Midterm Exam Emat 213

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Marked out of 36

[10 points] Problem 1.

Write the general solution of the following linear ODEs

$$\text{(a)}\quad y^{(5)} - y^{(4)} - y''' - y'' + 4y' - 2y = 0$$
$$\text{(b)}\quad x^4 y'''' + \frac{1}{2}x^2 y'' + \frac{1}{2}xy' - 2y = 0$$

You may find this useful:

$$m(m-1)(m-2)(m-3) + \frac{1}{2}m(m-1) + \frac{1}{2}m - 2 = (m-2)^2\left(m^2 - 2m - \frac{1}{2}\right)$$
$$m^5 - m^4 - m^3 - m^2 + 4m - 2 = (m-1)^3(m+1-i)(m+1+i)$$

Solution

For **(a)** the second hint gives the factorization of the auxiliary equation. The roots are $m = 1$ with multiplicity 3 and $-1 \pm i$. Therefore the complementary solution is

$$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 e^{-x}\cos(x) + c_5 e^{-x}\sin(x)$$

For (**b**) the first hint gives almost the factorization of the auxiliary equation: the quadratic polynomial has roots

$$m^2 - 2m - \frac{1}{2} = 0 \qquad m_{1,2} = 1 \pm \sqrt{1 + \frac{1}{2}} = 1 \pm \sqrt{6}/2$$

Therefore

$$y_c = c_1 x^2 + c_2 x^2 \ln(x) + c_3 x^{1+\sqrt{6}/2} + c_4 x^{1-\sqrt{6}/2}$$

[10 points] Problem 2.

Consider the linear ODE

$$y'' - 4y' + 4y = 25\cos(x)$$

(i) Find the **complementary solution** y_c (i.e. the general solution of the associated homogeneous equation).
(ii) Find a **particular solution** y_p (hint: Undetermined coefficients or Variation of Parameters (whichever)).
(iii) Use the above information to find the solution for the IVP

$$y'' - 4y' + 4y = 25\cos(x)$$
$$y(0) = 0,\ y'(0) = 0$$

Solution
Complementary: the aux. eq.

$$m^2 - 4m + 4 = (m-2)^2$$

Hence

$$y_c = c_1 \mathrm{e}^{2x} + c_2 x \mathrm{e}^{2x}$$

Particular: we use the Ansatz

$$\begin{aligned} y_p &= A\cos(x) + B\sin(x) \\ y_p' &= -A\sin(x) + B\cos(x) \\ y_p'' &= -A\cos(x) - B\sin(x)\ . \end{aligned}$$

Plugging into the equation we get

$$\begin{aligned} &-A\cos(x) - B\sin(x) - 4(-A\sin(x) + B\cos(x)) + 4(A\cos(x) + B\sin(x)) = \\ &(3A - 4B)\cos(x) + (4A + 3B)\sin(x) = 25\cos(x) \end{aligned}$$

$$\left\{ \begin{array}{c} 3A - 4B = 25 \\ 4A + 3B = 0 \end{array} \right. \mapsto \left\{ \begin{array}{c} 3A - 4B = 25 \\ A = -\frac{3}{4}B \end{array} \right. \mapsto \left\{ \begin{array}{c} -\frac{9}{4}B - 4B = 25 \\ A = -\frac{3}{4}B \end{array} \right.$$

$$\mapsto \left\{ \begin{array}{c} -\frac{25}{4}B = 25 \\ A = -\frac{3}{4}B \end{array} \right. \mapsto \left\{ \begin{array}{c} B = -4 \\ A = 3 \end{array} \right.$$

$$y_p = 3\cos(x) - 4\sin(x)$$

The general solution is thus

$$\begin{aligned} y &= 3\cos(x) - 4\sin(x) + c_1 \mathrm{e}^{2x} + c_2 x \mathrm{e}^{2x} \\ y' &= -3\sin(x) - 4\cos(x) + 2c_1 \mathrm{e}^{2x} + 2c_2 x \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} \end{aligned}$$

We have thus

$$\begin{aligned} &y(0) = 3 + c_1 = 0 \quad \Rightarrow \quad c_1 = -3 \\ &y'(0) = -4 - 6 + c_2 = 0 \quad \Rightarrow \quad c_2 = 10 \\ &y_{IVP} = 3\cos(x) - 4\sin(x) - 3\mathrm{e}^{2x} + 10x\mathrm{e}^{2x} \end{aligned}$$

[10 points] Problem 3.

