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Biorthogonal polynomials for 2-matrix models with semiclassical potentials

M. Bertola†‡[1][2]

† *Centre de recherches mathématiques, Université de Montréal*
C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7

‡ *Department of Mathematics and Statistics, Concordia University*
7141 Sherbrooke W., Montréal, Québec, Canada H4B 1R6

Abstract

We consider the biorthogonal polynomials associated to the two–matrix model where the potentials V_1, V_2 have arbitrary rational derivative and are constrained on an arbitrary union of intervals (hard-edges). We show that these polynomials satisfy certain recurrence relations with a number of terms d_i depending on the number of hard-edges and on the degree of the rational functions V_i'. Using these relations we derive Christoffel–Darboux identities satisfied by the biorthogonal polynomials: this enables us to give explicit formulæ for the differential equation satisfied by $d_i + 1$ consecutive polynomials, We also define certain integral transforms of the polynomials and use them to formulate a Riemann–Hilbert problem for $(d_i + 1) \times (d_i + 1)$ matrices constructed out of the polynomials and these transforms. Moreover we prove that the Christoffel–Darboux pairing can be interpreted as a pairing between two dual Riemann–Hilbert problems.

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[2]bertola@crm.umontreal.ca

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1 Introduction and setting

In this paper we consider the biorthogonal polynomials associated to the two–matrix model. The model is defined by a measure on the space of pairs of Hermitean matrices M_1, M_2 of the form

$$\mathrm{d}\mu(M_1, M_2) := \mathrm{d}M_1\mathrm{d}M_2\mathrm{e}^{-\mathrm{Tr}(V_1(M_1)-V_2(M_1)+M_1M_2)}\ . \tag{1-1}$$

Using Itzykson–Zuber/Harish-Chandra's formula, the model can be reduced to the study of biorthogonal polynomials [10] (BOPs for short), namely two sequences of polynomials $\{\pi_n(x)\}, \{\sigma_n(y)\}$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}x\mathrm{d}y \mathrm{e}^{-V_1(x)-V_2(y)+xy}\pi_n(x)\sigma_m(y) = \delta_{nm} \ . \tag{1-2}$$

For the model to have a probabilistic interpretation, the potentials should be real and satisfy certain growth conditions to ensure the convergence of the integrals. In order to introduce the setting of this paper we consider the following situation (which is strictly included in the more general setting to be expounded later)

1. There is a finite collection of disjoint intervals $I = \bigcup I_j \subseteq \mathbb{R}_x$ and $J = \bigcup_j J_j \subseteq \mathbb{R}_y$ ($\mathbb{R}_x$ denotes the real axis of the x-variable), in the complement of which the potentials are $+\infty$: in other words the matrices M_1, M_2 have spectrum confined to these multi-intervals, so that the associated BOPs satisfy

$$\int_I\int_J \mathrm{d}x\mathrm{d}y \mathrm{e}^{-V_1(x)-V_2(y)+xy}\pi_n(x)\sigma_m(y) = \delta_{nm} \tag{1-3}$$

2. The two potentials $V_1(x)$ and $V_2(y)$ are the restriction to I, J (respectively) of real-analytic functions with rational derivative (with poles symmetrically placed off the real axis, or on the complement of the intervals on the real axis) together with the necessary growth condition if the intervals are unbounded.

This situation has been addressed in [2] within the general context of bilinear moment functionals. Indeed it is convenient to recast the orthogonality condition in a more abstract setting where one considers a **bimoment functional** $\mathcal{L} : \mathbb{C}[x]\otimes\mathbb{C}[y] \to \mathbb{C}$ defined by

$$\mathcal{L}(x^i|y^j) := \int_I\int_J \mathrm{d}x\mathrm{d}y\, x^i y^j \mathrm{e}^{-V_1(x)-V_2(y)+xy} = \mu_{ij} \ . \tag{1-4}$$

and then extended by linearity to arbitrary polynomials. The biorthogonality condition then reads

$$\mathcal{L}(\pi_n|\sigma_m) = \delta_{nm} \ . \tag{1-5}$$

The properties of the potentials V_1, V_2 and the supports of integration can be dealt with on the same footing by purely algebraic methods: to this end one introduce four polynomials $A_i, B_i,\ i = 1, 2$ according to the strategy outlined hereafter. Let (x_j, m_j) be the location of the poles of $V_1'(x)$ with their order (we include all of the poles, in this case also the complex conjugates, which clearly come in with the same multiplicities) and let a_j be the endpoints of I. We define then A_1, B_1 (and similar expressions for A_2, B_2) as follows

$$B_1(x) = \prod(x-x_j)^{m_j}\prod(x-a_j) \ , \qquad A_1 := V_1'B_1 - B_1' \ , \tag{1-6}$$

so that now $V_i' = \frac{A_i + B_i'}{B_i}$. It is a straightforward exercise to verify (using integration by parts) that the bimoment functional satisfies the following distributional identities for arbitrary $p(x) \in \mathbb{C}[x], s(y) \in \mathbb{C}[y]$

$$\mathcal{L}\Big(- B_1(x)p'(x) + A_1(x)p(x) \Big| s(y)\Big) = \mathcal{L}\Big(B_1(x)p(x)\Big| ys(y)\Big) \ , \tag{1-7}$$

$$\mathcal{L}\Big(p(x)\Big| - B_2(y)s'(y) + A_2(y)s(y)\Big) = \mathcal{L}\Big(xp(x)\Big|B_2(y)s(y)\Big) \ . \tag{1-8}$$

Abstracting formulæ (1-7,1-8) from the specific context, we will say that a bimoment functional $\mathcal{L}$ is **semiclassical** if it satisfies those same relations (1-7,1-8) for some given (and fixed) polynomials A_i, B_i. The name comes from a similar usage in the context of ordinary orthogonal polynomials [9].

Such functionals have been studied in [2], where it was shown that

Proposition 1.1 *For given A_i, B_i, $i = 1, 2$, a semiclassical moment functional $\mathcal{L}$ is the linear combination of $s_1 s_2$ independent functionals $\mathcal{L}_{\nu,\mu}$, $\mu = 1\ldots, s_1$, $\nu = 1, \ldots, s_2$, where $s_i = \max(\deg A_i, \deg B_i + 1)$*

More importantly (at least in the case $\deg A_i \geqslant \deg B_i + 1$) all of these moment functionals $\mathcal{L}_{\mu,\nu}$ can be given an integral representation completely analog to (1-4), but without any restriction on the reality of the potentials or of the contours of integration: this is the setting of the present paper.

1.1 Connection to other orthogonal polynomials

The algebraic properties of semiclassical bilinear moment functionals apply to a slightly different class of orthogonal polynomials. Let us consider in fact orthogonal polynomials in the complex plane with respect to a measure of the form

$$\mathrm{d}\mu(z, \overline{z}) := \mathrm{e}^{-|z|^2 + 2\Re V(z)} \mathrm{d}^2 z \tag{1-9}$$

where $V(z)$ is a holomorphic function such that $V'(z)$ is rational. The convergence of the measure mandates that the residues of $V'(z)\mathrm{d}z$ must have real part greater than $-\frac{1}{2}$ and that the behavior at ∞ of V cannot exceed the second power (and also a certain open condition on the coefficient of this quadratic term which we do not specify here). Orthogonal polynomials are defined as a holomorphic basis of $L^2(\mathbb{C}, \mathrm{d}\mu)$. It is amusing to note that the moment functional

$$\mathcal{L}(z^j | \overline{z}^k) := \iint_{\mathbb{C}} z^i \overline{z}^k \mathrm{d}\mu(z, \overline{z}) =: \mu_{j\overline{k}} \tag{1-10}$$

is a semiclassical moment functional (using Stokes' thm. *in vece* of integration by parts) with just some (obvious) reality constraint on the bimoments. Therefore all the algebraic manipulations that rely on the semiclassicity alone carry out verbatim to this case. In particular (with very minor and trivial modifications) Section 2 almost entirely generalizes (in particular Thm. 2.1. Significant differences (sufficient to require a different analysis to appear elsewhere) arise in the construction of the fundamental systems and the Riemann–Hilbert problem.

1.2 Connection to 2-Toda equations

The framework of this paper is connected to the general theory of 2-Toda equations [16, 1]. This is the theory of a pair of (semi)-infinite matrices P, Q (in our notation) where Q is lower-Hessenberg and P is upper–Hessenberg[3] which evolve under a bi-infinite set of commuting flows $\{t_j, \widetilde{t}_j\}_{j\in\mathbb{N}}$

$$\partial_{t_J} Q = -\frac{1}{J}[Q,(Q^J)_{-0}] \ , \qquad \partial_{\widetilde{t}_J} Q = -\frac{1}{J}[Q,(P^J)_{-0}] \tag{1-11}$$

$$\partial_{t_J} P = -\frac{1}{J}[P,(Q^J)_{+0}] \ , \qquad \partial_{\widetilde{t}_J} P = -\frac{1}{J}[P,(P^J)_{+0}] \tag{1-12}$$

where the subscript $_{\pm 0}$ denotes the upper/lower triangular part plus half of the diagonal (we are assuming the normalization such that the upper triangular part of Q coincides with the transposed of the lower-triangular part of P).

Let now Q, P be semi-infinite matrices. We can use Q, P to denote the matrices expressing the multiplicative recurrence relations of a sequence of polynomials,

$$x\pi_n = \sum_{j=0}^{n+1} Q_{nj}\pi_j \ , \qquad y\sigma_n = \sum_{j=0}^{n+1} P_{nj}\sigma_n \ , \tag{1-13}$$

where the polynomials are recursively *defined* by this relation. Using the generalization of Favard's theorem proved in our [2] we prove the existence of (unique) a bimoment functional $\mathcal{L} : \mathbb{C}[x]\otimes\mathbb{C}[y] \to \mathbb{C}$ such that

$$\mathcal{L}(\pi_n|\sigma_m) = \delta_{nm} \ . \tag{1-14}$$

It then follows easily that the 2-Toda flows are **linearized** by this moment map, in the sense that the solutions $Q(\mathbf{t}, \widetilde{\mathbf{t}}), P(\mathbf{t}, \widetilde{\mathbf{t}})$ are simply the multiplication matrices for the biorthogonal polynomials of the moment functional

$$\mathcal{L}_{\mathbf{t},\widetilde{\mathbf{t}}}(\bullet|\bullet) := \mathcal{L}\left(\mathrm{e}^{-\sum \frac{t_J}{J}x^J}\bullet|\mathrm{e}^{-\sum \frac{\widetilde{t}_J}{J}y^J}\bullet\right) \ . \tag{1-15}$$

The moment functionals of semiclassical type (eqs. 1-7, 1-8) that we are going to analyze form a particular class of **reductions** of the above-mentioned 2-Toda hierarchy. The simplest situation is the one of bimoment semiclassical functionals with polynomial potentials as the ones considered in [3], where the matrices P, Q are also *finite band*. Moreover the solutions which arise in the context of semiclassical bilinear functionals also satisfy the (compatible) constraint of the **string equation**

$$[P, Q] = \hbar\mathbf{1} \tag{1-16}$$

[3] We say that a matrix is lower Hessenberg its $(i, i+1+k)$ entries vanish ($\forall k = 1, 2, \ldots$) and also all $(i, i+1)$-entries are nonzero. A matrix is upper Hessenberg if its transposed is lower-Hessenberg.

(the constant $\hbar$ can be disposed of by a rescaling). The parameter of the (finite–dimensional) reduction are the coefficients of the potentials: for more general semiclassical moment functionals as the ones considered in this paper, the parameters involve not only the coefficients of the partial fraction expansions of the (derivatives of the) potentials, but also the position of the poles and the position of the end-points of the supports of the measure (the hard–edge endpoints).

The paper is organized as follows

1. In Section 2 we derive the recurrence relation satisfied by the biorthogonal polynomials of a semiclassical moment functional. There are two types of recurrence relations: one which involves the multiplication by the spectral parameter (and plays the rôle of the more standard three–term recurrence relation for orthogonal polynomials) and one which involves a differential operator acting on the polynomials.

2. In Section 3 we recall some possibly not well known facts about a certain class of linear homogeneous ODEs. These equations are next in simplicity to the class of constant coefficients ODEs, inasmuch as the coefficients are allowed to be linear functions of the independent variable. When considering the formal adjoint equation then the classical bilinear concomitant provides a nondegenerate pairing between the solution spaces of the pair of mutually adjoint ODEs. In this case we give an interpretation of it in terms of an **intersection pairing** between certain contours used in the representation of the solutions as contour-integrals. This part of the paper is logically quite independent on the rest but it is nevertheless necessary in order to understand certain constructs of the following section.

3. In Section 4 we define the auxiliary wave vectors for our functionals, using a certain multiple integral transform which relies upon the form of the bilinear concomitant associated to our semiclassical moment functional (extending the some of the results of [5]). These expression will prove crucial in the formulation of a first order ODE of rank $d_i = \deg(A_i)$ satisfied by the biorthogonal polynomials. We also derive the analog of the Christoffel–Darboux identities satisfied by standard orthogonal polynomials to our case of biorthogonal polynomials: similar expression were extensively used in [3, 5] for the case where the potentials V_i are polynomials (which is a subcase strictly included in our present setting) and in absence of hard-edge endpoints. The novel feature is that these new identities involve not only the biorthogonal polynomials of the moment functional $\mathcal{L}$ itself, but also those of the **associated** bilinear semiclassical moment functionals

$$\check{\mathcal{L}} := \mathcal{L}(B_1 \bullet |\bullet) \ ; \qquad \widehat{\mathcal{L}} := \mathcal{L}(\bullet|B_2\bullet) \ . \tag{1-17}$$

This feature appears prominently in the *perfect duality* of the Riemann–Hilbert problems appearing in the next section.

4. In Section 5 we define a pair[4] of piecewise–analytic matrices constructed out of the entries of the wave-vectors and their auxiliary wave-vectors. They satisfy certain jump conditions on contours in the complex plane and some asymptotic behavior at the zeroes of B_1. Moreover they satisfy rational first order ODEs with poles at the zeroes of B_1. The Christoffel–Darboux identity, when written as a bilinear expression for these matrices becomes a *perfect pairing* (Thm. 5.1) in the sense that establishes a nondegenerate constant (in x) duality-pairing between the two solution spaces. This pairing is should be thought of as the "dressed" form of the bilinear concomitant pairing introduced in Sect. 3. Similar Riemann–Hilbert problems have appeared elsewhere in the literature, e.g. [14, 13, 5, 3].

In order to facilitate the navigation through the paper all proofs of more technical nature are collected in appendix and only those that may help the understanding are left in the main body of the paper.

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2 Semiclassical bilinear moment functionals of type BB

We consider an arbitrary bilinear semiclassical moment functional (as defined in the introduction) [2], i.e. satisfying (1-7, 1-8). Let $q_i = \deg(B_i)$ and $d_i = \deg(A_i)$: we assume that $d_i \geqslant q_i+1$ ("type BB" in the terminology of [2]). We also make the assumption that the two pairs of polynomials A_i, B_i are **reduced** in the sense that the only common zeroes of A_i and B_i $(i = 1, 2)$ are simple zeroes of B_j. Any moment functional coming from a representation like the one in the introduction (1-4) has this property of reducedness. In [2] the case of non-reduced moment functional is also considered, and it corresponds to functionals which may be expressed as delta functions (or derivatives thereof): we refer ibidem for details.

It is known [2] that any such reduced moment functional can be expressed in integral form

$$\mu_{ij} := \mathcal{L}(x^i|y^j) = \sum_{\mu=1}^{d_1}\sum_{\nu=1}^{d_2} \varkappa_{\mu,\nu}\mathcal{L}_{\mu,\nu}(x^i|y^j) \tag{2-1}$$

$$\mathcal{L}_{\mu\nu}(x^i|y^j) = \int_{\Gamma_{x,\mu}}\int_{\Gamma_{y,\nu}} \mathrm{e}^{-V_1(x)-V_2(y)+xy}\mathrm{d}x\mathrm{d}y \tag{2-2}$$

$$V_i'(y) = \frac{A_i + B_i'}{B_i} \tag{2-3}$$

$$\mathcal{L} = \iint_{\varkappa} x^i y^j \mathrm{e}^{-V_1(x)-V_2(y)+xy}\mathrm{d}x\mathrm{d}y \tag{2-4}$$

$$\tag{2-5}$$

[4] In fact there are two such pairs, the other being obtained by interchanging the rôles of x, y, B_1, B_2 etc.

The two sets of contours of integration $\Gamma_{x,\mu}$ and $\Gamma_{y,\nu}$ are defined in the x and y complex planes respectively and in completely parallel fashion: we will define them in Section 3.1. We have also introduced the short-hand notation

$$\iint_{\varkappa} := \sum_{\mu=1}^{d_1} \sum_{\nu=1}^{d_2} \varkappa_{\mu,\nu} \int_{\Gamma_{x,\mu}} \int_{\Gamma_{y,\nu}} \tag{2-6}$$

Note that the case of hard-edges is included (in this case the fraction defining $V_i's$ has common divisors vanishing at the endpoints of the hard-edges).

The constants $\varkappa_{\mu,\nu} \in \mathbb{C}$ are arbitrary (not all zero). In the paper we will often invoke "genericity" conditions for the moment functional $\mathcal{L}$: by this we mean that the genericity is in the choice of the $\varkappa$-constants and not in the choice of A_i, B_i which we consider as given once and for all. All of the genericity conditions that we will use can be translated into the nonvanishing of certain infinite sequences of minors of the matrix of bimoments $M = [\mu_{ij}]$: since the moments μ_{ij} are linear in $\varkappa$ as per (2-1), this genericity boils down to avoiding an at-most-denumerable collection of divisors of homogeneous polynomials in the $\varkappa$-space.

