}(p)$.

Question 20. Let $L(A)$ be "the basic Cohen model" as in section 6.1 in the notes. Fix some $a \in A$. Assume ϕ is a formula such that $\phi^{L(A)}(A,a)$ holds. Show that there is a finite partial function $p\colon \mathrm{dom}(p) \to \{0,1\}$ such that for any $b \in A$, if b agrees with p then $\phi^{L(A)}(A,b)$ holds.

Question 21 (ZFC)**.** Let κ be a regular cardinal. Say that a poset $\mathbb{P}$ satisfies the κ-chain condition ($\mathbb{P}$ is κ-c.c.) if there are no antichains in $\mathbb{P}$ of size $\geq \kappa$. So c.c.c. is $\aleph_1$-c.c.

Let $G \subseteq \mathbb{P}$ be a generic filter over V.

Show that if $\mathbb{P}$ is κ-c.c. and $\lambda \geq \kappa$ is a cardinal in V, then λ is a cardinal in $V[G]$ as well.

[If you prefer, you may replace V with a countable transitive model M, then the assumption is that κ is some cardinal in M and $M \models \mathbb{P}$ is κ-c.c. Now replace "generic over V" by "generic over M" and $V[G]$ by $M[G]$.]

Question 22. Let $(\mathbb{P}, \leq^{\mathbb{P}})$ and $(\mathbb{Q}, \leq^{\mathbb{Q}})$ be posets. A map $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a **complete projection** if

- for any $p_1, p_2 \in \mathbb{P}$, if $p_1 \leq^{\mathbb{P}} p_2$ then $\pi(p_1) \leq^{\mathbb{Q}} \pi(p_2)$;
- for any $p \in \mathbb{P}$ and for any $q \leq^{\mathbb{Q}} \pi(p)$ there is a $p' \leq^{\mathbb{P}} p$ such that $\pi(p') \leq^{\mathbb{Q}} q$.

(1) Show that the image of π, $\{\pi(p) : p \in \mathbb{P}\}$, is a dense in $\mathbb{Q}$ below $\pi(1_{\mathbb{P}})$.
(2) Show that for any $G \subseteq \mathbb{P}$ generic over V, there is a filter $H \subseteq \mathbb{Q}$ generic over V with $H \in V[G]$.
(3) (a) Let $\mathbb{P}$ and $\mathbb{Q}$ be any posets. Show that the map $\mathbb{P} \times \mathbb{Q} \to \mathbb{P}$ defined by $(p,q) \mapsto p$ is a complete projection.
  (b) Let $\mathbb{P}$ be the poset of all finite approximations to a subset of ω. Define a map $\pi\colon \mathbb{P} \to \mathbb{P}$ as follows. The domain of $\pi(p)$ is the set of all k such that $2k$ is in the domain of p, and $\pi(p)(k) = p(2k)$ when defined. Show that π is a complete projection.
(4) Let $\mathbb{P}$ be the poset of finite approximations to a subset of ω. Suppose $G \times H \subseteq \mathbb{P} \times \mathbb{P}$ is generic over V. Let $g = \bigcup G$ and $h = \bigcup H$. Define $f_1\colon \omega \to \{0,1\}$ by $f_1(n) = \max\{g(n), h(n)\}$. Define $f_2\colon \omega \to \{0,1\}$ by $f_2(n) = g(n) + h(n) \mod 2$.
  (a) Find a filter $F_1 \subseteq \mathbb{P}$, generic over V such that $f_1 = \bigcup F_1$.
  (b) Find a filter $F_2 \subseteq \mathbb{P}$, generic over V such that $f_2 = \bigcup F_2$.

(If it helps, you may replace V with a countable model M and change everything to "generic over M", where $\mathbb{P}$, $\mathbb{Q}$ and π are in M.)

