Midterm Exam Emat 233

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Material allowed: non programmable, non-graphing calculators.

Solve five problems of your choice

Self-serve formula sheet. For a vector-valued functions $\vec{r}(t)$ in three dimension defining a smooth curve for $t \in [a,b]$

$\kappa(t) =$	$a_T(t) =$	$s(t) =$	$\vec{T}(t) =$	$a_N(t) =$	$\vec{N}(t) =$	$\vec{B}(t) =$

$\frac{\lVert\vec{r}'(t)\times\vec{r}''(t)\rVert}{\lVert\vec{r}'(t)\rVert}$	$\frac{\vec{r}'(t)}{\lVert\vec{r}'(t)\rVert}$	$\int_a^t \lVert\vec{r}'(\tau)\rVert \mathrm{d}\tau$	$\frac{\lVert\vec{r}'(t)\times\vec{r}''(t)\rVert}{\lVert\vec{r}'(t)\rVert^3}$	$\vec{T}(t)\times\vec{N}(t)$	$\frac{\vec{T}'(t)}{\lVert\vec{T}'(t)\rVert}$	$\frac{\vec{r}'(t)\cdot\vec{r}''(t)}{\lVert\vec{r}'(t)\rVert}$
(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**

[10 points] Problem 1.

(i) Find the velocity and the acceleration vectors $\vec{v}(t)$, $\vec{a}(t)$ for a particle that moves as described by the vector-valued function

$$\vec{r}(t) = t\mathbf{i} + [t\sin(t) + \cos(t)]\,\mathbf{j} + [t\cos(t) - \sin(t)]\,\mathbf{k}\ .$$

(ii) Compute the unit tangent $\vec{T}(t)$.

(iii) Compute the tangent component of the acceleration a_T.

[10 points] Problem 2.

Using the chain rule, compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ for the function f

$$f(u,v) = \cos(uv),$$

where the variables u, v depend on the variables x, y as follows

$$u = \ln(1+x^2)\ ; \qquad v = x^2 y.$$

[10 points] Problem 3.

For the given curve find the arclength parameter $s(t)$ and reparametrize the curve in terms of it

$$\vec{r}(t) = t^2\mathbf{i} + [t\sin(t) + \cos(t)]\,\mathbf{j} + [t\cos(t) - \sin(t)]\,\mathbf{k}\ , \qquad t \geq 0\ .$$

[**Advice: the ensuing integral is immediate if your derivatives are done properly**]

[10 points] Problem 4.

(i) Compute the gradient of the function

$$F(x,y,z) = xy^2\cos(yz).$$

(ii) Find a direction along which the function F increases as fast as possible at the point $(0,1,\pi)$.

(iii) Compute the directional derivative at the point $(0,1,\pi)$ along the direction of the vector $\vec{u} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$.

[10 points] Problem 5.

Find points on the surface $x^2 + 3y^2 + 4z^2 - 2xy = 16$ at which the tangent plane is parallel to the zx-plane.

[10 points] Problem 6.

Compute the divergence of the gradient of the following function of three variables

$$F(x,y,z) = x^2yz - \frac{1}{3}y^3z + z^2$$

Solution to Problem 1

We have

$$\vec{v}(t) = \mathbf{i} + t\cos(t)\mathbf{j} - t\sin(t)\mathbf{k}$$
$$\vec{a}(t) = (\cos(t) - t\sin(t))\,\mathbf{j} + (-t\cos(t) - \sin(t))\,\mathbf{k}$$
$$\vec{T} = \frac{\vec{v}}{v} = \frac{1}{\sqrt{1+t^2}}\,(\mathbf{i} + t\cos(t)\mathbf{j} - t\sin(t)\mathbf{k})$$

The tangent component is

$$a_T(t) = \vec{T}\cdot\vec{a} = \frac{t\cos^2(t) - t^2\sin(t)\cos(t) + t\sin^2(t) + t^2\sin(t)\cos(t)}{\sqrt{1+t^2}} = \frac{t}{\sqrt{1+t^2}}$$

