Midterm Exam Emat 213

March 2006

The best four problems will be marked

[10 points] Problem 1. Find the general solution (complementary + particular) of the linear ODE (for y_p use the method of Undetermined Coefficients)

$$y^{(7)} - 6y^{(6)} + 14y^{(5)} - 20y^{(4)} + 25y''' - 22y'' + 12y' - 8y = x^2$$

You may find this useful:

$$m^7 - 6m^6 + 14m^5 - 20m^4 + 25m^3 - 22m^2 + 12m - 8 = (m-2)^3(m+i)^2(m-i)^2$$

Solution From the multiplicities of the roots

$$y_c = (c_1 + c_2 x + c_3 x^2)e^{2x} + (c_4 + c_5 x)\cos(x) + (c_6 + c_7 x)\sin(x)$$

For y_p we use "Undetermined Coefficients"

$$\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \\ y_p''' &= ... = 0 \end{aligned}$$

$$\begin{aligned} -44A + 24Ax + 12B - 8Ax^2 - 8Bx - 8C &= x^2 \\ A &= -\frac{1}{8} \\ -3 - 8B = 0 \ \Rightarrow \ B &= -\frac{3}{8} \\ \frac{44}{8} - \frac{36}{8} - 8C = 0 \ \Rightarrow \ C &= \frac{1}{8} \end{aligned}$$

[10 points] Problem 2. Find the general solution (complementary + particular) of the linear ODE

$$y'' - y' + \frac{1}{4}y = 3 + \mathrm{e}^x$$

[**Hint: for y_p you can use variation of parameters or undetermined coefficients, the latter is faster but it's your choice**].

Solution

$$\begin{aligned} m^2 - m + \frac{1}{4} &= (m - 1/2)^2 \\ y_c &= (c_1 + xc_2)\mathrm{e}^{-x/2} \\ y_p &= A + Be^x \\ (B - B + \frac{1}{4}B)e^x + \frac{A}{4} &= 3 + e^x \\ B = 4\ ,\ A &= 12 \\ y = (c_1 + xc_2)\mathrm{e}^{-x/2} + 4e^x &+ 12 \end{aligned}$$

[10 points] Problem 3.
Solve the following nonlinear second order ODE by the substitution $u = y'$ $u\frac{\mathrm{d}u}{\mathrm{d}y} = y''$

$$y'' + 2y(y')^3 = 0$$

Solution: problem 7 pag. 146.

[10 points] Problem 4.
When a mass of 2 Kilograms is attached to a spring whose constant is 32 N/mm, it comes to rest in the equilibrium position. Starting at $t = 0$, a force equal to $F_{ext}(t) = \mathrm{e}^{-2t}$ is applied to the system. Find the equation of motion in absence of damping.

Solution

$$\begin{aligned}
2x'' + 32x &= e^{-2t} \\
x(0) = 0,\ x'(0) &= 0 \\
x_c(t) &= c_1\cos(4t) + c_2\sin(4t) \\
x_p &= Ae^{-2t} \\
8A + 32A &= 1 \\
A &= \frac{1}{40} \\
x(t) = c_1\cos(4t) + c_2\sin(4t) &+ \frac{1}{40}e^{-2t} \\
x(0) = c_1 + \frac{1}{40} = 0\ \Rightarrow c_1 &= -\frac{1}{4}0 \\
x'(0) = 4c_2 - \frac{1}{20} = 0\ \Rightarrow c_2 &= \frac{1}{80}
\end{aligned}$$

So the solution of the motion is

$$x(t) = -\frac{1}{40}\cos(4t) + \frac{1}{80}\sin(4t) + \frac{1}{40}e^{-2t}$$

[10points] Problem 5.
Solve the eigenvalue problem for the CE equation below with the prescribed boundary value conditions

$$\begin{gathered}
x^2y'' + xy' + \lambda y = 0 \\
y'(1) = 0\ ,\quad y'(\mathrm{e}^2) = 0
\end{gathered}$$

Solution: problem 21 pag. 169 from the book.

[10 pts] Problem 6.
Using the method of **variation of parameters** find the general solution (complementary + particular) of the ODE

$$2y'' + 2y' + y = 4\sqrt{x} \qquad (x > 0)$$

Note: the antiderivatives for u_1, u_2 are transcendental (cannot be computed explicitly). Leave them indicated.

Solution Problem 16 pag. 136.