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Duality of spectral curves arising in two–matrix models[1]

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Abstract

The two matrix model is considered with measure given by the exponential of a sum of polynomials in two different variables. It is shown how to derive a sequence of pairs of "dual" finite size systems of ODEs for the corresponding biorthonormal polynomials. An inverse theorem is proved showing how to reconstruct such measures from pairs of semi-infinite finite band matrices defining the recursion relations and satisfying the string equation. A proof is given in the $N \to \infty$ limit that the dual systems obtained share the same spectral curve.

1 Introduction

We consider the two–matrix model [5, 7, 8, 9, 6], which involves an ensemble consisting of pairs of $N \times N$ hermitian matrices M_1 and M_2, with a $U(N)$ invariant probability measure of the form:

$$\frac{1}{\tau_N} d\mu(M_1, M_2) := \frac{1}{\tau_N} \exp K \mathrm{tr}\, (-V_1(M_1) - V_2(M_2) + M_1 M_2) dM_1 dM_2 \ . \tag{1-1}$$

Here $dM_1 dM_2$ is the standard Lebesgue measure for pairs of Hermitian matrices and V_1 and V_2 are chosen to be polynomials of degrees $d_1 + 1$, $d_2 + 1$ respectively, and are referred to as the called the potentials. The overall positive scaling factor K in the exponential is taken as having order N when considering the large N limit. We also assume that both potentials are real and bounded from below (for reasons of convergence) .

The normalization factor (partition function)

$$\tau_N = \int_{M_1} \int_{M_2} d\mu \tag{1-2}$$

is known to be a KP τ-function in each set of deformation parameters (the coefficients of the two polynomials V_1, V_2), as well as providing solutions to the two-Toda equations [11, 1, 2]. The key objects of the theory are the correlation functions for the eigenvalues of the two matrices. Analogously to the one–matrix models, such correlation functions can be recovered by means of certain Fredholm integral kernels. We recall here briefly that in one-matrix models with measure

$$\frac{1}{\tau_N} d\mu(M) := \frac{1}{\tau_N} \exp \mathrm{tr}\, (-V(M)) dM \tag{1-3}$$

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the computation of such a kernel is reduced to the construction of orthonormal polynomials $P_n(x)$ for the space $L^2\left(\mathbb{R}, \mathrm{e}^{-V(x)}\mathrm{d}x\right)$. In terms of these polynomials, the kernel is given by

$$\overset{N}{K}(x,x') = \sum_{n=0}^{N-1} P_n(x)\mathrm{e}^{-\frac{1}{2}V(x)}P_n(x')\mathrm{e}^{-\frac{1}{2}V(x')} \ . \tag{1-4}$$

In 2-matrix models there are four relevant kernels needed to compute the statistical correlations of eigenvalues. For m–matrix models there are m^2 such kernels.

These kernels are expressible in terms of suitably defined sequences of *biorthogonal* polynomials. By this we mean two sequences of monic polynomials

$$\pi_n(x) = x^n + \cdots, \qquad \sigma_n(y) = y^n + \cdots, \qquad n = 0, 1, \ldots \tag{1-5}$$

which are orthogonal with respect to a coupled measure on the product space:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}y \ \pi_n(x)\sigma_m(y)\mathrm{e}^{-KV_1(x)-KV_2(y)+Kxy} = h_n\delta_{mn}, \tag{1-6}$$

where $V_1(x)$ and $V_2(y)$ are the polynomials appearing in the two-matrix model measure (1-1). The orthogonality relations determine the two families uniquely, if they exist[6]. The four relevant kernels are expressed as follows in terms of these biorthogonal polynomials

$$\overset{N}{K}_{12}(x,y) = \sum_{n=0}^{N-1}\frac{1}{h_n}\pi_n(x)\sigma_n(y)\mathrm{e}^{-KV_1(x)}\mathrm{e}^{-KV_2(y)} \ , \qquad \overset{N}{K}_{11}(x,x') = \int_{\mathbb{R}}\mathrm{d}y \ \overset{N}{K}_{12}(x,y)\,\mathrm{e}^{Kx'y}, \tag{1-7}$$

$$\overset{N}{K}_{22}(y',y) = \int_{\mathbb{R}}\mathrm{d}x \ \overset{N}{K}_{12}(x,y)\,\mathrm{e}^{Kxy'} \ , \qquad \overset{N}{K}_{21}(y',x') = \int_{\mathbb{R}}\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{d}y\overset{N}{K}_{12}(x,y)\,\mathrm{e}^{Kxy'}\mathrm{e}^{Kx'y} \ . \tag{1-8}$$

All the statistical properties of the spectra of the 2-matrix ensemble may then be expressed in terms of these kernels [9] and the corresponding Fredholm integral operators $\overset{N}{\mathbf{K}}_{ij}, \ i,j = 1,2$. For instance the density of eigenvalues of the first matrix is:

$$\overset{N}{\rho}_1(x) = \frac{1}{N}\overset{N}{K}_{11}(x,x) \ , \tag{1-9}$$

the correlation function of two eigenvalues of the first matrix is:

$$\overset{N}{\rho}_{11}(x,x') = \frac{1}{N^2}\left(\overset{N}{K}_{11}(x,x)\overset{N}{K}_{11}(x',x') - \overset{N}{K}_{11}(x,x')\overset{N}{K}_{11}(x',x)\right) \ , \tag{1-10}$$

and the correlation function of two eigenvalues, one of the first matrix and one of the second is:

$$\overset{N}{\rho}_{12}(x,y) = \frac{1}{N^2}\left(\overset{N}{K}_{11}(x,x)\overset{N}{K}_{22}(y,y) - \overset{N}{K}_{12}(x,y)(\overset{N}{K}_{21}(y,x) - e^{Kxy})\right) \ . \tag{1-11}$$

Any other correlation function of m eigenvalues can similarly be written as a determinant involving these four kernels.