Solution to Problem 2

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \\
&= -v\sin(uv)\frac{2x}{1+x^2} - u\sin(uv)2xy = -\sin\left(x^2y\ln(1+x^2)\right)\left(\frac{2x^3y}{1+x^2} + 2xy\ln(1+x^2)\right) \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \\
&= -v\sin(uv)0 - u\sin(uv)x^2 = -\sin\left(x^2y\ln(1+x^2)\right)\ln(1+x^2)x^2
\end{aligned}$$

Solution to Problem 3

The arclength parameter is

$$s(t) = \int_0^t v(\tau)\mathrm{d}\tau$$

Here the speed $v(\tau)$ is

$$\vec{v}(\tau) = 2\tau\mathbf{i} + \tau\cos(\tau)\mathbf{j} - \tau\sin(\tau)\mathbf{k}$$
$$v(\tau) = \sqrt{4\tau^2 + \tau^2\cos^2(\tau) + \tau^2\sin^2(\tau)} = \sqrt{5\tau^2} = \tau\sqrt{5} \quad (\text{since } \tau \geq 0)$$

Therefore

$$s(t) = \int_0^t \tau\sqrt{5}\mathrm{d}\tau = \frac{\sqrt{5}}{2}t^2$$

To reparametrize the curve we have to express t in terms of s. Solving the above relation for t we have

$$t(s) = \sqrt{\frac{2}{\sqrt{5}}s}$$

We now plug this into the expression for $\vec{r}(t)$

$$\frac{2}{\sqrt{5}}s\mathbf{i} + \left[\sqrt{\frac{2}{\sqrt{5}}s}\sin(\sqrt{\frac{2}{\sqrt{5}}s}) + \cos(\sqrt{\frac{2}{\sqrt{5}}s})\right]\mathbf{j} + \left[\sqrt{\frac{2}{\sqrt{5}}s}\cos(\sqrt{\frac{2}{\sqrt{5}}s}) - \sin(\sqrt{\frac{2}{\sqrt{5}}s})\right]\mathbf{k}$$

Solution to problem 4

We have

$$\vec{\nabla}F(x,y,z) = y^2\cos(yz)\mathbf{i} + \left(2xy\cos(yz) - xy^2z\sin(yz)\right)\mathbf{j} - xy^3\sin(yz)\mathbf{k}$$

At the point $(0,1,\pi)$ the gradient is

$$\vec{\nabla}F(0,1,\pi) = \cos(\pi)\mathbf{i} = -\mathbf{i}.$$

Since it is already normalized (length one) the direction of max increase at this point is $-\mathbf{i}$ (the gradient itself). To compute the directional derivative along $\vec{u} = \frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})$ we have to compute

$$D_{\vec{u}}F|_{(0,1,\pi)} = \vec{\nabla}F(0,1,\pi)\cdot\vec{u} = (-\mathbf{i})\cdot\left(\frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})\right) = 0$$

Solution to Problem 5

For a plane to be parallel to the xz-plane it is necessary that the normal is -say- $\mathbf{j}$ (y-direction). That is the gradient of the function defining the surface must be parallel to $\mathbf{j}$ or equivalently must have zero components in the x and z direction. The gradient is

$$\vec{\nabla}F = (2x - 2y)\mathbf{i} + (6y - 2x)\mathbf{j} + 8z\mathbf{k}$$

Therefore we have to solve the system

$$\begin{cases} 2x - 2y = 0 \\ 8z = 0 \\ x^2 + 3y^2 + 4z^2 - 2xy = 16 \end{cases} \Rightarrow \begin{cases} x = y \\ z = 0 \\ 4x^2 - 2x^2 = 16 \ \Rightarrow x = \pm\sqrt{8} \end{cases}$$

There are thus two points as required, namely $(\sqrt{8}, \sqrt{8}, 0)$ and $(-\sqrt{8}, -\sqrt{8}, 0)$.

Solution to Problem 6

The gradient is

$$\vec{\nabla}F(x, y, z) = 2xyz\mathbf{i} - y^2z\mathbf{j} + \left(x^2y - \frac{1}{3}y^3 + 2z\right)\mathbf{k}$$

The divergence of the gradient is

$$\vec{\nabla} \cdot \vec{\nabla}F = \partial_x(\partial_x F) + \partial_y(\partial_y F) + \partial_z(\partial_z F) = 2yz - 2yz + 2 = 2.$$

Note that the end-result is a **scalar**