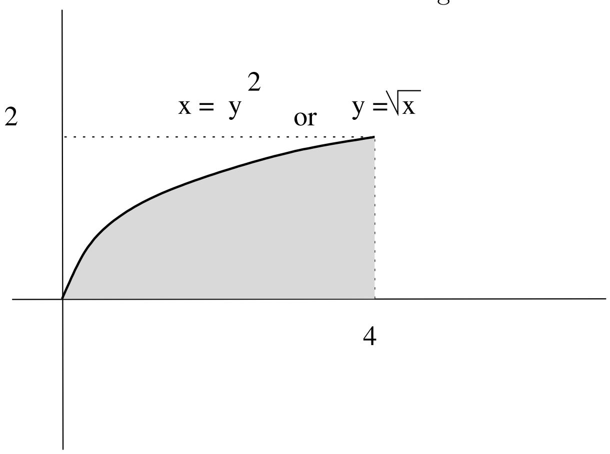
Midterm Exam Emat 233, March 18 2005

Special instructions: solve five problems of your choice

[10 points] Problem 1. Evaluate the following integral by reversing the order of integration

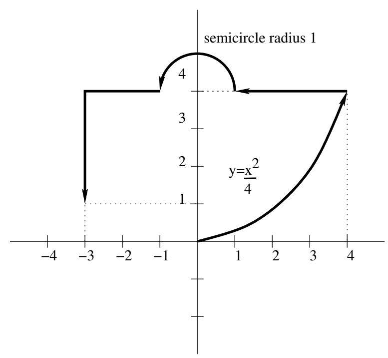
$$\int_0^2 \int_{y^2}^4 \cos(\sqrt{x^3}) \mathrm{d}x \mathrm{d}y$$

Solution to Problem 1. The region is



$$\int_0^4 \int_0^{\sqrt{x}} \cos(\sqrt{x^3}) \mathrm{d}y \mathrm{d}x = \int_0^4 \sqrt{x} \cos(\sqrt{x^3}) \mathrm{d}x = \frac{2}{3} \int_0^4 \cos(x^{\frac{3}{2}}) \mathrm{d}x^{\frac{3}{2}} = \frac{2}{3} \sin(x^{\frac{3}{2}}) \Big|_0^4 = \frac{2}{3} \sin(8)$$

[10 points] Problem 2. Consider the vector field in 2-dimensions



$$\vec{F}(x, y) = \left(x\mathrm{e}^{-x} + 2xy\right) \mathbf{i} + (y + x^2)\mathbf{j} \ .$$

Compute the line integral of $\vec{F}$ along the curve $\mathcal{C}$ in the picture. **Motivate** your answer if necessary.

Solution to Problem 2. The contour is rather complicated so I would suspect a shortcut to exist. Indeed we check that the integral is independent of the path

$$\partial_y \left(x\mathrm{e}^{-x} + 2xy\right) = 2x$$
$$\partial_x (y + x^2) = 2x \quad OK.$$

So we look for the potential ϕ.

$$\phi = \int \left(x\mathrm{e}^{-x} + 2xy\right) \mathrm{d}x = -x\mathrm{e}^{-x} + \int \mathrm{e}^{-x} \mathrm{d}x + x^2 y + h(y) = -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + x^2 y + h(y)$$

Then

$$\partial_y \phi = x^2 + h'(y) = x^2 + y \ \Rightarrow \ h'(y) = y \ \Rightarrow \ h = \frac{y^2}{2} + C$$

Thus a suitable potential is

$$\phi(x,y) = -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + x^2 y + \frac{y^2}{2}$$

The integral then evaluates to

$$\phi(x,y)\Big|_{(0,0)}^{(-3,1)} = \phi(-3,1) - \phi(0,0) = 2\mathrm{e}^3 + 9 + \frac{1}{2} + 1 = 2\mathrm{e}^3 + \frac{21}{2} \ .$$

[10 points] Problem 3. By using the appropriate theorem (which must be named) compute the circulation $\oint_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r}$ of the vector field

$$\vec{F}(x,y) = \left(3y - \mathrm{e}^{-x^2}\right)\mathbf{i} + \left(5x + \ln\left(3 + \cos^2(y^{45})\right)\right)\mathbf{j}$$

around the curve $\mathcal{C}$ given by the circle of radius 2 and center $(3,2)$ oriented counterclockwise.