The main objective of this paper is to derive and analyze certain differential systems of ODE's satisfied by the quasipolynomials $\psi_n(x) := \pi_n(x)\mathrm{e}^{-V_1(x)}$, $\phi_n(y) := \sigma_n(y)\mathrm{e}^{-V_2(y)}$ and their Fourier Laplace transforms. In section 2, we summarize the principal results for finite N. The details and proofs may be found in [3] and [4]. In section 3 we derive the corresponding results in the $N \to \infty$ limit in a simple way.

The proof of Prop. 2.1 and the non–abelian version of the transversality argument in Section 3 is based on joint work with J. Hurtubise, details of which will appear in [4].

2 Folding and systems of ODE in duality

Consider the normalized quasi-polynomials

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\pi_n(x)\mathrm{e}^{-KV_1(x)} \ , \qquad \phi_n(y) = \frac{1}{\sqrt{h_n}}\sigma_n(y)\mathrm{e}^{-KV_2(y)} \ , n = 0, \dots \infty \ . \tag{2-12}$$

Viewing these as components of a pair of semi–infinite column vectors

$$\underset{\infty}{\Psi} = (\psi_0, \psi_1, \dots \psi_n, \dots)^t \quad \text{and} \quad \underset{\infty}{\Phi} = (\phi_0, \phi_1, \dots \phi_n, \dots)^t \ , \tag{2-13}$$

we obtain a pair of semi-infinite matrices Q and P that implement multiplication of $\underset{\infty}{\Psi}$ by x and derivation $-\frac{1}{K}\frac{d}{dx}$, respectively. Equivalently, we obtain the transposes Q^t and P^t by applying $-\frac{1}{K}\frac{d}{dy}$ or multiplication by $-y$ to $\underset{\infty}{\Phi}$. By construction, these satisfy the Heisenberg commutation relations (or "string equation")

$$[P, Q] = -\frac{1}{K}\mathbf{1} \ . \tag{2-14}$$

Along with these quasipolynomials we need their Fourier-Laplace transforms and the corresponding semi-infinite (row)-vectors with components

$$\underline{\psi}_n(y) := \int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{Kxy}\psi_n(x) \ , \quad \underline{\phi}_n(x) := \int_{\mathbb{R}} \mathrm{d}y \, \mathrm{e}^{Kxy}\phi_n(y) \tag{2-15}$$

$$\underset{\infty}{\underline{\Psi}}(y) := (\underline{\psi}_0, ..., \underline{\psi}_n, ...) \ ; \quad \underset{\infty}{\underline{\Phi}}(x) := (\underline{\phi}_0, ..., \underline{\phi}_n, ...) \ . \tag{2-16}$$

The multiplicative and derivative recursion relations for these sequences can be shown (by integration by parts) to be

$$x \underset{\infty}{\underline{\Phi}}(x) = \underset{\infty}{\underline{\Phi}}(x)Q \ ; \ \frac{1}{K}\frac{d}{dx}\underset{\infty}{\underline{\Phi}}(x) = \underset{\infty}{\underline{\Phi}}(x)P \tag{2-17}$$

$$y \underset{\infty}{\underline{\Psi}}(y) = \underset{\infty}{\underline{\Psi}}(y)Q^t \ ; \ \frac{1}{K}\frac{d}{dy}\underset{\infty}{\underline{\Psi}}(y) = \underset{\infty}{\underline{\Psi}}(y)P^t \ . \tag{2-18}$$

It also follows [3] from integration by parts that the two matrices P and Q have a finite band structure

$$Q := \begin{bmatrix} \alpha_0(0) & \gamma(0) & 0 & 0 & \cdots \\ \alpha_1(1) & \alpha_0(1) & \gamma(1) & 0 & \cdots \\ \vdots \ \ddots & \ddots & \ddots & \ddots & \\ \alpha_{d_2}(d_2) & \cdots & \alpha_0(d_2) & \gamma(d_2) & \\ 0 \ \ddots & \ddots & \ddots & \ddots & \end{bmatrix} \tag{2-19}$$

$$P := \begin{bmatrix} \beta_0(0) & \beta_1(1) & \cdots & \beta_{d_1}(d_1) & \cdots \\ \gamma(0) & \beta_0(1) & \beta_1(2) & \ddots & \beta_{d_1}(d_1{+}1) \\ 0 & \gamma(1) & \beta_0(2) & \ddots & \ddots \\ 0 & 0 & \gamma(2) & \beta_0(3) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2-20}$$

where $\gamma(n) \neq 0$ for all $n \in \mathbb{N}$. This structure essentially follows from the fact that the two matrices

$$(P - V_1'(Q)), \quad (Q - V_2'(P)) \tag{2-21}$$

are strictly lower and upper triangular respectively. Indeed, in the basis of quasipolynomials it is obvious that

$$\sum_{m=0}^{\infty}(P - V_1'(Q))_{nm}\psi_m(x) = \left(-\frac{1}{K}\frac{d}{dx} - V_1'(x)\right)\psi_n(x) = c\psi_{n-1}(x) + \text{lower } n\text{'s quasipolynomials.} \tag{2-22}$$

and that Q and P^t, representing the multiplication by x and y respectively, can have no more than one diagonal above the main diagonal. The converse is also true as will be detailed below.