2.1 Biorthogonal polynomials

Let us consider the biorthogonal polynomials associated to this bilinear moment functional, namely two sequences of monic polynomials satisfying the following conditions

$$\begin{aligned} &\{\pi_n(x), \sigma_n(y)\}_{n\in\mathbb{N}} \\ &\pi_n(x) = x^n + \mathcal{O}(x^{n-1}) \\ &\sigma_n(y) = y^n + \mathcal{O}(y^{n-1}) \\ &\mathcal{L}(\pi_n|\sigma_m) = h_n\delta_{nm} \ . \end{aligned} \tag{2-7}$$

The existence of these BOPs is guaranteed provided that the principal minors of the matrix of bimoments do not vanish

$$\Delta_n[\mathcal{L}] := \det[\mu_{ij}]_{0\leqslant i,j,\leqslant n-1} \neq 0 \qquad \forall n \in \mathbb{N} \ , \tag{2-8}$$

which also guarantees that $h_n \neq 0$,$\forall n \in \mathbb{N}$ ([2]). We find it more convenient to deal with the normalized BOPs;

$$p_n := \frac{\pi_n}{\sqrt{h_n}} \ , \qquad s_n := \frac{\sigma_n}{\sqrt{h_n}} \tag{2-9}$$

We will use the following **quasipolynomials**

$$\psi_n := p_n e^{-V_1(x)} \ , \qquad \phi_n := s_n e^{-V_2(y)} \tag{2-10}$$

and the following semi-infinite vectors (**wave vectors**)

$$\mathbf{p}(x) := [p_0, p_1, \ldots, p_n, \ldots]^t \ , \qquad \mathbf{s}(y) := [s_0, s_1, \ldots, s_n, \ldots]^t \tag{2-11}$$

$$\underset{\infty}{\Psi} := \mathbf{p}(x)\mathrm{e}^{-V_1(x)} \ , \qquad \underset{\infty}{\Phi} := \mathbf{s}(y)\mathrm{e}^{-V_2(y)} \tag{2-12}$$

It will become necessary to consider the following **associated** semiclassical functionals defined by the relations

$$\widehat{\mathcal{L}}(p|s) := \mathcal{L}(p|B_2\, s) \ , \qquad \check{\mathcal{L}}(p|s) := \mathcal{L}(B_1\, p|s) \ . \tag{2-13}$$

We leave to the reader the simple check that these are also semiclassical moment functionals where the potentials are replaced –respectively– by

$$\widehat{V}_2(y) := V_2(y) - \ln B_2(y) \ , \qquad \check{V}_1(x) := V_1(x) - \ln B_1(x) \ . \tag{2-14}$$

Note, however, that they have the same $\varkappa$'s and are defined along the same contours as $\mathcal{L}$.

2.2 Multiplicative recurrence relations

We now prove

Theorem 2.1 *The BOPs satisfy the following finite-term recurrence relations:*

$$x\left(p_n + \sum_{j=1}^{q_2} \ell_j(n) p_{n-j}\right) = \sum_{j=-1}^{d_2} \alpha_j(n) p_{n-j} \tag{2-15}$$

$$y\left(s_n + \sum_{j=1}^{q_1} m_j(n) s_{n-j}\right) = \sum_{j=-1}^{d_1} \beta_j(n) s_{n-j} \tag{2-16}$$

$$q_i = \deg(B_i),\ d_i = \deg(A_i) \ ,$$

where $\ell_j(n) = 0$ for $n \leqslant d_2$ and $m_j(n) = 0$ for $n \leqslant d_1$, under a genericity assumption specified in the proof. The coefficients $\alpha_{-1}(n)$ and $\beta_{-1}(n)$ are nonzero for any n; furthermore, under the same genericity assumptions letting a_i, b_i be the leading coefficients of A_i, B_i we have

$$\begin{aligned} b_2\alpha_{d_2}(n)\sqrt{h_{n-d_2}} &= a_2\ell_{q_2}(n)\sqrt{h_{n-q_2}} \neq 0 \ , \quad n \geqslant d_2 \\ b_1\beta_{d_1}(n)\sqrt{h_{n-d_1}} &= a_1 m_{q_1}(n)\sqrt{h_{n-q_1}} \neq 0 \ , \quad n \geqslant d_1 \end{aligned} \tag{2-17}$$

Proof. We prove only one relation, the other being proved by interchanging the rôles. The statement $\alpha_{-1}(n) \neq 0$ follows from the form of the recurrence relation by comparison of the leading coefficients, which gives

$$\alpha_{-1}(n) = \sqrt{\frac{h_{n+1}}{h_n}} \neq 0 \ . \tag{2-18}$$

The fact that $\ell_j(n) = 0$ for $n \leqslant d_2$ is a choice of convenience: indeed, since $d_2 > q_2$ any xp_n can be written as a linear combination of the same BOPs of degrees $m = 0, \dots, n+1$ for $n \leqslant d_2$.
Consider $xp_n(x)$: by "integration by parts" (i.e. using relation 1-8 from right to left), we immediately conclude that

$$xp_n(x) \ \perp \ B_2(y)\, \mathbb{C}\{1, y, \dots, y^{n-d_2-1}\} =: V_n^{(2)} \tag{2-19}$$

Therefore $V_{n-q_2}^{(2)}$ is in the common annihilator of $xp_n(x), \dots, xp_{n-q_2}(x)$. We now show that it is generically possible to fix the coefficients $\ell_n(j)$ of a linear combination as the left hand side of eq. (2-15) such that the result is perpendicular to *any* polynomial $q(y)$ of degree $\deg(q) < n - d_2$. Let

$$q(y) = B_2(y)a(y) + b(y) \tag{2-20}$$

be the long division of q by B_2 with remainder b: then

$$\mathcal{L}\left(xp_n(x)\middle|q(y)\right) = \mathcal{L}\left(xp_n(x)\middle|B_2(y)a(y) + b(y)\right) = \mathcal{L}\left(xp_n(x)\middle|b(y)\right) \tag{2-21}$$

Since the remainder $b(y)$ is of degree at most $q_2 - 1$, we can find the aforementioned linear combination by solving the system

$$0 = \mathcal{L}\left(x\left(p_n + \sum_{j=1}^{q_2} \ell_j(n)p_{n-j}\right)\middle|y^k\right), \quad k = 0, \dots, q_2 - 1\ . \tag{2-22}$$

After doing so we have that a suitable linear combination in $x\mathbb{C}\{p_n, \dots, p_{n-q_1}\}$ is perpendicular to any $q = B_2 a + b$ with $\deg(a) < n - d_2 - q_2$, $\deg(b) \leqslant q_2 - 1$, or -in other words - to any $q(y)$ of degree less than $n - d_2$, thus proving the shape of the recurrence relation.

In order to clarify the genericity assumption we are imposing we express the above condition as a nonvanishing condition of certain submatrices of the matrix of moments. Indeed the polynomials $\widetilde{p}_n := p_n + \sum^{q_2} \ell_n(j)p_{n-j}$ are uniquely determined by the condition that (for $n \geqslant q_2$)

1. The degree of $\widetilde{p}_n$ is n;

2. The polynomial $\widetilde{p}_n$ is $\mathcal{L}$-orthogonal to $1, y, \dots, y^{n-q_2-1}$

3. The polynomial $x\widetilde{p}_n$ is $\mathcal{L}$-orthogonal to $1, y, \dots, y^{q_2-1}$.

This determines them as the following determinant (up to a nonzero multiplicative constant)

$$\widetilde{p}_n := c_n \det \left[\begin{array}{ccccc} \mu_{01} & & & & \mu_{0,n+1} \\ \mu_{11} & & & & \mu_{1,n+1} \\ \vdots & & & & \vdots \\ \mu_{q_2-1,1} & & & & \mu_{q_2-1,n+1} \\ \hline \mu_{00} & & & & \mu_{0n} \\ \mu_{10} & & & & \mu_{1n} \\ \vdots & & & & \vdots \\ \mu_{n-q_2-1,0} & & & & \mu_{n-q_2-1,n} \\ \hline 1 & x & x^2 & \dots x^{n-1} & x^n \end{array}\right] \tag{2-23}$$

The genericity condition is then the nonvanishing of the principal minor of size n of the above expression, namely the nonvanishing of the following matrices

$$\Delta_{n,2} := \det \left[\begin{array}{cc} \mu_{01} & \mu_{0,n} \\ \mu_{11} & \mu_{1,n} \\ \vdots & \vdots \\ \mu_{q_2-1,1} & \mu_{q_2-1,n} \\ \hline \mu_{00} & \mu_{0n-1} \\ \mu_{10} & \mu_{1n-1} \\ \vdots & \vdots \\ \mu_{n-q_2-1,0} & \mu_{n-q_2-1,n-1} \end{array}\right], n \in \mathbb{N}. \tag{2-24}$$

The normalization that $\widetilde{p}_n = p_n + (lower\ degree)$ gives for the c_n of eq. (2-23)

$$c_n = \frac{1}{\Delta_{n,2}\sqrt{h_n}} \tag{2-25}$$

Let us now check that this genericity assumption is actually equivalent to requiring $\alpha_n(d_2) \neq 0$, $\forall n$. Denoting by a_2, b_2 the leading coefficients of $A_2(y), B_2(y)$ we find

$$b_2\alpha_{d_2}(n)\sqrt{h_{n-d_2}} = \mathcal{L}(x\widetilde{p}_n|B_2y^{n-d_2-q_2}) = \mathcal{L}(\widetilde{p}_n|A_2y^{n-d_2-q_2} - \mathcal{O}(y^{n-d_2-1})) = \mathcal{L}(\widetilde{p}_n|a_2y^{n-q_2}) = \ell_{q_2}(n)a_2\sqrt{h_{n-q_2}} \tag{2-26}$$

This proves the identity (2-17): to prove that it does not vanish under our genericity conditions we compute

$$\mathcal{L}(\widetilde{p}_n|a_2 y^{n-q_2}) = \frac{a_2}{\Delta_{n,2}\sqrt{h_n}} \det \left[\begin{array}{cc} \mu_{01} & \mu_{0,n+1} \\ \mu_{11} & \mu_{1,n+1} \\ \vdots & \vdots \\ \mu_{q_2-1,1} & \mu_{q_2-1,n+1} \\ \hline \mu_{00} & \mu_{0n} \\ \mu_{10} & \mu_{1n} \\ \vdots & \vdots \\ \mu_{n-q_2,0} & \mu_{n-q_2,n} \end{array}\right] = \frac{a_2 \Delta_{n+1,2}}{\Delta_{n,2}\sqrt{h_n}} \neq 0 \qquad (2\text{-}27)$$

Q.E.D.

We can represent the previous recurrence relations in matrix form as follows

Proposition 2.1 *The wave vectors satisfy the following recurrence relations*

$$x(\mathbf{1}+L)\underset{\infty}{\Psi} = A\underset{\infty}{\Psi}\ , \qquad y(\mathbf{1}+M)\underset{\infty}{\Phi} = B\underset{\infty}{\Phi} \qquad (2\text{-}28)$$

where L is the lower triangular matrix with q_2 subdiagonals whose matrix entries are $L_{nm} = \ell_n(n-m)$ and A is a lower Hessenberg matrix with entries $A_{nm} = \alpha_n(m-n)$ (similarly for M, B). The entries in the lowest and highest diagonals in $\mathbf{1}+L, A$ are non vanishing.

2.3 Differential recurrence relations

Proposition 2.2 *Under a genericity assumption for the moment functional (specified in the proof) the BOPs satisfy the following differential finite–term recurrence relations*

$$\nabla_x \left(p_n + \sum_1^{q_1} \breve{m}_j(n+j) p_{n+j} \right) = -\sum_{j=-1}^{d_1} \breve{\beta}_j(n+j) p_{n+j} \qquad (2\text{-}29)$$

$$\nabla_y \left(s_n + \sum_1^{q_2} \widehat{\ell}_j(n+j) s_{n+j} \right) = -\sum_{j=-1}^{d_2} \widehat{\alpha}_j(n+j) s_{n+j} \qquad (2\text{-}30)$$

$$\nabla_x := \partial_x - V_1'(x)\ , \qquad \nabla_y := \partial_y - V_2'(y)\ . \qquad (2\text{-}31)$$

In matrix form we have

$$\begin{aligned} \nabla_x(\mathbf{1}+\breve{M}^t)\mathbf{p} &= -\breve{B}^t\mathbf{p} \\ \nabla_y(\mathbf{1}+\widehat{L}^t)\mathbf{s} &= -\widehat{A}^t\mathbf{s}\ , \end{aligned} \qquad (2\text{-}32)$$

where the matrices above are defined by

$$
\begin{aligned}
\breve{M}_{nk} = \breve{m}_{n-k}(n)\ , \qquad & \breve{B}_{nk} = \breve{\beta}_{n-k}(n) \\
\widehat{L}_{nk} = \widehat{\ell}_{n-k}(n)\ , \qquad & \widehat{A}_{nk} = \widehat{\alpha}_{n-k}(n)\ .
\end{aligned}
\tag{2-33}
$$

Note that they have the same shape as M, B, L, A respectively (whence the mnemonics of the symbols).

Proof. We prove only the first of the two relations, the other being proved analogously. Consider the unique (generically existing) vector $\tilde{p}_n$ in $\mathbb{C}[p_n, \ldots, p_{n+q_1}]$ which is divisible by $B_1(x)$ and "monic" w.r.t. p_n in the sense that $\tilde{p}_n = p_n + \mathbb{C}[p_{n+1}, \ldots, p_{n+q_1}]$. Writing then $\tilde{p}_n = B_1 q_n$ we find

$$
\mathrm{e}^{V_1}\partial_x \tilde{p}_n \mathrm{e}^{-V_1} = B_1' q_n + B_1' q_n' - V_1' B_1 q_n = B_1 q_n' - A_1 q_n. \tag{2-34}
$$

This implies that $(-\partial_x + V_1')\tilde{p}_n$ is a *polynomial* of degree $n + d_1$ in spite of the fact that V_1' is rational. Moreover

$$
\mathcal{L}\bigg((-\partial_x + V_1')\tilde{p}_n \bigg| y^k\bigg) = \mathcal{L}\bigg(-B_1 q_n' + A_1 q_n \bigg| y^k\bigg) = \mathcal{L}\bigg(B_1 q_n \bigg| y^{k+1}\bigg) = \mathcal{L}\bigg(\tilde{p}_n \bigg| y^{k+1}\bigg) \equiv 0 \tag{2-35}
$$

$$
k < n-1 \tag{2-36}
$$

This concludes the proof. The genericity condition that we are using now is the nonvanishing of the principal minors of the associated moment functional $\breve{\mathcal{L}}$ or (which is the same) the existence of biorthogonal polynomials for $\breve{\mathcal{L}}$. **Q.E.D.**

For later convenience we remark that the genericity condition we are invoking now is also equivalent to requiring that the vectors (the superscript $^{(r)}$ denoting the r-th derivative)

$$
\left[p_n^{(r)}(x_j), \ldots, p_{n+q_1-1}^{(r)}(x_j)\right]\ , \tag{2-37}
$$

$$
B_1(x_j) = 0,\ r = 0 \ldots r_j\ , \qquad B_1(x) = b_1 \prod_{j=1}^{s} (x - x_j)^{r_j} \tag{2-38}
$$

be linearly independent: indeed

$$
p_n + \sum_1^{q_1} \breve{m}_j(n+j) p_{n+j} = e_n \left[\begin{array}{cccc}
p_n(x_1) & & & p_{n+q_1}(x_1) \\
\vdots & & & \vdots \\
p_n^{(r_1)}(x_1) & & & p_{n+q_1}^{(r_1)}(x_1) \\
\hline
\vdots & & & \vdots \\
p_n^{(r_s)}(x_s) & & & p_{n+q_1}^{(r_s)}(x_s) \\
\hline
p_n(x) & p_{n+1}(x) & \ldots & p_{n+q_1}(x)
\end{array}\right] \tag{2-39}
$$

where e_n is the inverse of the $(q_1 + 1, 1)$–cofactor of the above matrix. The proposition can be rewritten for the wave vectors as follows

Proposition 2.3 *The wave vectors satisfy the following differential equations*

$$\partial_x(\mathbf{1} + \breve{M}^t)\underset{\infty}{\Psi} = -\breve{B}^t\underset{\infty}{\Psi} \ , \qquad \partial_y(\mathbf{1} + \widehat{L}^t)\underset{\infty}{\Phi} = -\widehat{A}^t\underset{\infty}{\Phi} \ , \tag{2-40}$$

where $\widehat{M}_{nk} = \widehat{m}_{k-n}(n)$ *and* $\widehat{A}_{nk} = \widehat{\alpha}_{k-n}(n)$ *(and similar expressions for* $\widehat{M}, \widehat{B}$*).*

The matrices $\breve{M}, \breve{B}, \widehat{L}, \widehat{A}$ play the same role of M, B and L, A for the moment functionals $\breve{\mathcal{L}}$ and $\widehat{\mathcal{L}}$ respectively.

