Catching all those vagrant roots

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Abstract The roots we are after are those of families of polynomials that satisfy "orthogonality" relations of some sort. For orthogonal polynomials w.r.t a positive measure on the real line the problem is at least one century old, but new results appear daily. Classical theory establishes that all the zeroes are real and simple and in the limit of large degrees they form some "nice" distribution. The reason of the interest is the relation of orthogonal polynomials to random-matrix theory, integrable systems, analysis, geometry.

If we replace the positive measure by a complex-valued analytic weight then the zeroes are no longer real, but they lie in the complex plane: yet, in the large-degree limit they arrange themselves in nice patterns (trees of Jordan arcs typically).

So the question arises: how do we find all those "flying zeroes"? Is there something to say about the geometry of these "trees"?

I will touch upon these issues, showing some nice pictures and animations, and also on some recent conjectures on the zeroes of orthogonal polynomials for area-integrals in the plane

Taylor polynomials

Where are the zeroes of the Taylor polynomials of e^z?

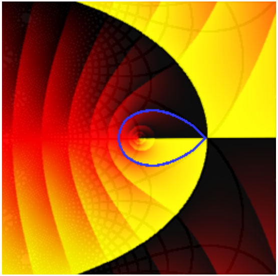
$$e^z \longrightarrow P_n(z) = \sum_{j=0}^{n} \frac{1}{j!} z^j$$

The exponential function has no zeroes! $P_n(z)$ has n zeroes!. They **fly away** : upon rescaling

$$p_n(z) := P_n(nz)$$

the zeroes of $p_n(z)$ become **dense on the curve**

$$\gamma := \{z : |ze^{1-z}| = 1 , \ |z| \le 1\} \quad \textbf{(Szegö, 1924)}$$



From Hans Lundmark's web-page:
http://www.mai.liu.se/ halun/complex/taylor/

Orthogonal polynomials

Given a positive measure $\mu(x)\mathrm{d}x$ on $\mathbb{R}$, O.P.s are a sequence of **monic** polynomials $p_n(x) = x^n + \dots$ such that

$$\int_{\mathbb{R}} p_n(x) p_m(x) \mu(x) \mathrm{d}x = h_n \delta_{nm}$$

They are **Padé approximants**

$$\int_{\mathbb{R}} \frac{\mu(s)\mathrm{d}s}{x-s} = \frac{q_n(x)}{p_n(x)} + \mathcal{O}(x^{-2n-1}) \ , \quad q_n(x) = \int_{\mathbb{R}} \frac{p_n(x) - p_n(s)}{x-s} \mu(s)\mathrm{d}s$$

Weyl function

• "Pade' approximants are to the Weyl functions what continued fractions are to real numbers"

They enter in an approximation scheme similarly to Taylor polynomials.

What are the interesting questions

- Where are the zeroes of OPs? More precisely, how to describe the (normalized) counting measure of zeroes

$$\nu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j^{(n)}} , \qquad p_n(x_j^{(n)}) = 0$$

- Can we describe "$\lim_{n\to\infty}$"$p_n(x)$? Yes: we need however advanced analysis like
  - Riemann–Hilbert problems;
  - algebraic geometry of hyperelliptic curves;
  - Θ–functions and half-differentials.

Known properties (Szegö)

- The zeroes of $p_n(x)$ are all **real**, **simple** and in the convex hull of $supp(\mu)$.
- The zeroes of $p_n(x)$ and $p_{n+1}(x)$ are **interlaced**.