Proposition 2.1 Suppose that P and Q have the above band structure and that the highest diagonal of Q and the lowest of P have nonzero entries. Then the two following conditions are equivalent

- (i) The commutator $[P, Q]$ is diagonal.
- (ii) There exist two polynomials of degrees d_1 and d_2 respectively which we denote by $V_1'(x)$ and $V_2'(y)$ such that

$$(P - V_1'(Q))_{\geq 0} = 0\ ,\quad (Q - V_2'(P))_{\leq 0} = 0\ , \tag{2-23}$$

where the subscripts ${}_{\leq}0$ or ${}_{\geq}0$ denote the lower or upper part.

Proof. The detailed proof of this result may be found in [4]. Here we just note that, given the band structure of the two semi-infinite matrices P and Q, the polynomial $V_1'(x)$ may be uniquely determined from the relation

$$(P - V_1'(Q)) \cdot e_0 = 0\ ,\quad e_0 := (1, 0, 0, 0, ...)^t \tag{2-24}$$

and its existence rests upon the assumption that $\gamma(n) \neq 0$. A similar relation uniquely determines $V_2'(y)$. It may then be shown that all the relation contained in eq. (2-23) are satisfied by these polynomials. Conversely, if two polynomials V_1' and V_2' satisfying eq. (2-23) exist, then

$$[P, Q - V_2(P)] = [P, Q] = [P - V_1(Q), Q]\ . \tag{2-25}$$

But the LHS is upper triangular and the RHS is lower triangular (not strictly), so that $[P, Q]$ must be diagonal. Q.E.D.

The structure (2-19), (2-20) of the two matrices P and Q means that the four sequences $\psi_n, \underline{\psi}_n, \phi_n, \underline{\phi}_n$ satisfy both multiplicative and derivative recursion relations

$$x\psi_n = \gamma(n)\psi_{n+1} + \sum_{j=0}^{d_2} \alpha_j(n)\psi_{n-j},\quad -\frac{1}{K}\frac{d}{dx}\psi_n = \gamma(n-1)\psi_{n-1} + \sum_{j=0}^{d_1} \beta_j(n+j)\psi_{n+j}\ , \tag{2-26}$$

$$y\phi_n = \gamma(n)\phi_{n+1} + \sum_{j=0}^{d_1} \beta_j(n)\phi_{n-j},\quad -\frac{1}{K}\frac{d}{dx}\phi_n = \gamma(n-1)\phi_{n-1} + \sum_{j=0}^{d_2} \alpha_j(n+j)\phi_{n+j}\ . \tag{2-27}$$

From the finite recursion relations satisfied by the quasi-polynomials $\{\psi_n(x)\}$ and $\{\phi_n(y)\}$ follows a set of "generalized Christoffel–Darboux relations [12, 8], which imply that the kernels $\overset{N}{K}_{11}(x, x')$ and $\overset{N}{K}_{22}(y', y)$ may be expressed as:

$$\overset{N}{K}_{11}(x, x') = \frac{\left(\overset{N-1}{\underline{\Phi}}(x'), \overset{N}{\mathbb{A}} \underset{N}{\Psi}(x)\right)}{x' - x}\ ,\quad \overset{N}{\mathbb{A}} := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \hline \alpha_{d_2}(N) & \cdots & \alpha_2(N) & \alpha_1(N) & 0 \\ 0 & \alpha_{d_2}(N+1) & \cdots & \alpha_1(N+1) & 0 \\ 0 & 0 & \alpha_{d_2}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \alpha_{d_2}(N+d_2-1) & 0 \end{array}\right] \tag{2-28}$$

$$\overset{N}{K}_{22}(y', y) = \frac{\left(\overset{N-1}{\underline{\Psi}}(y'), \overset{N}{\mathbb{B}} \underset{N}{\Phi}(y)\right)}{y' - y}\ ,\quad \overset{N}{\mathbb{B}} := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -\gamma(N-1) \\ \hline \beta_{d_1}(N) & \cdots & \beta_2(N) & \beta_1(N) & 0 \\ 0 & \beta_{d_1}(N+1) & \cdots & \beta_1(N+1) & 0 \\ 0 & 0 & \beta_{d_1}(N+2) & \cdots & 0 \\ 0 & 0 & 0 & \beta_{d_1}(N+d_1-1) & 0 \end{array}\right] \tag{2-29}$$

where $\underset{N}{\Psi}(x)$, $\underset{N}{\Phi}(y)$, $\overset{N-1}{\underline{\Psi}}(y)$ and $\overset{N-1}{\underline{\Phi}}(x)$ are the column or row vectors of dimension $(d_1 + 1)$ and $(d_2 + 1)$ defined by

$$\underset{N}{\Psi}(x) = [\psi_{N-d_2}, \ldots, \psi_N]^t\ ,\quad \underset{N}{\Phi}(y) = [\phi_{N-d_2}, \ldots, \phi_N]^t, \tag{2-30}$$

$$\overset{N-1}{\underline{\Psi}}(y) = [\underline{\psi}_{N-1}, \ldots, \underline{\psi}_{N+d_2-1}]\ ,\quad \overset{N-1}{\underline{\Phi}}(x) = [\underline{\phi}_{N-1}, \ldots, \underline{\phi}_{N+d_1-1}]. \tag{2-31}$$