Proposition 2.4 *The vectors of polynomials*

$$\widehat{\mathbf{p}}(x) := (\mathbf{1} + \widehat{L})^{-1}\mathbf{p} \ , \qquad \widehat{\mathbf{s}}(y) := \frac{1}{B_2(y)}(\mathbf{1} + \widehat{L}^t)\mathbf{s}(y) \tag{2-41}$$

(where $\widehat{L}$ *(and* $\breve{M}$*) are defined by eqs.(2-32) of Prop. 2.2) are the biorthogonal polynomials for* $\widehat{\mathcal{L}}$*. Similarly the vectors of polynomials*

$$\breve{\mathbf{p}}(x) := \frac{1}{B_1(x)}(\mathbf{1} + \breve{M})^{-1}\mathbf{p} \ , \qquad \breve{\mathbf{s}}(y) := (\mathbf{1} + \breve{M}^t)\mathbf{s} \tag{2-42}$$

are the biorthogonal polynomials for $\breve{\mathcal{L}}$

Proof. The two statements are completely parallel and hence we prove only the first.

By definition of the matrix $\widehat{L}$ in Prop. 2.2 the polynomial entries of $(\mathbf{1} + \widehat{L}^t)\mathbf{s}$ are all divisible by B_2, therefore $\widehat{\mathbf{s}}$ is indeed a vector of polynomials. Next we have (using an obvious matrix notation)

$$\widehat{\mathcal{L}}\left(\widehat{\mathbf{p}}\middle|\widehat{\mathbf{s}}^t\right) = \mathcal{L}\left((\mathbf{1} + \widehat{L})^{-1}\mathbf{p}\middle|\mathbf{s}^t(\mathbf{1} + \widehat{L})\right) = (\mathbf{1} + \widehat{L})^{-1}\mathcal{L}\left(\mathbf{p}\middle|\mathbf{s}^t\right)(\mathbf{1} + \widehat{L}) = \mathbf{1} \quad \textbf{Q.E.D} \tag{2-43}$$

We also have

Lemma 2.1 *The matrices* L, A, M, B *and the matrices* $\widehat{L}, \widehat{A}, \breve{M}, \breve{B}$ *are related by*

$$A(\mathbf{1} + \widehat{L}) = (\mathbf{1} + L)\widehat{A} \ , \qquad B(\mathbf{1} + \breve{M}) = (\mathbf{1} + M)\breve{B} \ . \tag{2-44}$$

Proof. Once more we prove only the first.

$$\begin{aligned} A(\mathbf{1} + \widehat{L}) &= \mathcal{L}\left(A\mathbf{p}\middle|\mathbf{s}^t(\mathbf{1} + \widehat{L})\right) = \mathcal{L}\left(x(\mathbf{1} + L)\mathbf{p}\middle|\mathbf{s}^t(\mathbf{1} + \widehat{L})\right) = \\ = \mathcal{L}\left(x(\mathbf{1} + L)\mathbf{p}\middle|B_2\widehat{\mathbf{s}}^t\right) &= \mathcal{L}\left((\mathbf{1} + L)\mathbf{p}\middle|(-B_2\partial_y + A_2)\widehat{\mathbf{s}}^t\right) = \mathcal{L}\left((\mathbf{1} + L)\mathbf{p}\middle| - \nabla_y B_2\widehat{\mathbf{s}}^t\right) = \\ = \mathcal{L}\left((\mathbf{1} + L)\mathbf{p}\middle| - \nabla_y\mathbf{s}^t(\mathbf{1} + \widehat{L})\right) &= \mathcal{L}\left((\mathbf{1} + L)\mathbf{p}\middle|\mathbf{s}^tA\right) = (\mathbf{1} + L)\widehat{A} \qquad \textbf{Q.E.D.} \end{aligned} \tag{2-45}$$

Lemma 2.2 *The associated wave vectors* $\widehat{\mathbf{p}}, \widehat{\mathbf{s}}$ *and* $\check{\mathbf{p}}, \check{\mathbf{s}}$ *satisfy*

$$x(\mathbf{1} + \widehat{L})\widehat{\mathbf{p}} = \widehat{A}\widehat{\mathbf{p}}$$
$$y(\mathbf{1} + \widecheck{M})\check{\mathbf{s}} = \check{B}\check{\mathbf{s}} \tag{2-46}$$

Moreover, under the same genericity assumptions

$$\check{m}_{q_1}(n) \neq 0 \neq \widehat{\ell}_{q_2}(n) \qquad \forall n \tag{2-47}$$

$$b_2\widehat{\alpha}_{d_2}(n)\sqrt{\widehat{h}_{n-d_2}} = a_2\widehat{\ell}_{q_2}(n)\sqrt{\widehat{h}_{n-q_2}} \neq 0 \tag{2-48}$$

Proof. Recalling that $\widehat{\mathbf{p}} = (\mathbf{1} + \widehat{L})^{-1}\mathbf{p}$ (by definition), we find

$$x(\mathbf{1} + \widehat{L})\widehat{\mathbf{p}} = x\mathbf{p} = (\mathbf{1} + L)^{-1}A\mathbf{p} = \widehat{A}(\mathbf{1} + \widehat{L})^{-1}\mathbf{p} = \widehat{A}\widehat{\mathbf{p}}\ . \tag{2-49}$$

The relations (2-48) for the moment functionals $\widehat{\mathcal{L}}, \check{\mathcal{L}}$ are proved in exactly the same way relations (2-17) are proved for $\mathcal{L}$. **Q.E.D.**

We can summarize all the relations collected so far in the following table:

Functional	BOPs	Mult. rec.	Diff. Rec.
$\mathcal{L}(\bullet\vert\bullet)$	$\mathbf{p}, \mathbf{s}$	$x(\mathbf{1} + L)\mathbf{p} = A\mathbf{p}$ $y(\mathbf{1} + M)\mathbf{s} = B\mathbf{s}$	$\nabla_x(\mathbf{1} + \widecheck{M}^t)\mathbf{p} = -\check{B}^t\mathbf{p}$ $\nabla_y(\mathbf{1} + \widehat{L}^t)\mathbf{s} = -\widehat{A}^t\mathbf{s}$
$\widehat{\mathcal{L}}(\bullet\vert\bullet) = \mathcal{L}(\bullet\vert B_2\bullet)$	$\widehat{\mathbf{p}} := (\mathbf{1} + \widehat{L})^{-1}\mathbf{p}$ $\widehat{\mathbf{s}} := \frac{1}{B_2}(\mathbf{1} + \widehat{L}^t)\mathbf{s}$	$x(\mathbf{1} + \widehat{L})\widehat{\mathbf{p}} = \widehat{A}\widehat{\mathbf{p}}$ $\star$	$\nabla_x(\mathbf{1} + \widetilde{M}^t)\widehat{\mathbf{p}} = -\widetilde{B}^t\widehat{\mathbf{p}}$ $\star$
$\check{\mathcal{L}}(\bullet\vert\bullet) = \mathcal{L}(B_1\bullet\vert\bullet)$	$\check{\mathbf{p}} = \frac{1}{B_1}(\mathbf{1} + \widecheck{M}^t)\mathbf{p}$ $\check{\mathbf{s}} = (\mathbf{1} + \widecheck{M})^{-1}\mathbf{s}$	$\star$ $y(\mathbf{1} + \widecheck{M})\check{\mathbf{s}} = \check{B}\check{\mathbf{s}}$	$\star$ $\nabla_y(\mathbf{1} + \widetilde{L}^t)\check{\mathbf{s}} = -\widetilde{A}^t\check{\mathbf{s}}$
$\widetilde{\mathcal{L}}(\bullet\vert\bullet) = \mathcal{L}(B_1\bullet\vert B_2\bullet)$	$\widetilde{\mathbf{p}} = \begin{cases} \frac{1}{B_1}(\mathbf{1} + \widetilde{M}^t)\widehat{\mathbf{p}} \\ (\mathbf{1} + \widetilde{L})^{-1}\check{\mathbf{p}} \end{cases}$ $\widetilde{\mathbf{s}} = \begin{cases} (\mathbf{1} + \widetilde{M})^{-1}\widehat{\mathbf{s}} \\ \frac{1}{B_2}(\mathbf{1} + \widetilde{L}^t)\check{\mathbf{s}} \end{cases}$	$x(\mathbf{1} + \widetilde{L})\widetilde{\mathbf{p}} = \widetilde{A}\widetilde{\mathbf{p}}$ $y(\mathbf{1} + \widetilde{M})\widetilde{\mathbf{s}} = \widetilde{B}\widetilde{\mathbf{s}}$	$\star$

$$(\mathbf{1} + L)\widehat{A} = A(\mathbf{1} + \widehat{L}) \tag{2-50}$$

$$(\mathbf{1} + \widehat{L})\widetilde{A} = \widehat{A}(\mathbf{1} + \widetilde{L}) \tag{2-51}$$

$$(\mathbf{1} + M)\check{B} = B(\mathbf{1} + \widecheck{M}) \tag{2-52}$$

$$(\mathbf{1} + \widecheck{M})\widetilde{B} = \check{B}(\mathbf{1} + \widetilde{M}) \tag{2-53}$$

Here the matrices $A, \widehat{A}, \widetilde{A}$ are lower–Hessenberg matrices with d_2 nontrivial sub-diagonals, $B, \widehat{B}, \widetilde{B}$ are lower–Hessenberg with d_1 nontrivial sub–diagonals. The matrices $L, \widehat{L}, \widetilde{L}$ and $M, \widehat{M}, \widetilde{M}$ are strictly lower triangular matrices with q_2 or q_1 nontrivial subdiagonals respectively.

The $\star$'s mean that there are (possibly under similar genericity requirements for the corresponding functional) similar relations as in the corresponding box on the first line, for which however we do not need to define symbols for our purposes.

3 Adjoint differential equations and the bilinear concomitant

In this section we recall some results which –although simple– I was not able to find in the literature. We consider a $n-th$ order differential equations of the form

$$\left(A(\partial_x) - xB(\partial_x)\right) f(x) = 0 \tag{3-1}$$

where $A(D)$ and $B(D)$ are polynomials and $n = \max(\deg(A), \deg(B))$: the reader should keep in mind the polynomials A_i, B_i of our matrix model. If we look for solutions written as "Fourier–Laplace" transforms

$$f_\Gamma(x) := \int_\Gamma \mathrm{d}y\, \mathrm{e}^{xy - V(y)} \ , \tag{3-2}$$

–where the contour of integration is so far unspecified–, simple formal manipulations involving integration by parts show that

$$V'(y) = \frac{A(y) + B'(y)}{B(y)} \ , \tag{3-3}$$

where the relation between A, B, V and A_i, B_i, V_i should be now completely clear.

In the situation of interest to us we will have A, B **reduced**

Definition 3.1 *Two polynomials A, B are called* **reduced** *if they share at most a simple zero of B*

It is a simple exercise to see that

Lemma 3.1 *Two polynomials A, B are reduced if and only if $A \pm B'$ and B are.*

This "duality" of the notion of reducedness will be important when considering the adjoint differential operator.

We now remark that V' is a rational function with poles at a **subset** of the zeroes of B

$$B(y) = c\prod_{j=1}^{r}(y-b_j)^{m_j+1} \ , c \neq 0 \ , \deg B = \sum_{j=1}^{r} m_j \ , \quad m_j \in \mathbb{N} \ . \tag{3-4}$$

$$V'(y) = \sum_{\ell=0}^{d} v_{\ell+1} y^\ell - \sum_{j\in J\subseteq 1,\dots,r} \sum_{k=0}^{m_j} \frac{t_{k,j}}{(y-b_j)^{k+1}} \tag{3-5}$$

$$\mathrm{e}^{-V(y)} = \prod_{j\in J}(y-b_j)^{t_{0,j}} \exp\left[\sum_{\ell=0}^{d} \frac{v_{\ell+1}}{\ell+1} y^{\ell+1} + \sum_{j\in J}\sum_{k=1}^{m_j} \frac{t_{k,j}}{k(y-b_j)^k}\right] \tag{3-6}$$

$$W(y) := \mathrm{e}^{-V(y)} \tag{3-7}$$

$$d := \deg(A) - \deg(B) \ . \tag{3-8}$$

[Here it is understood that if $\deg(A) < \deg(B)$ then the first sum in V' is absent.]

Some of the zeroes of $B(y)$ may appear also as zeroes of $A(y)+B'(y)$ and hence in the partial fraction expansion of V' those points do not appear. Since A, B are reduced, all multiple zeroes of B are not shared with $A+B'$. We will call the zeroes of B which are common with $A+B'$ the **hard-edge points** (note that not all simple zeroes of B are hard-edge points, but all hard-edge points are simple zeroes).

We now define some sectors $S_k^{(j)}$, $j=1,\dots p_1$, $k=0,\dots m_j-1$. around the multiple zeroes of B (b_j for which $m_j>0$) in such a way that

$$\Re\left(V(y)\right) \underset{\substack{y\to b_j,\\ y\in S_k^{(j)}}}{\longrightarrow} +\infty \ . \tag{3-9}$$

The number of sectors for each pole is the degree of that pole in the exponential part of $W(x)$, that is $d+1$ for the pole at infinity and g_j for the j-th pole. Explicitly

$$\begin{aligned}
S_k^{(0)} &:= \left\{ y :\in \mathbb{C};\ \ \frac{2k\pi - \frac{\pi}{2}+\epsilon}{d+1} < \arg(y) + \frac{\arg(v_{d+1})}{d+1} < \frac{2k\pi + \frac{\pi}{2}-\epsilon}{d+1}\right\},\\
&\quad k = 0\dots d\ ;\\
S_k^{(j)} &:= \left\{ y :\in \mathbb{C};\ \ \frac{2k\pi - \frac{\pi}{2}+\epsilon}{m_j} < \arg(y-b_j) + \frac{\arg(t_{m_j,j})}{m_j} < \frac{2k\pi + \frac{\pi}{2}-\epsilon}{m_j}\right\},\\
&\quad k = 0,\dots,m_j-1,\quad j\in J\ .
\end{aligned} \tag{3-10}$$

These sectors are defined precisely in such a way that approaching any of the essential singularities (i.e. an b_j such that $m_j>0$) the function $W(y)$ tends to zero faster than any power.

3.1 Definition of the contours

The contours we are going to define are precisely the type of contours $\Gamma_{x,\mu},\Gamma_{y,\nu}$ entering the definition of the bimoment functional $\mathcal{L}$. Let A, B be reduced: we then define $n=\max(\deg(A),\deg(B))$ contours. The definition of the contours follows directly [2, 15]. We first remark that the weight $W(y)$ is –in general– multivalued since it contains powers like $(y-c)^t$ with non-integer t; the multivaluedness is multiplicative and in fact is not very important which branch one chooses in the definition of the integrals (3-2) since different choices correspond to multiplying the same function by a nonzero constant. Nonetheless it will be convenient at some point to have a reference normalization for the integrals and hence we define some cuts so as to have a simply connected domain where $W(y)$ is single-valued. We do so by removing semi-infinite arcs extending from each branch-point of $W(y)$ to infinity: for convenience we choose the cuts approaching each singularity in one of the sectors, for example $S_0^{(j)}$, and approaching infinity within $S_0^{(0)}$. If $\deg(A)\leqslant\deg(B)-1$ there no sector is defined at ∞ and then we just choose arbitrarily an asymptotic direction for these cuts.

Note that if $\deg(A) \leqslant \deg(B) - 2$ then the sum of the finite residues of $V'\mathrm{d}y$ is zero, hence we could define the cuts as finite arcs joining in a chain the finite branch-points of $W(y)$: the resulting domain is not simply connected, however $W(y)$ is single valued in such domain precisely because of the vanishing of the sum of the residues of its logarithmic derivative. We will denote by $\mathcal{D}$ the connected domain obtained after such surgery.

In the following our primary focus is on the case $\det(A) \geqslant \deg(B)+1$ and we leave to the reader to check the literature [15] for the remaining cases (only minor modifications are needed).

1. For any zero b_j of B for which there is *no essential singularity* in W we have two cases
    (a) If b_j is a branch point (i.e. $t_{0,j} \in \mathbb{C}\backslash\mathbb{Z}$) we take a loop (referred to as a $\mathbf{lasso}$) starting at infinity in some fixed sector (e.g. $S_0^{(0)}$) encircling the singularity and going back to infinity in the same sector.
    (b) If b_j is a pole of W (i.e. $t_{0,j} \in -\mathbb{N}^{\times}$) then we take a small circle around it.
    (c) If b_j's which is a regular points ($t_{0,j} \in \mathbb{N}$) we take a line joining b_j to infinity and approaching ∞ in the same sector $S_0^{(0)}$ as before (this case includes the hard-edge points for which we may say that $t_{0,j} = 0$).
2. For any multiple zero b_j for which there is an essential singularity (i.e. for which $m_j > 0$) we define m_j contours (which we call the $\mathbf{petals}$) starting from b_j in the sector $S_0^{(j)}$ and returning to b_j in the next (counterclockwise) sector. Finally we join the singularity b_j to ∞ by a path (called the $\mathbf{stem}$) approaching ∞ within the sector $S_0^{(0)}$ chosen at point 1(a).
3. If $\deg(A) \geqslant \deg(B) + 1$ we define $b_0 := \infty$ and we take $d := \deg(A) - \deg(B)$ contours starting at X_0 in the sector $S_k^{(0)}$ and returning at X_0 in the sector $S_{k+1}^{(0)}$.

The reasons for the "floral" names should be clear by looking at an example like the one in Fig. 1. Cauchy's theorem grants us large freedom in the choices of such contours; we use this freedom so that the contours do not intersect each other in $\mathbb{C}\backslash\{b_j\}_{j=1,\dots,\deg(B)}$ and do not cross the chosen cuts.