Question 23. Let $\mathbb{P} = \mathbb{P}_\omega$ the poset of all finite partial functions $p\colon \mathrm{dom}(p) \to \{0,1\}$ where $\mathrm{dom}(p)$ is a finite subset of $\omega \times \omega$. Let $G \subseteq \mathbb{P}$ be generic over V and define $a_n = \{k \in \omega : (\exists p \in G)p(n,k) = 1\}$. Define $A_n = \{a_n \Delta X : X \subseteq \omega \text{ is finite}\}$. Let $A = \{A_n : n \in \omega\}$, and consider the model $L(A)$.

(1) Prove that for any set $B \in L(A)$ there is a formula ϕ, a parameter $v \in L$ and finitely many $z_1, ..., z_k \in \bigcup A$ such that $b \in B \iff \phi^{L(A)}(b, A, z_1, ..., z_k, v)$.
(2) Prove that if $B \in L(A)$, $B \subseteq A$, then either B is finite or $A \setminus B$ is finite.

(3) Prove in ZF that if $(X, \leq)$ is a linearly ordered set then there is some set $Y \subseteq X$ such that Y and $X \setminus Y$ are not finite. Conclude that in $L(A)$ there is not linear ordering of the set A.

Homework 6

Question 24 (ZFC)**.** Let λ be an ordinal. Let $\mathbb{P}$ be the poset of all functions $p\colon \operatorname{dom} p \to \{0,1\}$ where $\operatorname{dom} p \subseteq \lambda \times \omega_1$ is a countable subset. Assume that the Continuum Hypothesis holds, that is, $2^{\aleph_0} = \aleph_1$.

(1) Show that $|\mathbb{P}| = (|\lambda| + \aleph_1)^{\aleph_0}$.
(2) Assume CH holds. Let θ be a regular cardinal greater than $\aleph_1$ (e.g. $\theta = \aleph_2$). Prove that for any set $X \in H(\theta)$ there is an elementary substructure $M \prec H(\theta)$ such that
   - $\omega_1 \subseteq M$, $X \in M$;
   - $|M| = \aleph_1$;
   - M is closed under countable sequences. That is, if $h\colon \omega \to M$ is a function, then $h \in M$. (In particular, if $a \subseteq M$ and $|a| = \aleph_0$ then $a \in M$.)
(3) Show that the poset $\mathbb{P}$ satisfies the $\aleph_2$-c.c. condition. That is, any antichain $A \subseteq \mathbb{P}$ is of size at most $\aleph_1$. [Hint, assume A is an antichain of size $\aleph_2$, find a model M as above with A in M, and argue as we did before.]
(4) Show that $\mathbb{P}$ is σ-closed.
(5) Let λ be $\aleph_3$ (as calculated in V). Suppose $G \subseteq \mathbb{P}$ is generic over V. Show that in the model $M[G]$:
   - CH holds;
   - the powerset of $\aleph_1^{V[G]}$ has size $\geq \aleph_3^{V[G]}$.
(6) Suppose also that in V, $2^{\aleph_1} = \aleph_2$. Show that in $V[G]$ the powerset of $\aleph_1^{V[G]}$ has size precisely $\aleph_3^{V[G]}$.

Question 25. Let $\mathbb{S}$ be Sacks forcing as defined in section 10 in the notes. Let $G \subseteq \mathbb{S}$ be generic over V.

(1) Show that $\mathbb{S}$ is not c.c.c.
(2) Show that $\mathbb{S}$ is not σ-closed.
(3) [Assume ZFC] Suppose that $g\colon \omega \to \theta$ is in $V[G]$, for some ordinal θ. Prove that there is a set $X \in V$ such that $X \subseteq \theta$, X is countable, and $g(n) \in X$ for all $n \in \omega$. [Hint: more specifically, find in V sets $F_0, F_1, F_2, ...$ such that $|F_n| = 2^n$ and $g(n)$ is in F_n.]
(4) Conclude that forcing with $\mathbb{S}$ does not collapse $\aleph_1$. That is, the ordinal ω_1^V (the first uncountable ordinal as calculated in V) is still uncountable in $V[G]$.