The matrices $\overset{N}{\mathbb{A}}, \overset{N}{\mathbb{B}}$ entering eqs. 2-29 define two pairings (which we will refer to as the Christoffel-Darboux pairings) between $\underset{N}{\Psi}$ and $\overset{N-1}{\underline{\Phi}}$ and between $\underset{N}{\Phi}$ and $\overset{N-1}{\underline{\Psi}}$. We call these pairs *dual windows.*

The key observation is that any quasipolynomial $\psi_j(x)$ can be uniquely expressed, for any given $N \geq d_2$, in terms of linear combinations of any d_2+1 consecutive basis elements $\psi_{N-d_2}, .., \psi_N$ *with polynomial coefficients.* We call this procedure **folding** of the space onto the *window* spanned by $\Psi_N = [\psi_{N-d_2}, .., \psi_N]^t$. This is accomplished by means of the x-recursion relations for the quasipolynomials in eq. (2-27), which allow us to express the $(N+1)$st quasipolynomial in terms of the d_2+1 preceding ones, but with coefficients that are polynomials in x. Iteration of this procedure defines the folding.

In matricial form the above can be expressed as follows

$$\underset{N}{\mathbf{a}}(x)\,\underset{N}{\Psi}(x) \quad := \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N)}{\gamma(N)} & \cdots & \frac{-\alpha_1(N)}{\gamma(N)} & \frac{(x-\alpha_0(N))}{\gamma(N)} \end{array}\right] \left[\begin{array}{c} \psi_{N-d_2} \\ \vdots \\ \vdots \\ \psi_N \end{array}\right] = \left[\begin{array}{c} \psi_{N-d_2+1} \\ \vdots \\ \vdots \\ \psi_{N+1} \end{array}\right] = \underset{N+1}{\Psi}(x)\ , \quad N \geq d_1\ . \tag{2-32}$$

The matrix $\underset{N}{\mathbf{a}}$ is invertible, since its determinant equals $\alpha_{d_2}(n)/\gamma(n)$ and $\alpha_{d_2}(n)$ can be shown not to vanish as a consequence of the relation $(Q - V_2'(P))_{\leq 0} = 0$. It will be referred to in the following as a "ladder" matrix. A completely analogous relation can also be shown for the quasipolynomials $\phi_n(y)$ (see eq. (2-41) below) and for the respective Fourier-Laplace transforms.

By means of this folding, one can also express the action of any operator of finite band size as a matrix polynomial in x of size d_2+1. The most relevant case is the folding of the operator $P = -\frac{1}{K}\frac{d}{dx}$. Introducing the notation $\Psi_N := [\psi_{N-d_2}, ..., \psi_N]^t$ we have

$$-\frac{1}{K}\frac{d}{dx}\underset{N}{\Psi} = P\,\underset{N}{\Psi} = \overset{N}{D}_1(x)\,\underset{N}{\Psi} := \left(\overset{N}{\gamma}\,(\underset{N-1}{\mathbf{a}}(x))^{-1} + \overset{N}{\beta}_0 + \sum_{j=1}^{d_1} \overset{N}{\beta}_j\,\underset{N+j-1}{\mathbf{a}}(x)\,\underset{N+j-2}{\mathbf{a}}(x)\cdots\underset{N}{\mathbf{a}}(x)\right)\underset{N}{\Psi}, \tag{2-33}$$

where

$$\overset{N}{\beta}_j := \mathrm{diag}\,[\beta_j(N+j-d_2), \beta_j(N+j-d_2+1), \ldots, \beta_j(N+j)]\ ,\ j = 0, \ldots d_1 \tag{2-34}$$

$$\overset{N}{\gamma} := \mathrm{diag}\,[\gamma(N-1-d_2), \gamma(N-d_2), \ldots, \gamma(N+d_2-1)]\ . \tag{2-35}$$