We will refer to these contours collectively as $\mathbf{admissible\ contours}$ for the differential $W(y)\mathrm{d}y$. Note that we have defined exactly $n = \max(\deg(A), \deg(B))$ contours.

It is a straightforward check to see that

$$f_\Gamma(x) := \int_\Gamma \mathrm{d}y\, \mathrm{e}^{xy - V(y)} = \int_\Gamma \mathrm{e}^{xy} W(y)\mathrm{d}y\ , \tag{3-11}$$

all satisfy the differential equation (3-1): in these checks one is always allowed to perform integration by parts discarding all boundary terms because of the properties of the contours. We leave this check to the reader.

The content of [15] (and of the fix contained in [2]) was to show that these functions are also linearly independent, hence providing a basis for the solution space.

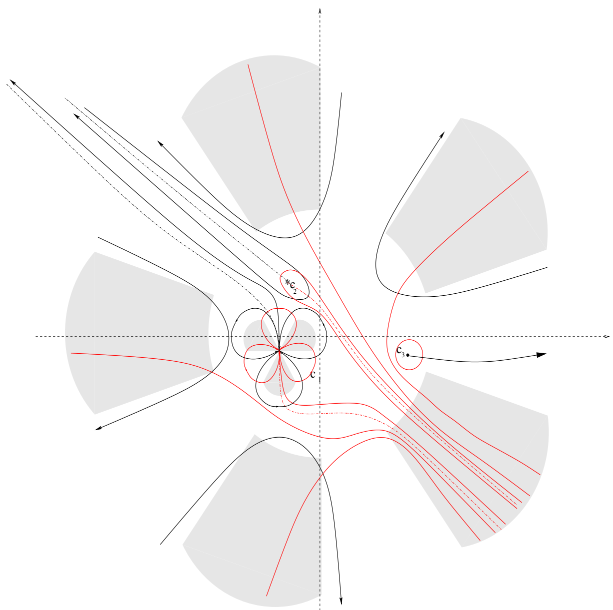


Figure 1: An example of contours Γ and $\widehat{\Gamma}$ for a pair of reduced adjoint differential operators. The black contours are the admissible ones for L while the red ones are the admissible ones for $L^\star$. Also shown in the picture are the cuts for $W(y)$ and $\widehat{W}(s)$ (line-dotted black lines and line-dotted red lines respectively).

3.2 Adjoint differential operators and the bilinear concomitant

In general, given a n-th order linear operator with polynomial coefficients

$$L := \sum_{j}^{n} a_j(x)\partial_x^j \ , \tag{3-12}$$

its classical adjoint is defined as

$$L^\star := \sum_{j}^{n} (-\partial_x)^j a_j(x) \ . \tag{3-13}$$

Between the solution spaces of a pair of adjoint such operators Legendre defined a nondegenerate pairing called the **bilinear concomitant**. We will show that this pairing for our class of reduced operators admits a natural interpretation as intersection pairing.

We begin by noticing that in our case the pair of adjoint operators is written

$$L := A(\partial_x) - xB(\partial_x) \ , \qquad L^\star := A(-\partial_x) - B(-\partial_x)x \ . \tag{3-14}$$

Since A, B are reduced then $L^\star$ is also reduced since

$$L^\star = A(-\partial_x) - B'(-\partial_x) - xB(-\partial_x) \tag{3-15}$$

in view of Lemma 3.1 (here the polynomials are $A(-y) - B'(-y)$ and $B(-y)$ which are clearly reduced iff $A(z) - B'(z)$ and $B(z)$ are). Therefore $L^\star$ is in the same class of operators as L and can be solved by contour integrals in the same way. The solutions of $L^\star g = 0$ are of the form

$$g = \int_{\widehat{\Gamma}} \mathrm{e}^{-xs+\hat{V}(s)} \mathrm{d}s \tag{3-16}$$

$$\widehat{V}(s)' := \frac{A(s)}{B(s)} = V'(s) - (\ln B(s))' \ . \tag{3-17}$$

A simple inspection shows that the sectors around the multiple zeroes of $B(s)$ where $\Re(\hat{V}(s)) \to -\infty$ are precisely the complementary sectors defined in (3-10) for V. We normalize $\widehat{V}(s)$ by choosing the integration constant in such a way that

$$\widehat{W}(s) := \mathrm{e}^{\hat{V}(s)} = \frac{1}{B(s)} \mathrm{e}^{V(s)} \tag{3-18}$$

(here e^V is supposed to be defined on the simply connected domain $\widehat{\mathcal{D}}$). One then proceeds in the definition of the admissible contours $\widehat{\Gamma}$ for the weight $\widehat{W}(s)$ and of the simply connected domain $\widehat{\mathcal{D}}$ in exactly the same way used for $W(y)$. We make the following important remarks:

1. If b_j is a hard-edge point for $W(y)$ (i.e. it is a zero of $B(y)$ but a regular point for $W(y)$ where W does not vanish) then b_j is a simple pole of $\widehat{W}(s)$.

2. If b_j is a zero of multiplicity m of $W(y)$ (i.e. a simple zero of $B(y)$ such that the residue of $(A+B')/B\mathrm{d}y$ is a negative integer) then it is a pole of order $m+1$ for $\widehat{W}(s)$.

3. In all other cases, the type of singularity of W and $\widehat{W}$ is the same (logarithmic branch-points or essential singularities of the same exponential type).

4. The intersection $\mathcal{D}\cap\widehat{\mathcal{D}}$ is the disjoint union of simply connected domains where $W(y)\widehat{W}(y)B(y)$ is constant. These constants depend only on the residues of $V'(y)\mathrm{d}y$ mod $\mathbb{Z}$.

These observations and the fact that $B(y)W(y)\widehat{W}(y)$ is locally constant (where they are both defined) follows immediately from their definition and eq. (3-18).

From the definitions of the contours it is not difficult to realize that dual contours can be chosen such that

1. For each flower (petal + stem) one can choose a dual flower whose elements intersect only the arcs of the given flower. (This includes the petals at ∞, in the case $\deg(A) \geqslant \deg(B)+1$).

2. For each pole c of $W(y)$ (whose corresponding admissible contours Γ is a small circle) the dual admissible contour for $\widehat{W}(s)$ is a semi-infinite arc starting at c and going to ∞ and can be chosen so that it intersects only its dual.

3. For each zero or hard-edge point a of $W(y)$ (whose corresponding admissible contour is a semi-infinite arc starting at a) the dual admissible contours for $\widehat{W}(s)$ (which is a small circle around a) intersects only Γ.

4. For each non-essential other singularity of $W(y)$ (i.e. a simple zero c of $B(y)$ such that the residue of $(A+B')/B\mathrm{d}y$ is in $\mathbb{C}\backslash\mathbb{Z}$), where the admissible contour Γ is a lasso around c, the dual loop $\widehat{\Gamma}$ (also a lasso around c) is also chosen so that it intersects only the dual lasso (at *two* points).

Lemma 3.2 *Consider the two adjoint differential equations*

$$\left(A(\partial_x) - xB(\partial_x)\right)f(x) = 0 \tag{3-19}$$

$$\left(A(-\partial_x) - B(-\partial_x)x\right)g(x) = 0\ . \tag{3-20}$$

The solutions are of the form

$$f(x) = f_\Gamma(x) := \int_\Gamma \mathrm{e}^{-V(y)+xy}\mathrm{d}x\ , \quad V := \int \frac{A(y)+B'(y)}{B(y)}\mathrm{d}y \tag{3-21}$$

$$g(x) = g_{\widehat\Gamma}(x) := \int_{\widehat\Gamma} \mathrm{e}^{\widehat V(s)-xs}ds\ , \quad \widehat V(s) := \int \frac{A(s)}{B(s)}\mathrm{d}s \tag{3-22}$$

Then the following expression is constant and defines a nondegenerate bilinear pairing (the **bilinear concomitant***) between the solutions spaces of the two adjoint equations:*

$$\mathcal{B}(f,g) := \int_{\widehat\Gamma}\int_\Gamma \left[\Big(B(y)-B(s)\Big)\left[\frac{x}{y-s} - \frac{1}{(y-s)^2}\right] - \frac{A(y)-A(s)-B'(s)}{y-s}\right]\mathrm{e}^{x(y-s)-V(y)+\widehat V(s)} \tag{3-23}$$

Proof. The integral representation of the solution is easily verified. We now write

$$0 \equiv g(x)\int_\Gamma (xB(y)-A(y))\,\mathrm{e}^{-V(y)+xy}\mathrm{d}y \tag{3-24}$$

$$0 \equiv f(x)\int_{\widehat\Gamma} \big(xB(s)-A(s)-B'(s)\big)\,\mathrm{e}^{\widehat V(s)-xs}\mathrm{d}s \tag{3-25}$$

We take the difference and obtain

$$0 \equiv \int_{\widehat\Gamma}\int_\Gamma \big(x(B(y)-B(s)) - (A(y)-B'(s)-A(s))\big)\,\mathrm{e}^{x(y-s)-V(y)+\widehat V(s)}\mathrm{d}y\mathrm{d}s \tag{3-26}$$

It is promptly seen that the integrand of this double integral is absolutely summable w.r.t. the arclength parameters along Γ and $\widehat\Gamma$, hence we can integrate w.r.t. x under the integral sign, thus obtaining the bilinear concomitant;

$$\int_\Gamma\int_{\widehat\Gamma}\left((B(y)-B(s))\left(\frac{x}{y-s}-\frac{1}{(y-s)^2}\right) - \frac{A(y)-B'(s)-A(s)}{y-s}\right)\mathrm{e}^{x(y-s)-V(y)+\widehat V(s)}\mathrm{d}s\mathrm{d}y \tag{3-27}$$

Note that the expression under integration is regular at $y=s$, and is –in fact– a **polynomial** in y,s

$$\left((B(y)-B(s))\left(\frac{x}{y-s}-\frac{1}{(y-s)^2}\right) - \frac{A(y)-B'(s)-A(s)}{y-s}\right) \underset{y\to s}{\sim} xB'(s) - \frac{1}{2}B''(s) - A'(s) + \mathcal{O}(y-s)$$

In particular the integrand is absolutely integrable w.r.t. the arclength parameters and hence the order of integrations is irrelevant. This concludes the proof. **Q.E.D.**

The bilinear concomitant is –in a certain sense– an integral representation of the intersection pairing of the contours of integration. To make this statement more precise we first prove the following standard

Lemma 3.3 *Let $\Omega(y,s)$ be an analytic function $\mathcal{D}\times\widehat{\mathcal{D}}$ where $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are simply connected domains. Suppose that in each connected component of $\mathcal{D}\cap\widehat{\mathcal{D}}$ there is a constant c such that*

$$\Omega(y,s) = \frac{c}{(y-s)} + \mathcal{O}(1) \tag{3-28}$$

as $y\to s$ within the intersection domain. Let $\Gamma\subset\mathcal{D}$ be a smooth curve such that

$$\int_\Gamma \Omega(y,s)\mathrm{d}y \equiv 0 \tag{3-29}$$

Let $\widehat{\Gamma}\subset\widehat{\mathcal{D}}$ be a curve of finite length intersecting once Γ at p and oriented positively w.r.t. Γ: then

$$\int_\Gamma \mathrm{d}y\int_{\widehat{\Gamma}}\mathrm{d}s\Omega(y,s) = 2i\pi c(p) \tag{3-30}$$

Proof. The integral

$$f(s) := \int_\Gamma \Omega\mathrm{d}y \tag{3-31}$$

defines –in principle– different holomorphic functions in the connected components of $\widehat{\mathcal{D}}\setminus\Gamma$: the difference among them -however- is the residue

$$\operatorname*{res}_{y=s}\Omega(y,s)\mathrm{d}y \tag{3-32}$$

which is zero by the assumption on Ω. Hence the analytic continuations of $f(s)$ from one component to the other all coincide. In our case they are all zero. The key fact is that, since Ω is singular on the diagonal, the orders of integration matters (otherwise (3-30) would give zero by interchanging the order of integration).

We compute the integral as a limit of regular integrals where we can interchange the order of integration

$$(3\text{-}30) = \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\mathrm{d}y\int_{\widehat{\Gamma}}\mathrm{d}s\Omega(y,s)\ , \tag{3-33}$$

where Γ_ϵ is the curve (or union of curves) obtained by removing a small ϵ-arc (which we denote by Γ^ϵ, i.e. an arc from $p-\epsilon$ to $p+\epsilon$, where these two points lie on the curve Γ at distance $|\epsilon|$ from the intersection and the direction of ϵ is the same as the orientation of Γ) around the intersection point p. This allows us to interchange the order of integration under the limit sign

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\mathrm{d}y\int_{\widehat{\Gamma}}\mathrm{d}s\Omega(y,s) &= \lim_{\epsilon\to 0}\int_{\widehat{\Gamma}}\mathrm{d}s\int_{\Gamma_\epsilon}\mathrm{d}y\Omega(y,s) = -\lim_{\epsilon\to 0}\int_{\widehat{\Gamma}}\mathrm{d}s\int_{\Gamma^\epsilon}\mathrm{d}y\Omega(y,s) = \\ = -\lim_{\epsilon\to 0}\int_{\widehat{\Gamma}}\mathrm{d}s\int_{\Gamma^\epsilon}\mathrm{d}y&\left(\frac{c(p)}{(y-s)^2}+\mathcal{O}(1)\right) = -\lim_{\epsilon\to 0}\int_{\widehat{\Gamma}}\mathrm{d}s\int_{\Gamma^\epsilon}\mathrm{d}y\frac{c(p)}{(y-s)^2}\end{aligned} \tag{3-34}$$

where we have dropped the $\mathcal{O}(1)$ part since the length of $\widehat{\Gamma}$ is finite and that of Γ^ϵ tends to zero. In the last expression the inner integral is –strictly speaking– defined only for $s \neq p$: however on the "left" and "right" the result is the same and gives

$$
\begin{aligned}
&-\lim_{\epsilon\to 0}\int_{\widehat{\Gamma}} \mathrm{d}s \int_{p-\epsilon}^{p+\epsilon} \mathrm{d}y \frac{c(p)}{(y-s)^2} = c(p)\lim_{\epsilon\to 0}\int_{\widehat{\Gamma}} \mathrm{d}s \left(\frac{1}{b-p-\epsilon} - \frac{1}{b-p+\epsilon}\right) = \\
&= c(p)\lim_{\epsilon\to 0} \ln\left(\frac{b-p-\epsilon}{a-p-\epsilon}\right) - \ln\left(\frac{b-p+\epsilon}{a-p+\epsilon}\right) \qquad (3\text{-}35)
\end{aligned}
$$

In this last limit the logarithms appearing have different branches: in particular the second differ by $2i\pi$ from the first, hence the result follows by taking the limit. **Q.E.D.**

We now come back to the computation of the concomitant: first of all, since we know that the result is independent of x we set $x = 0$, so that we have to compute

$$
\mathcal{B}(f,g) := \int_{\widehat{\Gamma}}\int_{\Gamma}\left[-\frac{B(y)-B(s)}{(y-s)^2} - \frac{A(y)-A(s)-B'(s)}{y-s}\right]\mathrm{e}^{-V(y)+\hat{V}(s)} \qquad (3\text{-}36)
$$

We have already remarked that this integral can be computed in either orders and gives the same result. We express it in terms of

$$
\begin{aligned}
\mathcal{B}(f,g) &= (2) - (1) & (3\text{-}37)\\
(1) &:= \int_{\Gamma}\mathrm{d}y\int_{\widehat{\Gamma}}\mathrm{d}s\left[\frac{B(y)}{(y-s)^2} - \frac{A(y)}{y-s}\right]W(y)\widehat{W}(s) & (3\text{-}38)\\
(2) &:= \int_{\Gamma}\mathrm{d}y\int_{\widehat{\Gamma}}\mathrm{d}s\left[\frac{B(s)}{(y-s)^2} - \frac{A(s)+B'(s)}{y-s}\right]W(y)\widehat{W}(s) & (3\text{-}39)
\end{aligned}
$$

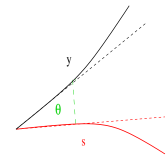
The integral (2) is zero because the inner integral w.r.t. s defines (for $y \notin \widehat{\Gamma}$) the identically zero function, as it is easily seen after an integration by parts. The integral (1) is computed using Lemma 3.3 after noticing that

$$
\Omega(y,s) := \left[\frac{B(y)}{(y-s)^2} - \frac{A(y)}{y-s}\right]W(y)\widehat{W}(s) = \frac{B(s)W(s)\widehat{W}(s)}{(y-s)^2} + \mathcal{O}(1)\ . \qquad (3\text{-}40)
$$

and hence satisfies the condition of the Lemma for Ω. The contour Γ satisfies the condition of the Lemma. The contour $\widehat{\Gamma}$ is not necessarily of finite length, but we can take only a small arc around the point of intersection and the remainder will be computed to be zero by interchanging the order of the integrals. To rigor one should also consider the common endpoints of contours like the petals and dual petals: it is easily seen, however that those points do not correspond to a singularity of the integrals (w.r.t. the arclength parameters) because of the fast decay of the weights W and $\widehat{W}$. For example, if the two contours $\Gamma, \widehat{\Gamma}$ form an angle $\theta \in [0+\epsilon, \pi-\epsilon]$ (asymptotically) near a point b

(where $W, \widehat{W}$ have an essential singularity) then

$$\left|\frac{W(y)\widehat{W}(s)}{(y-s)^2}\right| \leqslant \frac{\left|W(y)\widehat{W}(s)\right|}{\sin^2\theta |y-b|^2} \tag{3-41}$$



which is still jointly integrable w.r.t. the arc lengths (recall that the directions of approach of Γ and $\widehat{\Gamma}$ are such that the weights tend to zero faster than any power of the local coordinate).