Consider the Cauchy-Euler linear ODE

$$x^2 y'' - xy' + y = x^2$$

(i) Find the **complementary solution** y_c (i.e. the general solution of the associated homogeneous equation).
(ii) Find a **particular solution** y_p (hint: Variation of Parameters. **Warning***:*

must put in standard form first!). **Solution Complementary**: the auxiliary equation is

$$m(m-1) - m + 1 = m^2 - 2m + 1 = (m-1)^2$$

hence

$$y_c = c_1 x + c_2 x \ln x$$

Particular: using variation of parameters. Note that the standard form is

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 1$$

so $f(x)$ in the formula of variation of parameters is $f(x) = 1$.

$$\begin{aligned}
&y_p = u_1 y_1 + u_2 y_2 \\
&W := \det \begin{pmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{pmatrix} = x \\
&u_1' = \frac{-y_2 f}{W} = -\frac{x \ln x}{x} = -\ln(x) \;\Rightarrow\; u_1 = -\int \ln(x)\mathrm{d}x = -x\ln x + x \\
&u_2' = \frac{y_1 f}{W} = \frac{x}{x} = 1 \;\Rightarrow\; u_2 = x
\end{aligned}$$

So we have

$$\begin{aligned}
y_p &= (x - x\ln x)x + x^2 \ln x = x^2 \\
y &= x^2 + c_1 x + c_2 x \ln x
\end{aligned}$$

[10 points] Problem 4.

A mass of 2Kg is attached to a spring of constant $k = 1$ N/m. The medium offers a damping coefficient of 2 Ns/m. The system is subject to an external force equal to

$$F_{ext} = 13\cos(2t).$$

Find the motion of the mass if we know that it is released from equilibrium and at rest.

Solution

The equation of motion is

$$\begin{aligned}
&Mx'' + \beta x' + kx = F_{ext} \\
&2x'' + 2x' + x = 13\cos(2t).
\end{aligned}$$

We have the auxiliary equation

$$2m^2 + 2m + 1 = 0 \quad m_{1,2} = \frac{-1 \pm \sqrt{1-2}}{2} = \frac{-1 \pm i}{2}$$

$$x_c = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \cos(t/2) + c_2 \sin(t/2)\right)$$

The motion is **underdamped** (complex-conjugate roots).
The particular solution is

$$\begin{aligned} x_p &= A\cos(2t) + B\sin(2t) \\ x_p' &= -2A\sin(2t) + 2B\cos(2t) \\ x_p'' &= -4A\cos(2t) - 4B\sin(2t) \end{aligned}$$

Plugging into the equation we find

$$\begin{aligned} &2\left(-4A\cos(2t) - 4B\sin(2t)\right) + 2\left(-2A\sin(2t) + 2B\cos(2t)\right) + A\cos(2t) + B\sin(2t) = \\ &= (-7A + 4B)\cos(2t) + (-7B - 4A)\sin(2t) = 13\cos(2t) \end{aligned}$$

Therefore

$$\begin{cases} -7A + 4B = 13 \\ -7B - 4A = 0 \end{cases} \mapsto \begin{cases} -7A + 4B = 13 \\ A = -\frac{7}{4}B \end{cases} \mapsto \begin{cases} \frac{49}{4}B + 4B = 13 \\ A = -\frac{7}{4}B \end{cases} \mapsto$$

$$\begin{cases} \frac{65}{4}B = 13 \\ A = -\frac{7}{4}B \end{cases} \mapsto \begin{cases} B = \frac{4}{5} \\ A = -\frac{7}{5} \end{cases}$$

So

$$\begin{aligned} x(t) &= \mathrm{e}^{-\frac{t}{2}}\left(c_1\cos(t/2) + c_2\sin(t/2)\right) - \frac{7}{5}\cos(2t) + \frac{4}{5}\sin(2t) \\ x'(t) &= \frac{1}{2}\mathrm{e}^{-\frac{t}{2}}\left((c_2 - c_1)\cos(t/2) - (c_1 + c_2)\sin(t/2)\right) + \frac{14}{5}\sin(2t) + \frac{8}{5}\cos(2t) \end{aligned}$$

$$\begin{aligned} &0 = x(0) = c_1 - \frac{7}{5} \;\Rightarrow c_1 = \frac{7}{5} \\ &0 = x'(0) = \frac{c_2 - c_1}{2} + \frac{8}{5} = \frac{c_2}{2} - \frac{7}{10} + \frac{16}{10} = \frac{c_2}{2} + \frac{9}{10} \;\Rightarrow\; c_2 = -\frac{9}{5} \\ &x_{IVP} = \mathrm{e}^{-\frac{t}{2}}\left(\frac{7}{5}\cos(t/2) - \frac{9}{5}\sin(t/2)\right) - \frac{7}{5}\cos(2t) + \frac{4}{5}\sin(2t) \end{aligned}$$