The corresponding statement for the ϕ_n's is obtained by interchanging x and y, ψ_n and ϕ_n, d_1 and d_2, α_j and β_j etc. One obtains a similarly defined matrix $D_2(y)$ representing the action of the derivative on the quasipolynomials ϕ_n's. With the notations

$$\underset{N}{\Phi} := [\phi_{N-d_1}, ..., \phi_N]^t\ , \tag{2-36}$$

$$\overset{N}{\alpha}_j := \mathrm{diag}\,[\alpha_j(N+j-d_1), \alpha_j(N+j-d_1+1), \ldots, \alpha_j(N+j)]\ ,\ j = 0, \ldots d_2\ , \tag{2-37}$$

$$\overset{N}{\gamma} := \mathrm{diag}\,[\gamma(N-d_1-1), \ldots, \gamma(N-1)] \tag{2-38}$$

$$\underset{N}{\mathbf{b}}(y) := \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N)}{\gamma(N)} & \cdots & \frac{-\beta_1(N)}{\gamma(N)} & \frac{(y-\beta_0(N))}{\gamma(N)} \end{array}\right]\ , \quad N \geq d_1 \tag{2-39}$$

one finds

$$\underset{N}{\mathbf{b}}(y)\,\underset{N}{\Phi}(y) = \underset{N+1}{\Phi}(y) \tag{2-40}$$

$$-\frac{1}{K}\frac{d}{dy}\underset{N}{\Phi} = Q^t\,\underset{N}{\Phi} = \overset{N}{D}_2(y)\,\underset{N}{\Phi} := \left(\overset{N}{\gamma}\,(\underset{N-1}{\mathbf{b}})^{-1}(y) + \overset{N}{\alpha}_0 + \sum_{j=1}^{d_2} \overset{N}{\alpha}_j\,\underset{N+j-1}{\mathbf{b}}(y)\,\underset{N+j-2}{\mathbf{b}}(y)\cdots\underset{N}{\mathbf{b}}(y)\right)\underset{N}{\Phi}. \tag{2-41}$$

We can repeat a similar procedure for the respective Fourier-Laplace transforms. The relevant definitions and relations are given by the following formulæ

$$\overset{N}{\underline{\mathbf{a}}}(x) := \begin{bmatrix} \frac{x-\alpha_0(N)}{\gamma(N-1)} & 1 & 0 & 0 \\ \frac{-\alpha_1(N+1)}{\gamma(N-1)} & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ \frac{-\alpha_{d_2}(N+d_2)}{\gamma(N-1)} & 0 & 0 & 0 \end{bmatrix} \in gl_{d_2+1}[x] \ ; \ \overset{N}{\underline{\mathbf{b}}}(y) := \begin{bmatrix} \frac{y-\beta_0(N)}{\gamma(N-1)} & 1 & 0 & 0 \\ \frac{-\beta_1(N+1)}{\gamma(N-1)} & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ \frac{-\beta_{d_1}(N+d_1)}{\gamma(N-1)} & 0 & 0 & 0 \end{bmatrix} \in gl_{d_1+1}[y] \ , \tag{2-42}$$

$$\overset{N-1}{\underline{\Psi}} = \overset{N}{\underline{\Psi}}\,\overset{N}{\underline{\mathbf{a}}}(x) \ , \qquad \overset{N-1}{\underline{\Phi}} = \overset{N}{\underline{\Phi}}\,\overset{N}{\underline{\mathbf{b}}}(y) \ , \tag{2-43}$$

$$\frac{1}{K}\frac{d}{dy}\overset{N-1}{\underline{\Psi}}(y) = \overset{N-1}{\underline{\Psi}}(y)\overset{N}{\underline{D}}_2(y) \ , \quad N \ge d_1+1 \ , \tag{2-44}$$

$$\frac{1}{K}\frac{d}{dx}\overset{N-1}{\underline{\Phi}}(x) = \overset{N-1}{\underline{\Phi}}(x)\overset{N}{\underline{D}}_1(x) \ , \quad N \ge d_2+1 \ , \tag{2-45}$$

where

$$\overset{N}{\underline{D}}_2(y) := (\overset{N}{\underline{\mathbf{b}}})^{-1}\,\overset{N-1}{\underline{\gamma}} + \overset{N-1}{\underline{\alpha}}_0 + \sum_{j=1}^{d_2} \overset{N-1}{\underline{\mathbf{b}}}\,\overset{N-2}{\underline{\mathbf{b}}}\cdots\overset{N-j}{\underline{\mathbf{b}}}\,\overset{N-1}{\underline{\alpha}}_j \ , \tag{2-46}$$

$$\overset{N}{\underline{D}}_1(x) := (\overset{N}{\underline{\mathbf{a}}})^{-1}\,\overset{N-1}{\underline{\gamma}} + \overset{N-1}{\underline{\beta}}_0 + \sum_{j=1}^{d_1} \overset{N-1}{\underline{\mathbf{a}}}\,\overset{N-2}{\underline{\mathbf{a}}}\cdots\overset{N-j}{\underline{\mathbf{a}}}\,\overset{N-1}{\underline{\beta}}_j \tag{2-47}$$

$$\overset{N-1}{\underline{\alpha}}_j := \mathrm{diag}(\alpha_j(N-1),\dots,\alpha_j(N+d_1-1)) \ , \tag{2-48}$$

$$\overset{N-1}{\underline{\beta}}_j := \mathrm{diag}(\beta_j(N-1),\dots,\beta_j(N+d_2-1)) \tag{2-49}$$