Solution to Problem 3. The theorem is Green's theorem. So we have

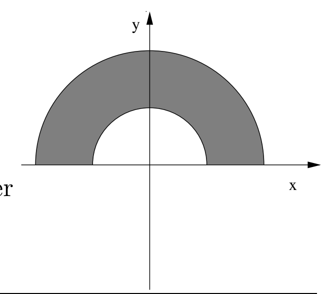
$$\oint_{\mathcal{C}} \left(3y - \mathrm{e}^{-x^2}\right)\mathrm{d}x + \left(5x + \ln\left(3 + \cos^2(y^{45})\right)\right)\mathrm{d}y =$$

$$= \int\int_{\mathcal{R}} \left[\partial_x\left(5x + \ln\left(3 + \cos^2(y^{45})\right)\right) - \partial_y\left(3y - \mathrm{e}^{-x^2}\right)\right]\mathrm{d}A = \int\int_{\mathcal{R}} (5-3)\mathrm{d}A = 2\int\int_{\mathcal{R}} \mathrm{d}A \ .$$

The area of the circle is $\pi r^2 = 4\pi$ so we have

$$2\int\int_{\mathcal{R}} \mathrm{d}A = 8\pi \ .$$

[10 points] Problem 4.



Compute the double integral

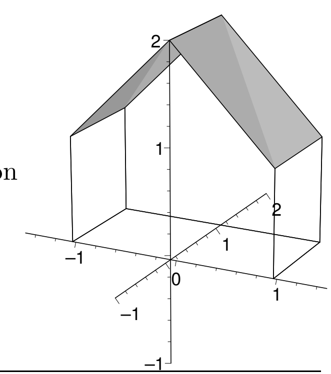
$$\iint_{\mathcal{R}} \mathrm{e}^{-x^2-y^2}\mathrm{d}A$$

over the region consisting of the half washer in the picture with inner radius 1 and outer radius 2.

Solution to Problem 4. The shape of the region suggests passing to polar coordinates, in which the integral reads

$$\int_0^\pi \mathrm{d}\theta \int_1^2 \mathrm{e}^{-r^2} r\mathrm{d}r = \frac{\pi}{2}\mathrm{e}^{-r^2}\Big|_1^2 = \frac{\pi}{2}\left(\mathrm{e}^{-4} - \mathrm{e}^{-1}\right)$$

[10 points] Problem 5.



Compute the flux of the vector field

$$\vec{F}(x,y,z) = (3yz)\,\mathbf{i} + \ln(1 + x^2y^2z^2)\mathbf{j} + z\mathbf{k}$$

across the "roof" surface $\mathcal{S}$ shown in the picture and described hereafter which is the union of portion of two planes, oriented upwards

$$z = 2 + x, \quad -1 \le x \le 0, \quad 0 \le y \le 1$$
$$z = 2 - x, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Solution to Problem 5. The normal of the first portion of plane is

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{k})$$

and the element of area is

$$\mathrm{d}S_1 = \sqrt{1+1}\mathrm{d}A = \sqrt{2}\mathrm{d}A\ .$$

Hence we have

$$\iint_{\mathcal{S}_1} \frac{(-3yz+z)}{\sqrt{2}}\mathrm{d}S_1 = \iint_{\mathcal{R}_1}(2+x)(1-3y)\mathrm{d}x\mathrm{d}y =$$

$$= \int_0^1\int_{-1}^0 (2+x)(1-3y)\mathrm{d}x\mathrm{d}y = \int_0^1 (1-3y)(2x+\frac{x^2}{2})\Big|_{-1}^0 = \int_0^1 (1-3y)\frac{3}{2} = \frac{3}{2}(y-\frac{3}{2}y^2)\Big|_0^1 = -\frac{3}{2}\frac{1}{2} = -\frac{3}{4}$$

For the second surface we have

$$\mathbf{n}_2 = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$$

and the element of area is

$$\mathrm{d}S_2 = \sqrt{1+1}\mathrm{d}A = \sqrt{2}\mathrm{d}A\ .$$

Hence we have

$$\iint_{\mathcal{S}_2} \frac{(3yz+z)}{\sqrt{2}}\mathrm{d}S_1 = \iint_{\mathcal{R}_2}(2-x)(1+3y)\mathrm{d}x\mathrm{d}y =$$

$$= \int_0^1\int_0^1 (2-x)(1+3y)\mathrm{d}x\mathrm{d}y = \int_0^1 (2-x)\mathrm{d}x \int_0^1 (1+3y)\mathrm{d}y = (2x-\frac{x^2}{2})\Big|_0^1 (y+\frac{3}{2}y^2)\Big|_0^1 = \frac{15}{4}$$

The total flux is thus

$$-\frac{3}{4}+\frac{15}{4} = 3$$

[10 points] Problem 6 Evaluate the following integral by using polar coordinates

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}}\int_0^{\sqrt{\pi-x^2}} \sin(x^2+y^2)\mathrm{d}y\mathrm{d}x$$

Solution to Problem 6. The region is the upper semidisk of radius $\sqrt{\pi}$ and hence we have

$$\int_0^\pi \mathrm{d}\theta \int_0^{\sqrt{\pi}} \sin(r^2)r\mathrm{d}r = \frac{\pi}{2}\int_0^{\sqrt{\pi}} \sin(r^2)\mathrm{d}r^2 = -\frac{\pi}{2}\cos(r^2)\Big|_0^{\sqrt{\pi}} = -\frac{\pi}{2}(\cos(\pi)-\cos(0)) = \pi.$$