Emat 213: a Classification

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1 First Order ODEs

Separable: prototype

$$y' = h(y)f(x) \; ; \quad \int \frac{\mathrm{d}y}{h(y)} = \int f(x)\mathrm{d}x$$

Exact: prototype

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \; ; \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$F(x,y) = C \; , \quad \frac{\partial F}{\partial x}(x,y) = M(x,y) \; ; \quad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

Homogeneous: prototype

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \; ; \quad M(tx,ty) = t^d M(x,y) \; ; \quad N(tx,ty) = t^d N(x,y)$$

Substitute: $y = x \cdot u(x)$

Bernoulli: prototype

$$y' + P(x)y = f(x)y^n \; ; \quad \textbf{Substitute: } y = u^{\frac{1}{1-n}}$$

By Substitution: prototype

$$y' = F(Ax + By + C) \; ; \quad \textbf{Substitute: } u = Ax + By + C$$

Linear: prototype

$$y' + P(x)y = f(x) \; ; \quad y = \mathrm{e}^{-\int P(x)\mathrm{d}x} \int f(x)\mathrm{e}^{\int P(x)\mathrm{d}x}\mathrm{d}x + C\mathrm{e}^{-\int P(x)\mathrm{d}x}$$

2 Second (and higher) order Linear ODEs

Const-Coeffs (CC): prototype

$$ay'' + by' + cy = f(x)$$

Auxiliary Eq. $am^2 + bm + c = 0$

$$y_c = \begin{cases} c_1 e^{m_1 x} + c_2 e^{m_2 x} & \text{Real and distinct } m_1, m_2 \\ c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x) & \text{Complex conjugate } m_{1,2} = \alpha \pm i\beta \\ c_1 e^{mx} + c_2\, x\, e^{mx} & \text{Only one root } m_1 = m_2 = m \end{cases}$$

$y_p =$ By undetermined coefficients or Variation of parameters

Cauchy–Euler: prototype

$$a\,x^2\,y'' + b\,x\,y' + c\,y = f(x)\ , \quad (x > 0)$$

Auxiliary Eq. $am(m-1) + bm + c = 0$

$$y_c = \begin{cases} c_1\, x^{m_1} + c_2\, x^{m_2} & \text{Real and distinct } m_1, m_2 \\ c_1\, x^{\alpha}\cos(\beta\ln(x)) + c_2\, x^{\alpha}\sin(\beta\ln(x)) & \text{Complex conjugate } m_{1,2} = \alpha \pm i\beta \\ c_1 x^m + c_2\, \ln(x)\, x^m & \text{Only one root } m_1 = m_2 = m \end{cases}$$

$y_p =$ Variation of parameters

Variation of Parms : put in normal form the equation

$$y'' + P(x)y' + Q(x)y = f(x)$$

Find complementary solutions y_1 and y_2 then

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1' = \frac{-y_2 f(x)}{W}\ ; \quad u_2' = \frac{y_1 f(x)}{W}$$

$$W := \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2$$

3 Series and solution by series (centered at x=0)

$$y'' + P(x)y + Q(x)y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

4 Homogeneous Systems

- Being able to convert from system form to matrix form and viceversa.

$$\begin{cases} x_1' = & a_{11}x_1 + \cdots + a_{1n}x_n \\ \ \vdots & \\ x_n' = & a_{n1}x_1 + \cdots + a_{nn}x_n \end{cases}$$

$$\mathbb{X}' = A\mathbb{X}\ , \qquad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\ ; \qquad \mathbb{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- Find Eigenvalues and Eigenvectors of A

$$\det(A - \lambda\mathbb{I}) = 0 \;\Rightarrow \lambda_1, \ldots,$$
$$\Big(A - \lambda\mathbb{I}\Big)\mathbb{K} = \lambda\mathbb{K}$$

(a) Matrix is diagonalizable (n eigenvectors) and eigenvalues are real

$$\mathbb{X} = c_1 \mathrm{e}^{\lambda_1 t}\mathbb{K}_1 + \ldots + c_n \mathrm{e}^{\lambda_n t}\mathbb{K}_n$$

(b) There are repeated eigenvalues and less than n eigenvectors; e.g. λ_1 has multiplicity 2 but there is only one eigenvector

$$\mathbb{X}_1 = c_1 \mathrm{e}^{\lambda_1 t}\mathbb{K} + c_2\ \mathrm{e}^{\lambda_1 t}\,(t\mathbb{K} + \mathbb{P})$$
$$\Big(A - \lambda_1 \mathbf{I}\Big)\mathbb{P} = \mathbb{K}$$

(c) There are complex conjugate eigenvalues: find complex eigenvector and split in real and imaginary part.

$$\lambda = \alpha \pm i\beta$$
$$\mathbb{K} = \mathbb{A} \pm i\mathbb{B}$$
$$\mathbb{X}_1 = \mathrm{e}^{\alpha t}\Big(\mathbb{A}\cos(\beta t) - \mathbb{B}\sin(\beta t)\Big)$$
$$\mathbb{X}_2 = \mathrm{e}^{\alpha t}\Big(\mathbb{A}\sin(\beta t) + \mathbb{B}\cos(\beta t)\Big)$$

- Solution by exponentiation

$$\Phi(t) = \mathrm{e}^{At}$$

5 Nonhomogeneous Systems

In matrix form

$$\mathbb{X}' = A\mathbb{X} + \mathbb{F}(t)$$

Complementary solution from above, particular solution by

1. Variation of parameters

$$\mathbb{X}_p = \Phi(t)\int \Phi(t)^{-1}\mathbb{F}(t)\,\mathrm{d}t$$

2. Diagonalization

$$\mathbb{X} = P\mathbb{Y}\ ; \quad \mathbb{Y}' = D\mathbb{Y} + P^{-1}\mathbb{F}(t)$$
$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$