Summarizing, we have four sequences of linear differential systems

$$\begin{array}{l|l} \text{Size } (d_2+1)\times(d_2+1) & \text{Size } (d_1+1)\times(d_1+1) \\ \hline -\dfrac{1}{K}\dfrac{d}{dx}\underset{N}{\Psi}(x) = \overset{N}{D}_1(x)\,\underset{N}{\Psi}(x) & \dfrac{1}{K}\dfrac{d}{dy}\overset{N-1}{\underline{\Psi}}(y) = \overset{N-1}{\underline{\Psi}}(y)\,\overset{N}{\underline{D}}_2(y) \\ \dfrac{1}{K}\dfrac{d}{dx}\overset{N-1}{\underline{\Phi}}(x) = \overset{N-1}{\underline{\Phi}}(x)\,\overset{N}{\underline{D}}_1(x) & -\dfrac{1}{K}\dfrac{d}{dy}\underset{N}{\Phi}(y) = \overset{N}{D}_2(y)\,\underset{N}{\Phi}(y) \end{array} \tag{2-50}$$

as well as the ladder relations (2-32), (2-40), (2-42), (2-43). We have not considered here the deformation equations, i.e. the differential equations obtained from infinitesimal variations of the coefficients of the potentials V_1 and V_2 entering the measure. The complete study of these deformations is carried out in [3]. In particular it is shown there that the resulting overdetermined system of PDEs is compatible. Here we will only recall that the mixed system of ODEs and difference equations is also compatible, as implied by the following:

Proposition 2.2 The ladder matrices $\underset{N}{\mathbf{a}}$ intertwine the differential systems D_1 with different N's, i.e.

$$\underset{N}{\mathbf{a}}(x)\left(\frac{d}{dx} + \overset{N}{D}_1(x)\right) = \left(\frac{d}{dx} + \overset{N+1}{D_1}(x)\right)\underset{N}{\mathbf{a}}(x) \tag{2-51}$$

Similar statements hold for the other three sequences of ODEs and ladder relations.

The next proposition explains how the four sequences of systems in the Table are related amongst themselves by means of the Christoffel–Darboux pairings.

Proposition 2.3 The following relations are satisfied

$$\overset{N}{\underline{D}_1}(x)\,\overset{N}{\mathbb{A}} = \overset{N}{\mathbb{A}}\,\overset{N}{D_1}(x)\ ;\quad \overset{N}{\underline{D}_2}(y)\,\overset{N}{\mathbb{B}} = \overset{N}{\mathbb{B}}\,\overset{N}{D_2}(y) \tag{2-52}$$

The spectra of the two matrices $D_1(x)$ and $\underline{D}_1(x)$ (i.e. their characteristic polynomials) coincide, as do the spectra of $D_2(y)$ and $\underline{D}_2(y)$. A less apparent spectral duality also holds. Indeed it is proven in [3] that

$$\det\left(y\mathbf{1} - \overset{N}{D_1}(x)\right) = c\det\left(x\mathbf{1} - \overset{N}{D_2}(y)\right)\ , \tag{2-53}$$

where c is the ratio of the leading coefficients of the two potentials V_1 and V_2. Notice that the two determinants involve square matrices of rank d_2+1 on the LHS and of rank d_1+1 on the RHS. In the following section we give a simple derivation of a "naïve" $N\to\infty$ limit of these results; namely one in which we treat the relevant recursion matrices as commuting.

3 The Abelian case

In this section we derive the spectral duality property in a particular limit $N\to\infty$, $K/N=\mathcal{O}(1)$. In such a limit the two matrices P and Q, while retaining their finite band structure, may be taken to commute because $[P,Q]=-\mathbf{1}/K\to 0$. In addition, we consider only the case in which the coefficients $\alpha_j(n)$, $\beta_j(n)$, $g(n)$ do not depend on n: this is a stronger requirement which occurs actually only for certain ranges of the coupling constants. This limit is studied in the literature and is referred to as the "one-cut case" or the "genus 0" case [7, 5].

A further simplification that is purely technical is obtained by considering the matrices as doubly-infinite, i.e. of size $\mathbb{Z}\times\mathbb{Z}$ instead of $\mathbb{N}\times\mathbb{N}$. We will show that the statement of spectral duality in this case reduces to a classical result in commutative algebra, namely the computation of the resultant of two Laurent polynomials.

The non-abelian case (i.e. for finite N) is detailed in [3] and the approach used there may be used to derive the result for the $N\to\infty$ case. However, we will present a proof here of a different nature, which can also be extended to the non-abelian case [4].

The equations $[P,Q]=-\frac{1}{K}\mathbf{1}$ in the limit $N\to\infty$, $K=\mathcal{O}(N)$, become commutativity equations $[P,Q]=0$. Moreover, since we are considering finite band matrices and we focus on the window at N, we can replace the semi–infinite matrices P,Q by doubly infinite matrices with the same band structure. For suitable scaling regimes it can be argued on physical grounds that the sequences $\gamma(n)$, $\alpha_j(n)$, $\beta_k(n)$ actually do not depend on n (provided $n=\mathcal{O}(N)$). It is precisely this very simple case that we want to address here.