It is then clear that if Γ $\widehat{\Gamma}$ are a circle and a semi-infinite arc (or vice-versa) the bilinear concomitant for the corresponding dual solutions is a nonzero constant (which depends on the choices of the branches of W and $\widehat{W}$). This is immediate for a pair of contours which intersect only once. For a pair of lassoes (which intersect twice and with opposite orientations), calling p_1, p_2 the points of intersection we have

$$\mathcal{B}(f_\Gamma, g_{\widehat{\Gamma}}) = \pm(W(p_1)\widehat{W}(p_1)B(p_1) - W(p_2)\widehat{W}(p_2)B(p_2)) \tag{3-42}$$

Since the local behavior at the singularity embraced by the lassoes is a noninteger power, let's say $(y-c)^t$, then the values of $BW\widehat{W}$ on the two intersection points (which lie on different sizes of the union of the cuts for W and $\widehat{W}$) satisfies

$$W(p_1)\widehat{W}(p_1)B(p_1) = \mathrm{e}^{2i\pi t}W(p_2)\widehat{W}(p_2)B(p_2) \tag{3-43}$$

so that

$$\mathcal{B}(f_\Gamma, g_{\widehat{\Gamma}}) = \pm(W(p_1)\widehat{W}(p_1)B(p_1)(1-\mathrm{e}^{2i\pi t}) \neq 0 \tag{3-44}$$

For dual flowers it is convenient to choose different paths for the dual contours as shown in Fig. 2, where the petals have been replaced by stems using a linear combination of the contour-integrals of the same petals and stem. It is easy to realize that the sub-block of the concomitant involving these contours is nondegenerate, since it can be given a diagonal form with nonzero entries on the diagonal. The precise values are not important since we are free to rescale each solution f_Γ and g_Γ. Summarizing we have proved that

Proposition 3.1 *There is a normalization of the integrals f_Γ and $g_{\widehat{\Gamma}}$ such that the bilinear concomitant is precisely the intersection pairing of the contours Γ and $\widehat{\Gamma}$. With appropriate choice and labeling of the contours the pairing is represented by the identity matrix.*

4 Auxiliary wave vectors

Caveat In this section we will make statements concerning the biorthogonal polynomials p_n, s_n and the corresponding quasipolynomials ψ_n, ϕ_n. It will be understood that

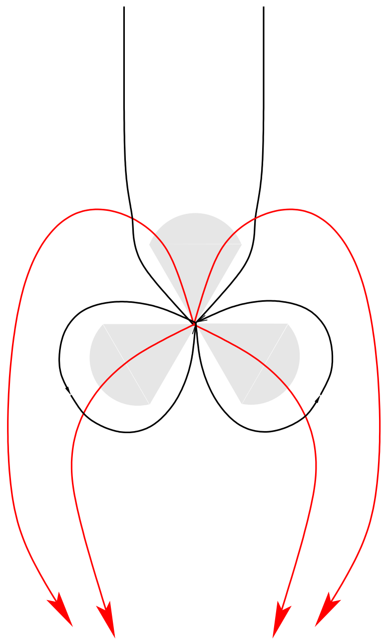


Figure 2: The equivalent choice of contours for the dual admissible petals.

1. Any statement made on the ψ_n's and the Fourier–Laplace transforms of the ϕ_n's admits a specular statement for the ϕ_n's and the F-L transforms of the ψ_n's.

2. Any statement made on the ψ_n's admits an analog statement for the $\widehat{\psi}_n$'s and $\check{\psi}_n$'s by replacing the moment functional $\mathcal{L}$ with $\widehat{\mathcal{L}}$ or $\check{\mathcal{L}}$, and specular statements for $\widehat{\phi}_n, \check{\phi}_n$.

Consider the functions

$$\mathcal{B}_2(x;y,s) := \left(\frac{B_2(y)-B_2(s)}{y-s}\left(x-\frac{1}{y-s}\right)-\frac{A_2(y)-B_2'(s)-A_2(s)}{y-s}\right) \tag{4-1}$$

$$\psi_n^{(\widehat{\Gamma})} := \frac{1}{2i\pi}\int_{\widehat{\Gamma}}\mathrm{d}s\iint_{\varkappa}\mathrm{d}\xi\mathrm{d}y\mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y-xs-V_2(y)+\widehat{V}_2(s)}\frac{\psi_n(\xi)}{x-\xi} \tag{4-2}$$

If x belongs to a contour $\Gamma_{x,\mu}$ of the integration $\int_{\varkappa}$ we obtain

$$\psi_n^{(\widehat{\Gamma})}(x)_+ = \psi_n^{(\widehat{\Gamma})}(x)_- + \sum_{\nu}\mathcal{B}_2(\widehat{\Gamma},\Gamma_{y,\nu})\varkappa_{\mu,\nu}\psi_n(\xi) \tag{4-3}$$

where the subscript $x_\pm$ denotes the boundary values from the left/right and $\mathcal{B}_2(\widehat{\Gamma},\Gamma_{y,\nu})$ stands for the constant (in x) bilinear concomitant

$$\mathcal{B}_2(\widehat{\Gamma},\Gamma_{y,\nu}) := \frac{1}{2i\pi}\int_{\widehat{\Gamma}}\mathrm{d}s\int_{\Gamma_{y,\nu}}\mathrm{d}y\mathcal{B}_2(x;y,s)\mathrm{e}^{\widehat{V}_2(s)-V_2(y)+x(y-s)}\ . \tag{4-4}$$

Therefore their jump across the contours of discontinuity is a constant multiple of $\psi_n(x)$.

We have

Proposition 4.1 *The sequences of functions $\{\psi_n^{(\widehat{\Gamma})}\}_{n\in\mathbb{N}}$ satisfy the same recurrence relations (for n large enough) as the quasipolynomials ψ_n*

$$x\left(\psi_n^{(\widehat{\Gamma})}+\sum_{j=1}^{q_2}\ell_j(n)\psi_{n-j}^{(\widehat{\Gamma})}\right)=\sum_{-1}^{d_2}\alpha_j(n)\psi_{n-j}^{(\widehat{\Gamma})}\ ,\qquad n\geqslant d_2+q_2 \tag{4-5}$$

$$\partial_x\left(\psi_n^{(\widehat{\Gamma})}+\sum_{j=1}^{q_1}\breve{m}_j(n+j)\psi_{n+j}^{(\widehat{\Gamma})}\right)=\sum_{-1}^{d_1}\breve{\beta}_j(n+j)\psi_{n+j}^{(\widehat{\Gamma})}\ ,\qquad n\geqslant 1 \tag{4-6}$$

(For the proof see App. A.1)

Definition 4.1 *Beside the wave vector $\underset{\infty}{\Psi}$ we define the following d_2 auxiliary wave-vectors*

$$\underset{\infty}{\Psi}^{(\nu)}(x):=\frac{1}{2i\pi}\int_{\widehat{\Gamma}_{y,\nu}}\mathrm{d}s\iint_{\varkappa}\mathrm{d}\xi\mathrm{d}y\mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y-xs-V_2(y)+\widehat{V}_2(s)}\frac{1}{x-\xi}\underset{\infty}{\Psi}(\xi)\ ,\ \nu=1,\ldots,d_2 \tag{4-7}$$

$$\underset{\infty}{\Psi}^{(0)}(x):=\underset{\infty}{\Psi}(x)\ . \tag{4-8}$$

We also define the **dual wave vectors**

$$\underset{\infty}{\Phi}^{(0)}(x):=\mathrm{e}^{V_1(x)}\iint_{\varkappa}\mathrm{e}^{\xi y-V_1(\xi)}\frac{1}{x-\xi}\underset{\infty}{\Phi}(y) \tag{4-9}$$

$$\underset{\infty}{\Phi}^{(\mu)}(x):=\int_{\Gamma_{y,\nu}}\mathrm{d}y\,\mathrm{e}^{xy}\underset{\infty}{\Phi}(y)\ ,\nu=1,\ldots,d_2 \tag{4-10}$$

Proposition 4.2 *The components of the dual wave vectors satisfy the recurrence relations*

$$x\left(\underline{\phi}_n^{(\nu)}+\sum_{j=1}^{q_2}\widehat{\ell}_j(n+j)\underline{\phi}_{n+j}^{(\nu)}\right)=\sum_{j=-1}^{d_2}\widehat{\alpha}_j(n+j)\underline{\phi}_{n+j}^{(\nu)}+\delta_{\nu0}\delta_{n0}\sqrt{h_0}\mathrm{e}^{V_1(x)}\ ,\quad \nu=0,\ldots,d_2 \tag{4-11}$$

$$\partial_x\left(\underline{\phi}_n^{(\nu)}+\sum_{j=1}^{q_1}m_j(n)\underline{\phi}_{n-j}^{(\nu)}\right)=\sum_{j=-1}^{d_1}\beta_j(n)\underline{\phi}_{n-j}^{(\nu)}\ ,\nu=1,\ldots,d_2\ . \tag{4-12}$$

Remark 4.1 *The wave vector $\underset{\infty}{\Phi}^{(0)}$ does not satisfy a finite-term differential recurrence relation: a formula can be derived but it is not useful for our purposes.*

Proof The formulæ for the Fourier–Laplace transforms follow from integration by parts from the relations satisfied by $\phi_n(y)$ (Prop. 2.3). We only point out that integration by parts does not give any boundary contribution because $s_n+\sum\widehat{\ell}_j(n+j)s_{n+j}(y)$ is divisible by $B_2(y)$ and hence vanishes at the hard-edge end-points.

The only relation that needs to be checked is the multiplicative relation for $\nu = 0$. Denoting temporarily by a tilde the linear combination

$$\widetilde{\phi}_n := \phi_n + \sum_1^{q_1} \widehat{\ell}_j(n+j)\phi_{n+j} \ , \tag{4-13}$$

we have

$$\begin{aligned} x\widetilde{\underline{\phi}}^{(0)}(x) =& \mathrm{e}^{V_1(x)} \iint_{\varkappa} \mathrm{e}^{\xi y - V_1(\xi)} \frac{x}{x-\xi} \widetilde{\phi}_n(y) = \\ =& \mathrm{e}^{V_1(x)} \iint_{\varkappa} \mathrm{e}^{\xi y - V_1(\xi)} \widetilde{\phi}_n(y) + \mathrm{e}^{V_1(x)} \iint_{\varkappa} \mathrm{e}^{\xi y - V_1(\xi)} \frac{\xi}{x-\xi} \widetilde{\phi}_n(y) = \\ =& \mathrm{e}^{V_1(x)} \delta_{n0}\sqrt{h_0} + \mathrm{e}^{V_1(x)} \iint_{\varkappa} \mathrm{e}^{\xi y - V_1(\xi)} \frac{-\partial_y}{x-\xi} \widetilde{\phi}_n(y) = \\ =& \mathrm{e}^{V_1(x)} \delta_{n0}\sqrt{h_0} + \sum_{j=-1}^{d_2} \widehat{\alpha}_j(n+j) \underline{\phi}^{(0)}_{n+j} \ . \quad \mathbf{Q.E.D.} \end{aligned} \tag{4-14}$$

4.1 Christoffel–Darboux identities

In the general theory of the two–matrix model the following kernels plays an essential rôle in the computation of statistical correlation functions

$$K_{12}^N(x,y) := \sum_{j=0}^{N-1} p_j(x) s_j(y) \mathrm{e}^{-V_1(x)-V_2(y)} = \sum_{j=0}^{N-1} \psi_j(x)\phi_j(y) \ . \tag{4-15}$$

In a previous paper by the author and collaborators [3, 5] the case of polynomial potentials V_i was considered (without hard-edges) and it was of capital importance the existence of a Christoffel–Darboux identity allowing to express K_{12}^N (or rather some transform of it) in terms of bilinear combinations of the BOPs involving only a number of BOPs depending only on the degrees of the potentials.

We look for a similar bilinear expression in this model.

Definition 4.2 *We define the* **windows** *of the wave vectors* $\underset{\infty}{\Psi}^{(\mu)}$ *and* $\underset{\infty}{\Phi}^{(\mu)}$, $\mu = 0, \dots, d_2$

$$\underline{\Phi}_n^{(\mu)}(x) := [\underline{\phi}^{(\mu)}_{n-1}, \dots, \underline{\phi}^{(\mu)}_{n+d_2-1}] \ , \qquad \Psi_n^{(\mu)}(x) := [\psi^{(\mu)}_{n-d_2}, \dots, \psi^{(\mu)}_n]^t \ . \tag{4-16}$$

We rewrite (4-15) in terms of the wave vectors

$$K_{12}^N = \underset{\infty}{\Phi}^t(y) \Pi_N \underset{\infty}{\Psi}(x) \ , \qquad \Pi_N := \left\{ \begin{array}{c} \delta_{ij} \ , \ 0 \leqslant i \leqslant N-1 \\ 0 \ \text{ otherwise} \ . \end{array} \right. \tag{4-17}$$

Recall the multiplicative and differential recurrence relations in Prop. 2.1 and Prop. 2.3 (which we rewrite here for the reader's convenience)

$$\partial_y \underset{\infty}{\Phi}^t(\mathbf{1}+\widehat{L}) = -\underset{\infty}{\Phi}^t\widehat{A}\ , \qquad x(\mathbf{1}+L)\underset{\infty}{\Psi} = A\underset{\infty}{\Psi}$$
$$(\mathbf{1}+L)^{-1}A = \widehat{A}(\mathbf{1}+\widehat{L})^{-1} =: Q\ .$$

Consider now the following expressions

$$\begin{aligned}
&(x+\partial_y)\underset{\infty}{\Phi}^t(y)(\mathbf{1}+\widehat{L})\Pi(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi}(x) = \\
&\quad = \underset{\infty}{\Phi}^t(y)(\mathbf{1}+\widehat{L})\Pi(\mathbf{1}+\widehat{L})^{-1}\widehat{A}(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi}(x) - \underset{\infty}{\Phi}^t(y)\widehat{A}\Pi(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi}(x) = \\
&\quad = \underset{\infty}{\Phi}^t\Pi\widehat{A}(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi} + \underset{\infty}{\Phi}[\widehat{L},\Pi](\mathbf{1}+\widehat{L})^{-1}\widehat{A}(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi} + \\
&\qquad\quad - \underset{\infty}{\Phi}^t\Pi\widehat{A}(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi} - \underset{\infty}{\Phi}^t[\widehat{A},\Pi](\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi} = \\
&\quad = \underset{\infty}{\Phi}[\widehat{L},\Pi]\widehat{Q}(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi} - \underset{\infty}{\Phi}^t[\widehat{A},\Pi](\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi} = \\
&\quad = \underset{\infty}{\Phi}[\widehat{L},\Pi]\widehat{Q}\underset{\infty}{\widehat{\Psi}} - \underset{\infty}{\Phi}^t[\widehat{A},\Pi]\underset{\infty}{\widehat{\Psi}}
\end{aligned} \tag{4-18}$$

where we have set $\widehat{Q} := (\mathbf{1}+\widehat{L})^{-1}\widehat{A}$. We now use the fact that $\widehat{Q}$ is the recurrence matrix for the associated $\underset{\infty}{\widehat{\Psi}}$ wave vector (see Prop. 2.2 where $\underset{\infty}{\widehat{\Psi}} := \widehat{\mathbf{p}}\mathrm{e}^{-V_1(x)}$) and obtain

$$(x+\partial_y)\underset{\infty}{\Phi}^t(y)(\mathbf{1}+\widehat{L})\Pi(\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi}(x) = \underset{\infty}{\Phi}(y)[x\widehat{L}-\widehat{A},\Pi]\underset{\infty}{\widehat{\Psi}}(x) \tag{4-19}$$

$$\underset{\infty}{\widehat{\Psi}} = (\mathbf{1}+\widehat{L})^{-1}\underset{\infty}{\Psi} = \widehat{\mathbf{p}}(x)\mathrm{e}^{-V_1(x)} \tag{4-20}$$

$$\underset{\infty}{\widehat{\Phi}} = (\mathbf{1}+\widehat{L}^t)\underset{\infty}{\Phi} = \widehat{\mathbf{s}}(y)B_2\mathrm{e}^{-V_2(y)} = \widehat{\mathbf{s}}(y)\mathrm{e}^{-\widehat{V}_2(y)} \tag{4-21}$$

With these notation we have

Theorem 4.1 (Christoffel–Darboux identity) *For the kernels*

$$\widehat{K}_{11}^{N,\nu}(x,x') = \int_{\Gamma_{y,\nu}} \mathrm{e}^{xy}\underset{\infty}{\widehat{\Phi}}^t(y)\Pi_N\underset{\infty}{\widehat{\Psi}}(x') = \underset{\infty}{\widehat{\Phi}}^{(j)}(x)^t\,\Pi_N\underset{\infty}{\widehat{\Psi}}(x')\ , \tag{4-22}$$

$$K_{11}^{N,\nu}(x,x') = \int_{\Gamma_{y,\nu}} \mathrm{e}^{xy}\underset{\infty}{\Phi}^t(y)\Pi_N\underset{\infty}{\Psi}(x') = \underset{\infty}{\Phi}^{(j)}(x)^t\,\Pi_N\underset{\infty}{\Psi}(x')\ , \qquad j=1,\dots,d_2 \tag{4-23}$$

we have the identities

$$(x'-x)\widehat{K}_{11}^{N,j}(x',x) = \underset{\infty}{\Phi}^{(j)}(x')^t\,\underset{N}{\widehat{\mathbb{A}}}(x)\underset{\infty}{\widehat{\Psi}}(x) \tag{4-24}$$

$$(x'-x)K_{11}^{N,j}(x',x) = \underset{\infty}{\Phi}^{(j)}(x')^t\,\underset{N}{\widehat{\mathbb{A}}}(x')\underset{\infty}{\widehat{\Psi}}(x)\ . \tag{4-25}$$

(note the argument of $\widehat{\mathbb{A}}_N$ in the two formulæ) where $\widehat{\mathbb{A}}_N(x) := \left[\widehat{A}-x\widehat{L},\Pi_N\right]$.