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An example: Hermite polynomials

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2} H_n(x) H_m(x) = \delta_{mn} 2^n n! \sqrt{n}$$

then (Plancherel–Rotach) for $x = \sqrt{2n+1}\cos\phi$

$$\mathrm{e}^{-x^2/2} H_n = \frac{2^{n/2+\frac{1}{4}} (n!)^{\frac{1}{2}}}{(\pi n)^{\frac{1}{4}} (\sin\phi)^{\frac{1}{2}}} \left[\sin\left(\left(\frac{n}{2} + \frac{1}{4} \right) (\sin 2\phi - 2\phi) + \frac{3\pi}{4} \right) + \mathcal{O}(n^{-1}) \right]$$

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YUCKY!
But we can read off the (scaled) position of the zeroes and see that their distribution is approaching the **Wigner semicircle law**

$$\rho(\xi) = \frac{1}{\pi}\sqrt{2-\xi^2}\ , \qquad \xi = \frac{x}{\sqrt{n}}$$

The density of zeroes

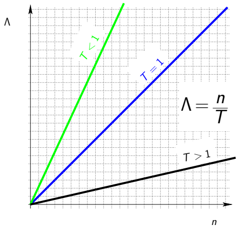
We scale the measure

(real) analytic

$$\mu_\Lambda(x)\mathrm{d}x = \mathrm{e}^{-\Lambda V(x)}\mathrm{d}x \ , \qquad \Lambda > 0$$

Then the OPs depend also on $\Lambda = \frac{n}{T}$ (i.e. $T = \frac{n}{\Lambda}$)

$$p_n(x;\Lambda) = \prod_{j=1}^{n}(x - x_j^{(n)}) \ \Rightarrow \ \nu_{n,\Lambda} = \frac{1}{n}\sum_{j=1}^{n}\delta_{x_j^{(n)}}$$



Theorem (a long list of people.... Pastur, Deift, Johansson,....)

The density $\nu_{n,\Lambda} \xrightarrow[\Lambda=\frac{n}{T},\ n\to\infty]{} \frac{1}{T}\rho(x)$ *where* ρ *is the equilibrium density of a distribution of charges on the real axis,* $\int \rho(x) = T$*, subject to the* external electrostatic potential $V(x)$.

Stepping out of reality: pseudo-OPs

Pseudo Orthogonal polynomials are monic polynomials $p_n(x) = x^n + \ldots$ such that

$$\int p_n(x) p_m(x) \mathrm{e}^{-\Lambda V(x)} \mathrm{d}x = h_n \delta_{nm}$$

where $V(x)$ is **COMPLEX!** (and also the contour of integration may be in the complex plane).

Q: where are the zeroes going now? (i.e. where are the poles of the Padé approximants)

Theorem

Suppose $V(x)$ is a complex polynomial. Then $\nu_{n,\Lambda} \to \rho$ where ρ is a positive measure of mass T supported on a **tree of Jordan arcs***. This tree grows as T grows.*

Finding the tree is a problem in **harmonic analysis** (harmonic function + BV conditions + free boundary) $\longrightarrow$ **Boutroux curves**

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Some nice animations

Example: $V(x) = -x^4 + x^2$ and T growing. Different evolutions correspond to different choices of contours of integration. The real axis is not an option even if V is real since $\mathrm{e}^{-x^4+x^2}$ is not integrable on the real axis.

Another growing tree ...

(Three hours on the four CPUs of Cuma....)

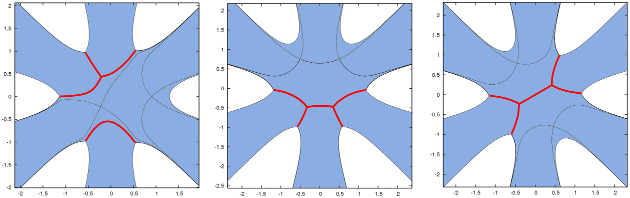
...and another ...

(Five hours on the four CPUs of Cuma....)

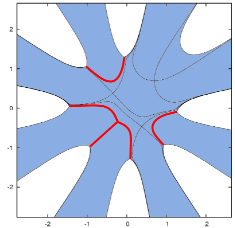
... and in full glory!

(A night on the four CPUs of Cuma....)