The pair of commuting matrices P and Q with the same band structure as before now just become polynomials in the shift matrix. All the matrices are taken to be $\mathbb{Z}\times\mathbb{Z}$ matrices and hence $\Lambda=[\delta_{i,i+1}]$ is actually invertible, the inverse being just the transpose Λ^t. With this in mind we can write

$$Q(\Lambda) := \gamma\Lambda + \alpha_0 + \sum_{i=1}^{d_2}\alpha_i\Lambda^{-i}\ ,\quad \gamma\neq 0\neq\alpha_{d_2} \tag{3-54}$$

$$P(\Lambda) := \gamma\Lambda^{-1} + \beta_0 + \sum_{i=1}^{d_1}\beta_i\Lambda^{i}\ ,\quad \gamma\neq 0\neq\beta_{d_1}\ , \tag{3-55}$$

with Q and P are viewed as Laurent polynomials in Λ,Λ^{-1}. It is convenient to introduce an indeterminate λ and represent Q and P as acting on the graded space

$$Q,P:\mathbb{C}[\lambda,\lambda^{-1}]\to\mathbb{C}[\lambda,\lambda^{-1}]\ , \tag{3-56}$$

determined by substituting Λ by λ in the relations (3-54,3-55). The shift matrix Λ is just multiplication by λ while $\Lambda^t=\Lambda^{-1}$ represents multiplication by λ^{-1}. The equivalent of a window is then the linear span of d_2+1 consecutive powers of λ

$$\mathbb{C}\{\psi_{N-d_2},\dots,\psi_N\}\leftrightarrow\mathbb{C}\{\lambda^{N-d_2},\dots,\lambda^N\}\ . \tag{3-57}$$

The folding of the previous sections here reduces to a very simple expression. Indeed, folding the graded space $W := \mathbb{C}[[\lambda]]$ onto the span of $\lambda^{N-d_2}, ..., \lambda^N$ simply means taking the quotient

$$\mathbb{C}[[\lambda]] \simeq \mathbb{C}[x] \otimes \mathbb{C}[[\lambda]] \bmod \langle x - Q(\lambda) = 0\rangle \ \simeq \mathbb{C}[x]\{\lambda^{N-d_2}, ..., \lambda^N\} \ . \tag{3-58}$$

In other words, the power λ^{N+1} can be re-expressed in terms of the powers $\lambda^{N-d_2}, ..., \lambda^N$ using the relation $x - Q(\lambda) = 0$. The equivalent of the ladder matrix is just the expression of multiplication by λ in the "folded" window $\mathbb{C}[x]\{\lambda^{N-d_2}, ..., \lambda^N\}$. It is defined so as to make the following diagram commutative

$$\begin{array}{ccc} \mathbb{C}[[\lambda]] & \xrightarrow{\ \lambda\ } & \mathbb{C}[[\lambda]] \\ {\scriptstyle \langle x - Q(\lambda) = 0\rangle}\downarrow & & \downarrow{\scriptstyle \langle x - Q(\lambda) = 0\rangle} \\ \mathbb{C}[x]\{\lambda^{N-d_2}, ..., \lambda^N\} & \xrightarrow{\mathbf{a}(x)} & \mathbb{C}[x]\{\lambda^{N-d_2}, ..., \lambda^N\} \end{array} \tag{3-59}$$

In principle, $\mathbf{a}$ could depend on N, but it is easy to see that in fact it is represented by the following N-independent companion-like matrix

$$\mathbf{a}(x) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{\alpha_{d_2}}{\gamma} & \cdots & -\frac{\alpha_1}{\gamma} & \frac{x-\alpha_0}{\gamma} \end{bmatrix} \tag{3-60}$$

Similarly, we could define another folding along P by means of the following diagram

$$\begin{array}{ccc} \mathbb{C}[[\lambda]] & \xrightarrow{\ \lambda\ } & \mathbb{C}[[\lambda]] \\ {\scriptstyle \langle y - P(\lambda) = 0\rangle}\downarrow & & \downarrow{\scriptstyle \langle y - P(\lambda) = 0\rangle} \\ \mathbb{C}[y]\{\lambda^{N-1}, ..., \lambda^{N+d_1-1}\} & \xleftarrow{\mathbf{b}(y)} & \mathbb{C}[y]\{\lambda^{N-1}, ..., \lambda^{N+d_1-1}\} \end{array} \tag{3-61}$$

where $\mathbf{b}$ is given by

$$\mathbf{b}(y) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{\beta_{d_1}}{\gamma} & \cdots & -\frac{\beta_1}{\gamma} & \frac{y-\beta_0}{\gamma} \end{bmatrix} \ . \tag{3-62}$$

In this framework the matrices $D_1(x)$ and $D_2(y)$ are simply

$$D_1(x) := P(\mathbf{a}(x)) = \gamma\, \mathbf{a}(x)^{-1} + \sum_{j=0}^{d_1} \beta_j \overset{j}{\mathbf{a}}(x) \tag{3-63}$$

$$D_2(y) := Q(\mathbf{b}(y)) = \gamma\, \mathbf{b}(y)^{-1} + \sum_{j=0}^{d_2} \alpha_j \overset{j}{\mathbf{b}}(y) \ . \tag{3-64}$$