Proof The identity for $\widehat{K}_{11}^{N,j}(x,x')$ follows by performing integration by parts on (4-19) and noticing that the boundary contributions vanish since $\widehat{\Phi}(y) = B_2(y)\widehat{\mathrm{s}}(y)\mathrm{e}^{-V_2(y)}$ and $B_2(y)$ vanishes at the hard-edges. The identity for $K_{11}^{N,j}(x,x')$ follows from the one for $\widehat{K}_{11}^{N,j}$ and this manipulation

$$
\begin{aligned}
(x'-x)\underset{\infty}{\underline{\widehat{\Phi}}}^{(j)}(x')^t\Pi\underset{\infty}{\widehat{\Psi}}(x) &= (x'-x)\underset{\infty}{\underline{\Phi}}^{(j)}(x')^t(\mathbf{1}+\widehat{L})\Pi\underset{\infty}{\widehat{\Psi}}(x) = \\
&= (x'-x)\left(\underset{\infty}{\underline{\Phi}}^{(j)}(x')^t[\widehat{L},\Pi]\underset{\infty}{\widehat{\Psi}}(x) + \underset{\infty}{\underline{\Phi}}^{(j)}(x')^t\Pi\underset{\infty}{\Psi}(x)\right) = \\
&= (x'-x)K_{11}^{N,j}(x',x) + (x'-x)\underset{\infty}{\underline{\Phi}}^{(j)}(x')^t[\widehat{L},\Pi]\underset{\infty}{\widehat{\Psi}}(x)
\end{aligned}
\tag{4-26}
$$

so that

$$
\begin{aligned}
&(x'-x)K_{11}^{N,j}(x',x) = \\
&= (x'-x)\widehat{K}_{11}^{N,j}(x',x) - (x'-x)\underset{\infty}{\underline{\Phi}}^{(j),t}(x')[\widehat{L},\Pi]\underset{\infty}{\widehat{\Psi}}(x) = \underset{\infty}{\underline{\Phi}}^{(j),t}(x')\underset{N}{\widehat{\mathbb{A}}}(x')\underset{\infty}{\widehat{\Psi}}(x)
\end{aligned}
\tag{4-27}
$$

Q.E.D.

Note that –with a slight abuse of notation– in the RHS of the CDIs we can replace the wave vectors $\underset{\infty}{\underline{\Phi}}$ by the corresponding window $\underline{\Phi}_n$ since the matrix $\widehat{\mathbb{A}}_n$ has a nonzero square block of size d_2+1 with top-right corner in the $(n-1,n)$ entry, and hence the bilinear expression $\underset{\infty}{\underline{\Phi}}\widehat{\mathbb{A}}\underset{\infty}{\Psi}$ only involves the terms in the dual windows $\underline{\Phi}_n$ and $\widehat{\Psi}_n$. We will denote from now on by $\widehat{\mathbb{A}}$ only the d_2+1 square matrix which is relevant to the pairing.

The importance of the theorem is that we can express the kernel K_{11} in terms of the dual quantities $\underline{\phi}_n(x)$ and $\widehat{\psi}_n(x')$ involving only the indexes $N-d_2 \leqslant n \leqslant N$.

Note, however, that we must introduce the orthogonal polynomials $\widehat{\mathbf{p}}$ for the associated moment functional $\widehat{\mathcal{L}}$ in order to find a Christoffel–Darboux relation similar to the standard one for orthogonal polynomials.

Theorem 4.2 (Auxiliary CDIs) *The auxiliary wave vectors enter in the following auxiliary Christoffel–Darboux identities*

$$
\begin{aligned}
\textbf{(a)}\ \ &(z-x)\underset{\infty}{\underline{\Phi}}^{(0)}(z)^t\Pi_n\underset{\infty}{\Psi}^{(0)}(x) = \underline{\Phi}_n^{(0)}(z)\widehat{\mathbb{A}}(z)\widehat{\Psi}_n(x) + \mathrm{e}^{V_1(z)-V_1(x)} \\
&(z-x)\underset{\infty}{\underline{\widehat{\Phi}}}^{(0)}(z)^t\Pi_n\underset{\infty}{\widehat{\Psi}}^{(0)}(x) = \underline{\Phi}_n^{(0)}(z)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(x) + \mathrm{e}^{V_1(z)-V_1(x)}
\end{aligned}
\tag{4-28}
$$

$$
\begin{aligned}
\textbf{(b)}\ \ &(z-x)\underset{\infty}{\underline{\Phi}}^{(j)}(z)^t\Pi_n\underset{\infty}{\Psi}^{(k)}(x) = \underline{\Phi}_n^{(j)}(z)\widehat{\mathbb{A}}(z)\widehat{\Psi}_n(x) - \frac{1}{2i\pi}\int_{\Gamma_{y,\nu}}\int_{\widehat{\Gamma}_k}\mathcal{B}_2(x;y,s)\mathrm{e}^{yz-xs+\widehat{V}_2(s)-V_2(y)}\ , \\
&(z-x)\underset{\infty}{\underline{\widehat{\Phi}}}^{(j)}(z)^t\Pi_n\underset{\infty}{\widehat{\Psi}}^{(k)}(x) = \underline{\Phi}_n^{(j)}(z)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(x) - \frac{1}{2i\pi}\int_{\Gamma_{y,\nu}}\int_{\widehat{\Gamma}_k}\mathcal{B}_2(x;y,s)\mathrm{e}^{yz-xs+\widehat{V}_2(s)-V_2(y)}\ , \\
&\qquad j,k = 1,\dots,d_2\ .
\end{aligned}
\tag{4-29}
$$

(For the proof see App. A.2).

4.2 Ladder matrices

In this section we derive an expression for the ODE satisfied by the polynomials in terms of the so-called "folding" (see [3]). This will have certain advantages when explaining the relations between the various ODEs that naturally appear in the problem: a different explicit representation of the ODE will be given in the next section as well, using a completely different approach based upon the explicit integral representations of the wave vectors and on the duality provided by the Christoffel–Darboux pairing.

We first have the simple lemma

Lemma 4.1 (Ladder matrices) *The multiplicative recurrence relations for the wave vectors* $\underset{\infty}{\Psi}^{(0)}, \underset{\infty}{\underline{\Phi}} = \underset{\infty}{\underline{\Phi}}^{(j)}$ *(*$j = 1, \ldots d_2$*)*

$$x(\mathbf{1} + L)\underset{\infty}{\Psi}^{(0)} = A\underset{\infty}{\Psi}^{(0)} , \qquad x(\mathbf{1} + \widehat{L}^t)\underset{\infty}{\underline{\Phi}}^{(j)} = \widehat{A}^t\underset{\infty}{\underline{\Phi}}^{(j)} \tag{4-30}$$

are equivalent to the relations

$$\Psi_{n+1}^{(0)}(x) = \mathbf{a}_n(x)\Psi_n^{(0)}(x) , \tag{4-31}$$

$$\underline{\Phi}_n^{(j)}(x) = \underline{\Phi}_{n+1}^{(j)}(x)\underline{\widehat{\mathbf{a}}}_n(x) \tag{4-32}$$

where

$$\mathbf{a}_n(x) = \Lambda - \frac{1}{\alpha_{-1}(n)}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}[\alpha_{d_2}(n), \ldots, \alpha_0(n)] + \frac{x}{\alpha_{-1}(n)}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}[0, \ldots, \ell_{q_2}(n), \ldots, \ell_1(n), 1] \tag{4-33}$$

$$\underline{\widehat{\mathbf{a}}}_n(x) = \Lambda - \frac{1}{\widehat{\alpha}_{-1}(n-1)}\begin{bmatrix} \widehat{\alpha}_0(n) \\ \widehat{\alpha}_1(n+1) \\ \vdots \\ \widehat{\alpha}_{d_2}(n+d_2) \end{bmatrix}[1, 0, \ldots, 0] + \frac{x}{\widehat{\alpha}_{-1}(n-1)}\begin{bmatrix} 1 \\ \widehat{\ell}_1(n+1) \\ \vdots \\ \widehat{\ell}_{q_2}(n+q_2) \\ 0 \\ \vdots \end{bmatrix}[1, 0, \ldots, 0] \tag{4-34}$$

and Λ *denotes the upper shift matrix (of size* $d_2 + 1$*). The relations (4-31) and (4-32) hold also for the other sequences of windows* $\Psi_n^{(j)}$ *and* $\underline{\Phi}_n^{(0)}$ *provided that* $n \geqslant d_2 + q_2$ *(*$n \geqslant 1$ *respectively).*

Proof. The proof follows immediately from the recurrence relations for the wave vectors $\underset{\infty}{\Psi}^{(0)}$ (the quasipolynomials) and $\underset{\infty}{\underline{\Phi}}^{(j)}$ (the Fourier–Laplace transforms) by solving for $\psi_{n+1}(x)$ (or $\underline{\phi}_{n-1}$) in terms of $\psi_{n-d_2}, \ldots, \psi_n$ ($\underline{\phi}_n, \ldots, \underline{\phi}_{n+d_2}$) and rewriting the relation in matrix form. The statement for the other sequences of windows follows from the fact that the corresponding wave vectors satisfy the same finite-term recurrence relations in the specified range (see Prop. 4.1 and Prop. 4.2). **Q.E.D.**

Lemma 4.2 (Folded recursion relations) *The differential recurrence relations for the wave-vector* $\underset{\infty}{\Psi}$

$$\partial_x(\mathbf{1} + \widetilde{M}^t)\underset{\infty}{\Psi} = -\breve{B}^t \underset{\infty}{\Psi} \tag{4-35}$$

are equivalent to the relations

$$\partial_x\left(\widetilde{\mathcal{M}}_n(x)\Psi_n\right) = -\breve{\mathcal{B}}_n(x)\Psi_n \tag{4-36}$$

$$\widetilde{\mathcal{M}}_n := \mathbf{1} + \sum_{j=1}^{q_1} \widetilde{\boldsymbol{m}}_j(n)\mathbf{a}_n \cdots \mathbf{a}_{n+j-1} \tag{4-37}$$

$$\widetilde{\boldsymbol{m}}_j(n) := \mathrm{diag}(\breve{m}_j(n+j-d_2), \ldots, \breve{m}_j(n+j)) \tag{4-38}$$

$$\breve{\mathcal{B}}_n := \breve{\boldsymbol{\beta}}_{-1}(n)(\mathbf{a}_{n-1})^{-1} + \breve{\boldsymbol{\beta}}_0(n) + \sum_{j=1}^{d_1} \breve{\boldsymbol{\beta}}_j(n)\mathbf{a}_n \cdots \mathbf{a}_{n+j-1} \tag{4-39}$$

$$\breve{\boldsymbol{\beta}}_j(n) := \mathrm{diag}(\breve{\beta}_j(n+j-d_2), \ldots, \breve{\beta}_j(n+j)) \tag{4-40}$$

Proof. The formula is an iterated application of the ladder recurrence relations (on a window of consecutive elements with indexes $n-d_2, \ldots, n$) to the differential recurrence relation for the wave vector (see [3] for more details). **Q.E.D.**

Remark 4.2 *A completely analogous statement can be derived for the windows of the dual vector* $\underline{\Phi}_n^{(j)}$*,* $j = 1, \ldots, d_2$*.*

Remark 4.3 *The matrices* $\mathbf{a}_n$ *have a companion-form and are* **invertible** *since the determinant is* $-\frac{\alpha_{d_2}(n)}{\alpha_{-1}(n)}$ *which has been proved nonvanishing in Thm. 2.1. Moreover the inverse is also linear in* x *(***Exercise***).*

Remark 4.4 *By the very definition* $\widetilde{\mathcal{M}}_n(x)\Psi_n = \breve{\Psi}_n$ *is the window of quasipolynomial (and associated functions) for the moment functional* $\breve{\mathcal{L}}$*.*

Corollary 4.1 *The* $d_2 + 1$ *columns provided by the windows of the auxiliary wave vectors* $\underset{\infty}{\Psi}^{(j)}(x)$ *provide a fundamental system for the ODE (4-36) for* $n \geqslant d_2 + q_2$*.*

Proof. From Prop. 4.1 we know that the components of the auxiliary wave vectors satisfy the same recurrence relations (both multiplicative and differential) as the quasipolynomials provided n is large enough. Moreover the recurrence relations always involve a fixed number of terms with indexes "around n": since the derivation of the ODE is entirely based on the recurrence relations the statement follows. **Q.E.D.**

Proposition 4.3 *The determinant of $\breve{\mathcal{M}}_n(x)$ is proportional to $B_1(x)$ by a nonzero constant.*

Proof. Consider the window of polynomials $\mathbf{p}_n := [p_{n-d_2}, \ldots, p_n]^t$: from the definition of the matrix $\breve{M}$ it follows that

$$\breve{\mathcal{M}}_n(x)\mathbf{p}_n(x) = B_1(x)\breve{\mathbf{p}}_n(x) \tag{4-41}$$

We first prove that $\det \breve{\mathcal{M}}_n$ (which is *a fortiori* a polynomial) is divisible by B_1. Let c be a zero of B_1 of multiplicity r: at least one component (say the ℓ-th) of $\mathbf{p}_n(c)$ is nonzero because of the very genericity assumption which guarantees the existence of $\breve{M}$ (2-38). Let $E(x)$ be the matrix obtained by replacing the ℓ-th column of the identity with $\mathbf{p}_n(x)$. Clearly $\det E(x)$ is nonzero in a neighborhood of $x = c$ by our definition of ℓ. It follows that the ℓ-th column of $\breve{\mathcal{M}}_n E$ is precisely $B_1\breve{\mathbf{p}}_n$ and hence each component vanishes at c of order r. Also

$$\det \breve{\mathcal{M}}_n E = p_{n-d_2+\ell-1}(x) \det \breve{\mathcal{M}}_n \tag{4-42}$$

and $p_{n-d_2-1+\ell}(c) \neq 0$. On the other hand $\det \breve{\mathcal{M}}_n E$ must vanish at $x = c$ of order r since the whole ℓ-th column does. Repeating this for all roots of B_1 we find the assertion of divisibility of $\det \breve{\mathcal{M}}_n$ by $B_1(x)$.

On the other hand, using a technique of evaluation of determinants used in [3],

$$\det \breve{\mathcal{M}}_n = \det \left[\mathbf{1}_{(d_2+1)(q_1+1)} - \left[\begin{array}{c|c|c|c} & \mathbf{a}_{n+q_1} & & \\ \hline & & \ddots & \\ \hline & & & \mathbf{a}_n \\ \hline \breve{\boldsymbol{m}}_{q_1}(n) & \cdots & \breve{\boldsymbol{m}}_1(n) & 0 \end{array} \right] \right] \tag{4-43}$$

Considering carefully the structure of the sparse matrix in the last identity, one realizes that the highest power in x is

$$\det \breve{\mathcal{M}}_n = x^{q_1} \frac{\breve{m}_{q_1}(n+q_1)}{\prod_{j=1}^{q_1} \alpha_{-1}(n+j)} + \mathcal{O}(x^{q_1-1}) \tag{4-44}$$

This shows that (since the coefficient does not vanish as per (2-17,2-48)) the determinant is of degree $q_1 = \deg B_1$; since it must be also divisible by B_1, this concludes the proof. **Q.E.D.**

Corollary 4.2 *The windows $\Psi_n,\ \check{\Psi}_n$ satisfy*

$$\partial_x \Psi_n = -\widetilde{\mathcal{M}}_n^{-1}\left(\check{\mathcal{B}}_n + \partial_x \widetilde{\mathcal{M}}_n\right)\Psi_n \tag{4-45}$$

$$\partial_x \check{\Psi}_n = -\check{\mathcal{B}}_n \widetilde{\mathcal{M}}_n^{-1}\check{\Psi}_n \tag{4-46}$$

where $\check{\mathcal{B}}_n, \widetilde{\mathcal{M}}_n$ are defined in ((4-37)–(4-40)). The ODEs have the same singularity structure as V_1'.

The first relation follows from (4-36) and the second from the fact that $\widetilde{\mathcal{M}}_n(x)\Psi_n(x) = \widehat{\Psi}_n(x)$.

This shows that the ODE's for Ψ_n and $\check{\Psi}_n$ are gauge-equivalent, the gauge being provided by the (polynomial) matrix $\widetilde{\mathcal{M}}_n$. Moreover formula (4-46) together with Prop. 4.3 shows that the singularities of the differential equation are at the zeroes of $B_1(x)$.