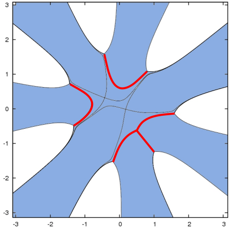
Other (static) examples of nice trees



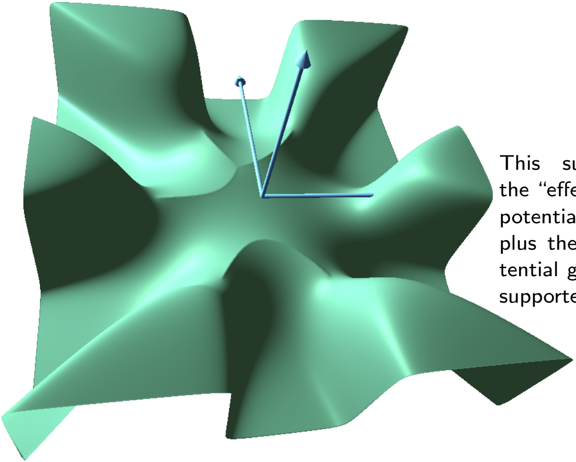
Three possible solution of cuts and hyperelliptic Boutroux curves for the potential $V(x) = \frac{x^6}{6}$ and total charge $T = 1$.



$V(x) = \frac{x^8}{4} + \frac{ix^6}{6} - \frac{x^2}{2} + 3x, \; T = 10.$



$V(x) = \frac{x^7}{7} + \frac{ix^5}{5} - \frac{x^3}{3} + \frac{x^2}{10}, \; T = 10.$



This surface represents the “effective electrostatic potential” of $V(x) = x^6$ plus the electrostatic potential generated by $\rho(x)$, supported on the creases.

Orthogonal Polynomias in the plane

A different situation here: consider **holomorphic** polynomials $p_n(z) = z^n + \dots$ such that

$$\iint_{\mathbb{C}} p_n(z)\overline{p_m(z)}\mathrm{e}^{-\Lambda\left(|z|^2+H(z)\right)}\mathrm{d}^2 z = h_n \delta_{nm}$$

Where are the zeroes now?
For example for the measure

$$\mathrm{e}^{-\Lambda|z|^2}|z-1|^{2\Lambda}\mathrm{d}^2 z$$

the zeroes of the OPs are in the animation on the right: $\Lambda = 40$ and $n = 1, 2, \dots, \Lambda$ ($T = \frac{n}{\Lambda}$).
There are some very natural (numerically supported) conjectures.

The region drawn around the zeroes (and the arc) are not fitted: they are analytical curves. The region describes (also) the (slow) growth of a bubble of air in a viscous fluid (in a 2D geometry) (**Laplacian growth**): the arc is the so-called **skeleton** (or **mother-body**) of the *generalized quadrature domain* (the smiling bean). There is a connection to two–dimensional fluid dinamics (zero-surface-tension, Darcyan). It is an open problem to prove what we plainly see....

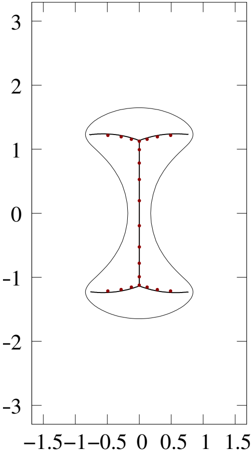
If we run time long enough a **transition** occurs: the domain closes on itself (the drawn domain is not accurate then). These transitions are interesting and can be studied in their own sake in the context of fluid-dynamics (see S. Y. Lee et al.)

Fido's bone's skeleton

For example for the measure

$$e^{-N|z|^2}|z-a|^{2N}|z+a|^{2N}\mathrm{d}^2 z$$

the
zeroes of the OPs are in the picture on the right: $N = 22 = n$ $(T = 1)$ and $a = 1.2588808\ldots$.



Conclusions

There are several connections to other areas of mathematics/physics I did not mention

- Padé approximations - Number Theory
- Orthogonal polynomials and Random matrix models
- Riemann–Hilbert analysis (nonlinear steepest descent method); a fantastically powerful method to obtain asymptotic expansions.
- Isomonodromic deformations and integrable systems.
- Algebraic geometry and harmonic analysis.
- String theory.
- Coherent states.
- Group theory and differential geometry
- ....

The punchline

It is fair to say that the topic has tentacular *ramifications*.