The previous statement about spectral duality now translates into the identity

$$\det(y\mathbf{1} - D_1(x)) \propto \det(x\mathbf{1} - D_2(y)) \ . \tag{3-65}$$

We will show that both determinants are in fact the resultants (w.r.t. λ) of the two Laurent polynomials $Q(\lambda) - x$ and $P(\lambda) - y$. The proof is actually quite standard for polynomials and here we just adapt it to the situation with Laurent polynomials (see e.g. [10]). This amounts to studying the following embedding

$$\mathbb{C}\{\lambda^{N-d_2}, ..., \lambda^N\} \oplus \mathbb{C}\{\lambda^{N-1}, ..., \lambda^{N+d_1-1}\} \xrightarrow{(P(\lambda)-y)\oplus(Q(\lambda)-x)} \mathbb{C}\{\lambda^{N-d_2-1}, ..., \lambda^{N+d_1}\} \ , \tag{3-66}$$

where x and y are treated as parameters of the embedding. Let us denote by $W, \underline{W}, U$ the three vector spaces

$$W := \mathbb{C}\{\lambda^{N-d_2}, ..., \lambda^N\} \ ; \ \underline{W} := \mathbb{C}\{\lambda^{N-1}, ..., \lambda^{N+d_1-1}\} \ ; \ \ U := \mathbb{C}\{\lambda^{N-d_2-1}, ..., \lambda^{N+d_1}\} \ . \tag{3-67}$$

The above embedding may be combined into a single map

$$W \oplus \underline{W} \xrightarrow{(P(\lambda)-y)\oplus(Q(\lambda)-x)} U \tag{3-68}$$

The two parts of this map give spaces generically transverse as x and y vary. If they are not transverse for a given pair (x, y), this means that

$$\exists\, w \in W, \ \exists\, \underline{w} \in \underline{W} \text{ such that } \ w \neq 0 \neq \underline{w} \ , \ (P - y)w = (Q - x)\underline{w} \ \ \in U \ . \tag{3-69}$$

Taking the quotient of this relation by the relation $Q(\lambda) - x = 0$ or $P(\lambda) - y = 0$ gives rise to the relation

$$(D_1(x) - y)w = 0 \tag{3-70}$$
$$(D_2(y) - x)\underline{w} = 0 \ , \tag{3-71}$$

which means that y is an eigenvalue of $D_1(x)$ and x an eigenvalue of $D_2(y)$. Conversely if either of the two equations (3-70) (3-71) holds, say the first, for nonzero vector w, this means that there exists $\underline{w} \in \underline{W}$ such that

$$(P - y)w = (Q - x)\underline{w} \ . \tag{3-72}$$

Notice that $(Q - x)\underline{w}$ cannot be zero since the map $P - y : W \to U$ is injective for all y and so $(P - y)w \neq 0$. The same result follows if we start from eq. (3-71). This proves that the embedding is not transverse if and only if x is an eigenvalue of $D_2(y)$ which is equivalent to y being an eigenvalue of $D_1(x)$.

The condition of transversality amounts to the nonvanishing of the determinant of the embedding (in any fixed basis). It is easy to see that such an embedding is represented by the Sylvester matrix

$$\left[\begin{array}{ccccccccc}
\gamma & \beta_0 - y & \beta_1 & \cdots & \cdots & \beta_{d_1} & 0 & 0 & 0 \\
0 & \gamma & \beta_0 - y & \beta_1 & \cdots & \cdots & \beta_{d_1} & 0 & 0 \\
0 & 0 & \gamma & \beta_0 - y & \beta_1 & \cdots & \cdots & \beta_{d_1} & 0 \\
0 & 0 & 0 & \gamma & \beta_0 - y & \beta_1 & \cdots & \cdots & \beta_{d_1} \\
\hline
\alpha_{d_2} & \cdots & \alpha_1 & \alpha_0 - x & \gamma & 0 & 0 & 0 & 0 \\
0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0 - x & \gamma & 0 & 0 & 0 \\
0 & 0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0 - x & \gamma & 0 & 0 \\
0 & 0 & 0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0 - x & \gamma & 0 \\
0 & 0 & 0 & 0 & \alpha_{d_2} & \cdots & \alpha_1 & \alpha_0 - x & \gamma
\end{array}\right] \tag{3-73}$$

of the two Laurent polynomials, whose determinant $\Delta(x, y)$ equals the resultant. A simple counting of degrees and inspection of the highest powers in x or y shows that

$$\alpha_{d_2}\gamma^{d_1} \det(y\mathbf{1} - D_1(x)) = \Delta(x, y) = \beta_{d_1}\gamma^{d_2} \det(x\mathbf{1} - D_2(y)) \ . \tag{3-74}$$

which defines the spectral curves as the non-transversality locus of the embeddings. The intersection of the two embeddings on this spectral curve is (generically) one-dimensional and projects to the eigenvectors of $D_1(x)$ and $D_2(y)$.

While this is very simple, and just a reformulation of standard algebraic results in this abelian setting, a very similar approach can also be used to prove spectral duality for the pair ${D_1}^N(x)$ and $\underline{D_2}^N(y)$ in the finite N setting, in which the matrices P and Q do not commute. A refinement and elaboration on this theme also leads to the other results of [3] in a more elegant and compact form [4], such as the compatibility of the deformation equations in the coupling constants of the potentials V_1, V_2 which, in particular imply the invariance of the generalized monodromy of the operators $\partial_x + \overset{N}{D}_1(x)$ and $\partial_y + \overset{N}{D}_2(y)$. This defines a sort of "noncommutative resultant" for finite band matrices whose properties will be developed in a subsequent publication.

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