4.3 Differential equations for the dual pair of systems

In this section we present an explicit formula for the ODE satisfied by the dual pair of fundamental systems, in particular the polynomials $\widehat{\psi}_n$ and the Fourier–Laplace transforms $\underline{\phi}_n$'s. The result generalizes those of [4] but the method of derivation is similar to the one adopted in [5], with additional complications deriving from the presence of boundary contributions in the integration by parts at various steps of the derivation.

Notation. In the proof of this and the following theorems we will encounter integrations by parts that yield nonzero boundary contributions. Typically we will encounter integrals of the form

$$\iint_{\varkappa} y \mathrm{e}^{\rho y - V_1(\rho)} F(\rho)\phi_m(y)\mathrm{d}y\mathrm{d}\rho\ , \tag{4-47}$$

where $F(\rho)$ is some expression (typically polynomial or rational in ρ) possibly depending on "external" variables. If we attempt an integration by parts on the term $y\mathrm{e}^{y\rho} = \partial_\rho \mathrm{e}^{y\rho}$, we obtain a certain number of boundary terms. In all cases they will be boundary evaluation on the various contours $\Gamma_{x,\mu}$; it is the nature of all these integrals that only the contours emanating from a **hard–edge** point give a contribution, due to the fast decay of $\mathrm{e}^{-V_1(\rho)}$ at all the boundary points of the other contours. In the above example and in all minute detail, we have

$$\iint_{\varkappa} y \mathrm{e}^{\rho y - V_1(\rho)} F(\rho)\phi_m(y) = -\iint_{\varkappa} \mathrm{e}^{\rho y - V_1(\rho)}(-\partial_\rho + V_1'(\rho))F(\rho)\phi_m(y) + (\text{Boundary terms})$$

$$(\text{Boundary terms}) = \sum_{\mu=1}^{d_1} \mathrm{e}^{-V_1(\rho)}F(\rho)\sum_{\nu=1}^{d_2}\varkappa_{\mu,\nu}\int_{\Gamma_{y,\nu}} \mathrm{e}^{\rho y}\phi_m(y)\Big|_{\rho\in\partial\Gamma_{x,\mu}} \tag{4-48}$$

The evaluation at the boundary points of the various contours $\Gamma_{x,\mu}$ is clearly to be understood as limits along the contours; the decay of $\mathrm{e}^{-V_1(\rho)}$ along the contours gives zero contributions except for the hard–edge contours, at the (finite) boundary of which $V_1(\rho)$ is regular. In order to economize on space, we introduce the following shorthand notation for the above boundary terms

$$F(\rho)\mathrm{e}^{-V_1(\rho)}\underline{\phi}^{(\varkappa)}(\rho)\bigg|_{\rho\in\partial_x\varkappa} := \text{(Boundary terms)} \tag{4-49}$$

Theorem 4.3 *The dual fundamental system.*

$$\underline{\mathbf{\Phi}}_n(x) := \left[\begin{array}{c}\Phi_n^{(0)}\\ \vdots\\ \Phi_n^{(d_2)}\end{array}\right] = \left[\begin{array}{cccc}\phi_{n-1}^{(0)} & \phi_n^{(0)} & \dots & \phi_{n+d_2-1}^{(d_2)}\\ \hline \phi_{n-1}^{(1)} & \phi_n^{(1)} & \dots & \phi_{n+d_2-1}^{(d_2)}\\ \vdots & & & \vdots\\ \phi_{n-1}^{(d_2)} & \phi_n^{(d_2)} & \dots & \phi_{n+d_2-1}^{(d_2)}\end{array}\right] \tag{4-50}$$

satisfies the ODE

$$\underline{\mathbf{\Phi}}_n^{-1}(x)\underline{\mathbf{\Phi}}_n'(x) = \left[\begin{array}{ccccc}V_1'(x) & 0 & \dots & & 0\\ P_{n,n-1} & P_{n,n} & \dots & & P_{n,n+d_2-1}\\ 0 & P_{n+1,n} & & & \vdots\\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & P_{n+d_2,n+d_2-1} & P_{n+d_2-1,n+d_2-1}\end{array}\right] +$$

$$+\,\mathrm{diag}(P_{n+d_2,n-1},\dots,P_{n+d_2,n+d_2-1})\underline{\mathbf{a}}_n{}^{-1}(x) + \widehat{\mathbb{A}}(x)\left[\frac{\widehat{\Psi}_n(\rho)\underline{\mathbf{\Phi}}_n^{(\varkappa)}(\rho)}{x-\rho}\right]_{\rho\in\partial_x\varkappa} - \widehat{\mathbb{A}}(x)W(x)$$

$$W_{ab}(x) := \mathcal{L}\left(\widehat{p}_{n-d_2+a}(\rho)\frac{V_1(\rho)-V_1(x)}{\rho-x}\bigg| s_{n-1+b}(y)\right)\ ,\ a,b=0,1,\dots,d_2 \tag{4-51}$$

$$P_{j,k} := \mathcal{L}(p_j|ys_k)\ . \tag{4-52}$$

where $\underline{\mathbf{a}}_n$ *is the ladder matrix for the dual wave vector (Note that* $P=((1+M)^{-1}B)^t$*)*

(For the proof see App. A.3).

Theorem 4.4 *The direct system*

$$\widehat{\mathbf{\Psi}}_n(x) := \left[\widehat{\Psi}_n^{(0)}|\widehat{\Psi}_{n-1}^{(1)}\cdots\widehat{\Psi}_n^{(d_2)}\right] = \left[\begin{array}{c|ccc}\widehat{\psi}_{n-d_2}^{(0)} & \widehat{\psi}_{n-d_2}^{(1)} & \dots & \widehat{\psi}_{n-d_2}^{(d_2)}\\ \vdots & & & \vdots\\ \widehat{\psi}_{n-1}^{(0)} & \widehat{\psi}_{n-1}^{(1)} & \dots & \widehat{\psi}_{n-1}^{(d_2)}\\ \widehat{\psi}_n^{(0)} & \widehat{\psi}_n^{(1)} & \dots & \widehat{\psi}_n^{(d_2)}\end{array}\right] \tag{4-53}$$

satisfies the ODE

$$\widehat{\mathbf{\Psi}}_n'\,\widehat{\mathbf{\Psi}}_n^{-1} = -\left[\begin{array}{ccccc}\widehat{P}_{n-d_2,n-d_2} & & \dots & \widehat{P}_{n-d_2,n-1} & 0\\ \widehat{P}_{n-d_2+1,n-d_2} & & & \vdots & 0\\ 0 & \ddots & & & \vdots\\ 0 & & \widehat{P}_{n-1,n-2} & \widehat{P}_{n-1,n-1} & 0\\ & & & \widehat{P}_{n,n-1} & V_1'(x)\end{array}\right] + \mathrm{diag}(\widehat{P}_{n+1,n-d_2},\dots,\widehat{P}_{n+1,n})\widehat{\mathbf{a}}_{n-1}^{-1} +$$

$$
+\left[\frac{\widehat{\Psi}_n(\xi)\underline{\Phi}_n^{(\varkappa)}(\xi)}{\xi-x}\right]_{\xi\in\partial_x\varkappa}\widehat{\mathbb{A}}(x)+W(x)\widehat{\mathbb{A}}(x)
$$

$$
\widehat{P}_{j,k}:=\widehat{\mathcal{L}}(\widehat{p}_j|y\widehat{s}_k)\ , \tag{4-54}
$$

where $W(x)$ was defined in the previous theorem and $\widehat{\mathbf{a}}_{n-1}$ is the ladder matrix implementing the multiplicative recurrence relations $\widehat{\Psi}_n = \widehat{\mathbf{a}}_{n-1}\Psi_{n-1}$ as per Lemma 4.1 (in particular eq. (4-31)) specified to the hat-wave vectors.

(For the proof see App. A.4).

5 Dual Riemann–Hilbert problems

The shape of the Christoffel–Darboux identity (Thm. 4.1) suggests that the duality of the Riemann–Hilbert problems (and of the differential equations) involves naturally the dual pair of fundamental systems $\underline{\mathbf{\Phi}}_n(x), \widehat{\mathbf{\Psi}}_n(x)$ defined in Thm. 4.3 and Thm. 4.4. Recall (from Section 3) that we can choose a basis in the relative homology of contours $\Gamma_{y,\nu}$ and $\widehat{\Gamma}_{y,\nu}$ (and a rescaling of the $\underset{\infty}{\widehat{\Psi}}^{(j)}$ wave vectors depending only on the residues of $V_2'(y)\mathrm{d}y$) which span the solution space of the two adjoint equations and with bilinear concomitant

$$
\mathcal{B}_2(\Gamma_{y,\nu},\widehat{\Gamma}_{y,\mu}):=\Gamma_{y,\nu}\sharp\widehat{\Gamma}_{y,\mu}=\delta_{\mu\nu}\ . \tag{5-1}
$$

We can rewrite (Thm. 4.1) as

$$
(x-x')\sum_{j=0}^{n-1}\underline{\widehat{\phi}}_j^{(\nu)}(x)\widehat{\psi}_j^{(0)}(x')=\underline{\Phi}_n^{(\nu)}(x)\widehat{\mathbb{A}}(x')\widehat{\Psi}_n^{(0)}(x') \tag{5-2}
$$

$$
(x-x')\sum_{j=0}^{n-1}\underline{\phi}_j^{(\nu)}(x)\psi_j^{(0)}(x')=\underline{\Phi}_n^{(\nu)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n^{(0)}(x') \tag{5-3}
$$

$\nu=1,\ldots,d_2$, where we stress the fact that on the LHS we have the quasipolynomials ψ_n whereas on the RHS we have the $\widehat{\psi}_n$'s.

Theorem 5.1 *The fundamental dual pair is put in* **perfect duality** *by the Christoffel–Darboux matrix $\widehat{\mathbb{A}}$*

$$
\underline{\mathbf{\Phi}}_n(x)\widehat{\mathbb{A}}_n(x)\widehat{\mathbf{\Psi}}_n(x)=\left[\begin{array}{c|c}1&0\\ \hline 0&\mathcal{B}_2(\bullet,\bullet)\end{array}\right] \tag{5-4}
$$

where $\mathcal{B}_2(\bullet,\bullet)$ represents the (constant in x) bilinear concomitant for the solutions of the adjoint ODEs along the contours $\Gamma_{y,\nu}$, $\widehat{\Gamma}_{y,\mu}$, $\mu,\nu=1,\ldots,d_2$. By suitable choice of the homology classes we have seen that we can always assume it to be diagonal. The entries on the diagonal are nonzero and *may be set to 1 by suitable rescaling of the d_2 left-most columns of $\mathbf{\Psi}_n$: these rescalings depend on the way we have performed the cuts in the definitions of V_2 and $\widehat{V}_2$ but depend* **only** *on the residues of V_2' mod $\mathbb{Z}$.*

(For the proof see App. A.5).

5.1 Riemann–Hilbert data

In this section we summarily indicate how to obtain the data of the Riemann–Hilbert problems solved by the dual fundamental systems. The details are considerably involved and not strictly necessary in this paper. They will appear in a different publication.

Since the two matrices $\underline{\mathbf{\Phi}}_n$ and $\widehat{\mathbf{\Psi}}_n$ are put in perfect duality by the Christoffel–Darboux pairing, it is -in principle- sufficient to describe the Riemann–Hilbert data of one of the two members of the pair, the data for its partner being completely determined by duality.

It is significantly simpler to analyze the RH data for the matrix $\underline{\mathbf{\Phi}}_n$. We recall that this means controlling the jump discontinuities and the asymptotic behaviors near the singularities.

Jump discontinuities. They are uniquely due to the first row in the definition of $\underline{\mathbf{\Phi}}_n$ and occur at the contours $\Gamma_{x,\nu}$:

$$\underline{\mathbf{\Phi}}_n(x_+) = \begin{bmatrix} 1 & 2i\pi\varkappa_{\nu,1} & 2i\pi\varkappa_{\nu,2} & \dots & 2i\pi\varkappa_{\nu,d_2} \\ & 1 & & & \\ & & \ddots & & \\ & & & & \\ & & & & 1 \end{bmatrix} \underline{\mathbf{\Phi}}_n(x_-) \tag{5-5}$$

where $x_\pm$ denote the boundary values on the left/right of the point $x \in \Gamma_{x,\nu}$.

Note that the fundamental matrix $\widehat{\mathbf{\Psi}}_n(x)$ satisfies a similar jump condition which can be read off eq. (4-3) (specified to the $\widehat{\psi}_n$ quasipolynomials).

Singularities The bottom d_2 rows (the Fourier–Laplace transforms) are entire functions. The only singularities in the finite part of the plane arise from the first row $\underline{\Phi}_n^{(0)}(x)$: apart from the jump discontinuities (discussed above) we have all the singularities of $\mathrm{e}^{V_1(x)}$ and the logarithmic branching singularities around the hard-edge endpoints. Note that the (piecewise analytic) function

$$F_n(x) := \iint_{\varkappa} \vec{\Phi}_n(y) \frac{\mathrm{e}^{-V_1(\xi)+\xi y}}{x-\xi} = \mathrm{e}^{-V_1(x)} \underline{\Phi}_n(x) \tag{5-6}$$

has a well defined limit as x approaches any of the non hard-edge endpoints (where it is understood that the approach occurs within one connected component of its domain of analyticity). Indeed, if c is such a point one finds

$$F_n(c) = \iint_{\varkappa} \vec{\Phi}_n(y) \frac{e^{-V_1(\xi)+\xi y}}{c-\xi} \tag{5-7}$$

which is a well-defined value. In other words, near a non hard-edge singularity one has

$$\underline{\mathbf{\Phi}}_n(x) \sim \mathrm{diag}\left(e^{V_{1,sing}(x)}, 1, \ldots, 1\right) Y_0(\mathbf{1} + \mathcal{O}(x-c)). \tag{5-8}$$

where Y_0 is just the evaluation of the Fourier–Laplace rows and the $F_n(x)$ defined above at the point c, and $V_{1,sing}$ denotes the singular part of V_1 at c.

Near a hard–edge point $x = a$, if Γ_{x,ν_a} is the the hard-edge contour originating from a, we find that the matrix

$$Y(x) := \begin{bmatrix} 1 & \ln(x-a)\varkappa_{\nu_a,1} & \ldots & \ln(x-a)\varkappa_{\nu_a,d_2} \\ & \ddots & & \\ & & & \\ & & & 1 \end{bmatrix} \underline{\mathbf{\Phi}}_n(x) \tag{5-9}$$

has a removable singularity at $x = a$ and from this we can obtain the asymptotic behavior near the hard–edge endpoints.

Stokes Phenomenon. Possibly the most intricate part is the description of the Stokes' phenomenon at $x = \infty$.

Indeed, apart from the aforementioned jump-discontinuities of $\underline{\Phi}_n^{(0)}$ in a neighborhood of ∞ (which may be interpreted as part of the Stokes data), the first row displays no Stokes' phenomenon, and has an asymptotic behavior which encodes the orthogonality

$$\underline{\phi}_n^{(0)}(x) = e^{V_1(x)} \iint_{\varkappa} e^{-V_1(\xi)+\xi y} \frac{\phi_n(y)}{x-\xi} \sim \sqrt{h_n} e^{V_1(x)} x^{-n-1}(1 + \mathcal{O}(1/x)) \tag{5-10}$$

The remaining part of the Stokes phenomenon is given by the asymptotic behavior of the d_2 Fourier–Laplace transforms: this is precisely the same Stokes' phenomenon displayed by the solutions of the ODE

$$(A_2(\partial_x) - xB_2(\partial_x))f = 0 \tag{5-11}$$

These solutions are described by contour integrals of the same kind as the ones appearing in the expressions for $\underline{\Phi}_n^{(\nu)}$; a standard steepest descent formal argument shows that the leading asymptotic is determined by the saddle-point equation

$$\frac{A_2(y) + B_2'(y)}{B_2(y)} = V_2'(y) = x \tag{5-12}$$

$(x \to \infty)$ which has $d_2 - H$ solutions (H being the number of hard-edge contours, i.e. the number of (simple) zeroes of B_2 which cancel against corresponding zeroes of the numerator in (5-12)).

Whereas it is not very difficult to analyze the formal properties of the asymptotic, it is considerably harder and outside of the intents of the present paper to present the Stokes matrices associated to this Stokes' phenomenon. We leave this topic to a different publication.

5.1.1 Isomonodromic deformations

The (generalized) 2-Toda equations for this reduction as explained in the introduction, determine the evolution of the biorthogonal polynomials under infinitesimal deformations of the parameters entering the semiclassical data A_i, B_i. It is more convenient to parametrize the polynomials A_i, B_i not by their coefficients but by the location of the zeroes of B_i and the coefficients in the partial fraction expansions of the derivative potentials V_i'. Following the strategy in our [3, 6, 8] one could easily write the pertinent 2-Toda flows corresponding to these infinitesimal deformations.

At the level of the pair of fundamental systems the flows will generate isomonodromic deformations for the ODEs satisfied by $\underline{\mathbf{\Phi}}_n$ and $\widehat{\mathbf{\Psi}}_n$, provided that the exponents of formal monodromy at the singularities remain unchanged. In this case these are precisely the residues of $V_1'(x)\mathrm{d}x$ and $V_2'(y)\mathrm{d}y$ at the various singularities.

The reason why the deformations are isomonodromic is that –by their very definition– the fundamental systems are functions of these deformation parameters and the matrices $\dot{\underline{\mathbf{\Phi}}}_n \underline{\mathbf{\Phi}}_n^{-1}$ (and $\dot{\widehat{\underline{\mathbf{\Psi}}}}_n \widehat{\underline{\mathbf{\Psi}}}_n^{-1}$, the dot representing a derivative w.r.t. one of the monodromy-preserving parameters) are rational (or polynomial) functions of x, which follows from the analysis of their behavior at the various singularities ([12, 7] for details on the general properties of isomonodromic deformations).

The details of this isomonodromic system could be derived from the complete Riemann–Hilbert characterization of the fundamental systems and are beyond the scope of this paper, although their derivation is -in principle- a straightforward computation.

A Proofs

In this appendix we report all proof of more technical nature. The expressions are rather long and hence to shorten them we have decided to suppress explicit reference to the variables of integration in the multiple integrals below, since which variables are integrated on which contour is unambiguously implied by the context. We have adhered to the following general naming scheme: the variables ξ, ρ are integrated along the contours $\Gamma_{x,\nu}$ appearing in the integral $\int_{\varkappa}$, the variables y and η are variables integrated on the $\Gamma_{y,\mu}$'s. The variable s is always running along the dual contours $\widehat{\Gamma}_{y,\mu}$ (the admissible contours for the differential $\widehat{W}(s)\mathrm{d}s = \mathrm{e}^{\widehat{V}_2(s)}\mathrm{d}s = \frac{\mathrm{e}^{V_2(s)}}{B_2(s)}\mathrm{d}s$).

A.1 Proof of Prop. 4.1

We temporarily denote by a tilde the following linear combination

$$\widetilde{\psi}_n = \psi_n + \sum_1^{q_2} \ell_j(n)\psi_{n-j} \tag{1-1}$$

and notice that

$$x\widetilde{\psi}_n = \sum_{-1}^{d_2} \alpha_j(n)\psi_{n-j}\ . \tag{1-2}$$

For the transformed functions $\psi_n^{(\widehat{\Gamma})}$ (denoting by a tilde the same linear combination)

$$x\widetilde{\psi}_n^{(\widehat{\Gamma})} = \frac{x}{2i\pi}\int_{\widehat{\Gamma}}\iint_{\varkappa} \mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y - xs - V_2(y) + \widehat{V}_2(s)}\frac{\widetilde{\psi}_n(\xi)}{x-\xi} = \tag{1-3}$$

$$= \frac{1}{2i\pi}\int_{\widehat{\Gamma}}\iint_{\varkappa} \mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y - xs - V_2(y) + \widehat{V}_2(s)}\left(\widetilde{\psi}_n(\xi) + \frac{\xi\widetilde{\psi}_n(\xi)}{x-\xi}\right) = \tag{1-4}$$

$$= \sum_{-1}^{d_2}\alpha_j(n)\psi_{n-j}^{(\widehat{\Gamma})} + \underbrace{\int_{\widehat{\Gamma}}\iint_{\varkappa} \mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y - xs - V_2(y) + \widehat{V}_2(s)}\widetilde{\psi}_n(\xi)}_{=0 \text{ for } n\geqslant d_2+q_2} \tag{1-5}$$

where the last term vanishes for $n \geqslant q_2 + d_2$ because the bilinear concomitant kernel $\mathcal{B}_2(x;y,s)$ is a polynomial in y of degree $d_2 - 1$ and the linear combination $\widetilde{\psi}_n$ contains the orthogonal function ψ_{n-q_2}.

For the differential equation we have (by definition of the $\check{\psi}_n$'s)

$$\check{\psi}_n := \psi_n + \sum_1^{q_1}\check{m}_j(n+j)\psi_{n+j} \tag{1-6}$$

We then have

$$\begin{aligned}
\partial_x\check{\psi}_n^{(\widehat{\Gamma})} &= \int_{\widehat{\Gamma}}\iint_{\varkappa} \mathrm{e}^{-xs}(\partial_x - s)\frac{\mathcal{B}_2(x;y,s)}{x-\xi}\mathrm{e}^{\xi y - V_2(y) + \widehat{V}_2(s)}\check{\psi}_n(\xi) = \\
&= \int_{\widehat{\Gamma}}\iint_{\varkappa} \mathrm{e}^{-xs}\frac{B_2(y) - B_2(s)}{y-s}\mathrm{e}^{\xi y - V_2(y) + \widehat{V}_2(s)}\frac{\check{\psi}_n(\xi)}{x-\xi} + \\
&\quad + \int_{\widehat{\Gamma}}\iint_{\varkappa} \mathrm{e}^{\xi y - xs - V_2(y) + \widehat{V}_2(s)}\check{\psi}_n(\xi)(-\partial_\xi - s)\frac{\mathcal{B}_2(x;y,s)}{x-\xi} = \\
&\stackrel{\star}{=} \int_{\widehat{\Gamma}}\iint_{\varkappa} \frac{B_2(y) - B_2(s)}{y-s}\mathrm{e}^{\xi y - xs - V_2(y) + \widehat{V}_2(s)}\frac{\check{\psi}_n(\xi)}{x-\xi} + \\
&\quad + \int_{\widehat{\Gamma}}\iint_{\varkappa} \frac{\mathcal{B}_2(x;y,s)}{x-\xi}(\partial_\xi - s)\mathrm{e}^{\xi y - xs - V_2(y) + \widehat{V}_2(s)}\check{\psi}_n(\xi) =
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{-1}^{d_1} \breve\beta_j(n+j)\psi^{(\breve\Gamma)}_{n+j} + \\
&\quad + \int_{\widehat\Gamma}\iint_{\varkappa} \frac{B_2(y)-B_2(s)}{y-s}\mathrm{e}^{\xi y - xs - V_2(y) + \widehat V_2(s)}\frac{\breve\psi_n(\xi)}{x-\xi} + \\
&\quad + \int_{\widehat\Gamma}\iint_{\varkappa} \mathcal{B}_2(x;y,s)(y-s)\mathrm{e}^{\xi y - xs - V_2(y) + \widehat V_2(s)}\frac{\breve\psi_n(\xi)}{x-\xi} = \\
&= -\sum_{-1}^{d_1} \breve\beta_j(n+j)\psi^{(\breve\Gamma)}_{n+j} + \\
&\quad + \int_{\widehat\Gamma}\iint_{\varkappa} (x(B_2(y)-B_2(s)) - A_2(y) + B_2'(s) + A_2(s))\mathrm{e}^{\xi y - xs - V_2(y) + \widehat V_2(s)}\frac{\breve\psi_n(\xi)}{x-\xi} \overset{\text{(the s-part is a total derivative)}}{=} \\
&= -\sum_{-1}^{d_1} \breve\beta_j(n+j)\psi^{(\breve\Gamma)}_{n+j} + \int_{\widehat\Gamma}\iint_{\varkappa} (xB_2(y) - A_2(y))\mathrm{e}^{\xi y - xs - V_2(y) + \widehat V_2(s)}\frac{\breve\psi_n(\xi)}{x-\xi} = \\
&= -\sum_{-1}^{d_1} \breve\beta_j(n+j)\psi^{(\breve\Gamma)}_{n+j} + \int_{\widehat\Gamma}\iint_{\varkappa} B_2(y)\mathrm{e}^{\xi y - xs - V_2(y) + \widehat V_2(s)}\breve\psi_n(\xi) + \\
&\quad + \overbrace{\int_{\widehat\Gamma}\iint_{\varkappa} (\xi B_2(y) - A_2(y))\mathrm{e}^{\xi y - xs - V_2(y) + \widehat V_2(s)}\frac{\breve\psi_n(\xi)}{x-\xi}}^{\text{total derivative in } y} = \\
&= -\sum_{-1}^{d_1} \breve\beta_j(n+j)\psi^{(\breve\Gamma)}_{n+j} + \underbrace{\int_{\widehat\Gamma}\iint_{\varkappa} B_2(y)\mathrm{e}^{\xi y - xs - V_2(y) + \widehat V_2(s)}\breve\psi_n(\xi)}_{=0 \text{ for } n\geqslant q_2+1}
\end{aligned}
\tag{1-7}
$$

In the step marked with $\star$ we have performed an integration by parts: in this integration we do not get any boundary contributions because the quasipolynomials $\breve\psi_n$ by definition are divisible by B_1 (which vanishes at all endpoints and in particular at the hard-edge ones). This concludes the proof. **Q.E.D.**

A.2 Proof of Thm. 4.2

During this and following proofs we use the notation

$$
\vec\Phi_n(y) := [\phi_{n-1}, \dots, \phi_{n+d_2-1}]\ , \tag{1-8}
$$

for the row-vector of quasipolynomials in y. Moreover, at the risk of marginal confusion, we omit all differentials of the integration variables since which variables are integrated and on which contour should be always uniquely determined by the context (the formulas become significantly longer otherwise). For $(\mathbf{a})$ we have (recall that $\widehat{\mathbb{A}}(\xi)$ is linear in ξ)

$$
(\mathbf{a}) = \sum_{j=0}^{n-1}\mathrm{e}^{V_1(z)}\iint_{\varkappa}\mathrm{e}^{-V_1(\xi)+\xi y}\frac{\phi_j(y)}{z-\xi}\psi_j(x)(z-x) =
$$

$$
\begin{aligned}
&= \mathrm{e}^{V_1(z)} \iint_{\varkappa} \mathrm{e}^{-V_1(\xi)+\xi y} \vec{\Phi}_n(y) \widehat{\mathbb{A}}(\xi) \widehat{\Psi}_n(x) \frac{z-x}{(z-\xi)(\xi-x)} = \\
&= \mathrm{e}^{V_1(z)} \iint_{\varkappa} \mathrm{e}^{-V_1(\xi)+\xi y} \vec{\Phi}_n(y) \widehat{\mathbb{A}}(\xi) \widehat{\Psi}_n(x) \left(\frac{1}{z-\xi} - \frac{1}{x-\xi} \right) = \\
&= \mathrm{e}^{V_1(z)} \iint_{\varkappa} \mathrm{e}^{-V_1(\xi)+\xi y} \frac{\vec{\Phi}_n(y)}{z-\xi} \widehat{\mathbb{A}}(\xi) \widehat{\Psi}_n(x) - \mathrm{e}^{V_1(z)} \iint_{\varkappa} \mathrm{e}^{-V_1(\xi)+\xi y} \vec{\Phi}_n(y) \widehat{\mathbb{A}}(\xi) \frac{\widehat{\Psi}_n(x)}{x-\xi} = \\
&\stackrel{\star}{=} \mathrm{e}^{V_1(z)} \iint_{\varkappa} \mathrm{e}^{-V_1(\xi)+\xi y} \frac{\vec{\Phi}_n(y)}{z-\xi} \widehat{\mathbb{A}}(z) \widehat{\Psi}_n(x) - \mathrm{e}^{V_1(z)} \iint_{\varkappa} \mathrm{e}^{-V_1(\xi)+\xi y} \vec{\Phi}_n(y) \widehat{\mathbb{A}}(x) \frac{\widehat{\Psi}_n(x)}{x-\xi} = \\
&= \underline{\Phi}_n^{(0)}(z) \widehat{\mathbb{A}}(z) \widehat{\Psi}_n(x) + \mathrm{e}^{V_1(z)} \sum_{j=0}^{n-1} \iint_{\varkappa} \mathrm{e}^{-V_1(\xi)+\xi y} \widehat{\phi}_j(y) \widehat{\psi}_j(x) = \\
&= \underline{\Phi}_n^{(0)}(z) \widehat{\mathbb{A}}(z) \widehat{\Psi}_n(x) + \mathrm{e}^{V_1(z)-V_1(x)} \ ,
\end{aligned} \tag{1-9}
$$

where in the identity marked $\star$ we have used the linearity of $\widehat{\mathbb{A}}$ which implies the following identity

$$
\frac{\widehat{\mathbb{A}}(\xi)}{z-\xi} - \frac{\widehat{\mathbb{A}}(\xi)}{x-\xi} = \frac{\widehat{\mathbb{A}}(z)}{z-\xi} - \frac{\widehat{\mathbb{A}}(x)}{x-\xi} \ . \tag{1-10}
$$

The second form of $(\mathbf{a})$ is proved along the same lines using the principal CDI for the kernel $\widehat{K}_{11}$ (in Thm. 4.1). For $(\mathbf{b})$ we have

$$
\begin{aligned}
(\mathbf{b}) &= \frac{z-x}{2i\pi} \sum_{r=0}^{n-1} \int_{\Gamma_j} \mathrm{e}^{zy} \phi_r(y) \int_{\widehat{\Gamma}_k} \iint_{\varkappa} \mathcal{B}_2 \mathrm{e}^{\cdots} \frac{\psi_j(\rho)}{x-\rho} = \\
&= \frac{1}{2i\pi} \int_{\Gamma_j} \mathrm{e}^{zy} \vec{\Phi}_n(y) \int_{\widehat{\Gamma}_k} \iint_{\varkappa} \mathcal{B}_2 \mathrm{e}^{\cdots} \frac{\widehat{\mathbb{A}}(z)(z-x)}{(z-\rho)(x-\rho)} \widehat{\Psi}_n(\rho) = \\
&= \frac{1}{2i\pi} \int_{\Gamma_j} \mathrm{e}^{zy} \vec{\Phi}_n(y) \int_{\widehat{\Gamma}_k} \iint_{\varkappa} \mathcal{B}_2 \mathrm{e}^{\cdots} \widehat{\mathbb{A}}(z) \left(\frac{1}{x-\rho} - \frac{1}{z-\rho} \right) \widehat{\Psi}_n(\rho) = \\
&= \underline{\Phi}_n^{(j)}(z) \widehat{\mathbb{A}}(z) \widehat{\Psi}_n^{(k)}(x) - \frac{1}{2i\pi} \sum_{r=0}^{n-1} \int_{\Gamma_j} \mathrm{e}^{zy} \phi_r(y) \int_{\widehat{\Gamma}_k} \iint_{\varkappa} \mathcal{B}_2(x;\eta,s) \mathrm{e}^{\eta\rho - xs + \widehat{V}_2(s) - V_2(\eta)} \psi_r(\rho) = \\
&\stackrel{\star}{=} \underline{\Phi}_n^{(j)}(z) \widehat{\mathbb{A}}(z) \widehat{\Psi}_n(x) - \frac{1}{2i\pi} \int_{\Gamma_j} \int_{\widehat{\Gamma}_k} \mathcal{B}_2(x;y,s) \mathrm{e}^{yz - xs + \widehat{V}_2(s) - V_2(y)} \ ,
\end{aligned} \tag{1-11}
$$

where the identity marked $\star$ is valid for $n \geqslant d_2$ (so that the kernel reproduces the polynomial $\mathcal{B}_2(x;\eta,s)$ of degree $d_2 - 1$).

The proof of the second form of $(\mathbf{b})$ is only marginally different in that we have to use the second form of the principal CDI for the kernel $\widehat{K}_{11}$ (in Thm. 4.1). **Q.E.D.**

A.3 Proof of Thm. 4.3

Let $n-1 \leqslant m \leqslant n+d_2-1$: in the following chain of equalities all the steps are "elementary" and hence the computation is straightforward. For reader's convenience we have tried to make annotations on the formula in order to highlight less obvious steps.

$$
\begin{aligned}
\partial_x \underline{\phi}_m^{(0)} =& V_1'(x)\underline{\phi}_m^{(0)} + \mathrm{e}^{V_1(x)}\iint_{\varkappa} \mathrm{e}^{\xi y}(-\partial_\xi)\frac{\mathrm{e}^{-V_1(\xi)}\phi_m(y)}{x-\xi} = \\
=& -\underbrace{\frac{\mathrm{e}^{V_1(x)-V_1(\xi)}}{x-\xi}\underline{\phi}_m^{(\varkappa)}(\xi)\bigg|_{\xi\in\partial_x\varkappa}}_{=:(B)} + \underbrace{\mathrm{e}^{V_1(x)}\iint_{\varkappa}\frac{(V_1'(x)-V_1'(\xi))\mathrm{e}^{-V_1(\xi)+\xi y}\phi_m(y)}{x-\xi}}_{=:(C)} + \mathrm{e}^{V_1(x)}\iint_{\varkappa}\frac{y\,\mathrm{e}^{-V_1(\xi)+\xi y}\phi_m(y)}{x-\xi} = \\
=&(-B+C) + \sum_{j=0}^{n+d_2}\underline{\phi}_j^{(0)}(x)\iint_{\varkappa}\psi_j(\rho)\eta\phi_m(\eta)\mathrm{e}^{\rho\eta} = \\
=&(-B+C) + \sum_{j=0}^{n-1}\underline{\phi}_j^{(0)}(x)\iint_{\varkappa}\psi_j(\rho)\eta\phi_m(\eta)\mathrm{e}^{\rho\eta} + \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)\overbrace{\iint_{\varkappa}\psi_j(\rho)\eta\phi_m(\eta)\mathrm{e}^{\rho\eta}}^{=:P_{jm}} = \\
=&(-B+C) + \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)P_{jm} + \sum_{j=0}^{n-1}\underline{\phi}_j^{(0)}(x)\left[\psi_j(\rho)\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} + \\
&-\sum_{j=0}^{n-1}\underline{\phi}_j^{(0)}(x)\iint_{\varkappa}\phi_m(\eta)\mathrm{e}^{\rho\eta-V_1(\rho)}(\partial_\rho - V_1'(\rho))\pi_j(\rho) = \\
=&(-B+C) + \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)P_{jm} + \left[\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\widehat{\mathbb{A}}}(x)\widehat{\Psi}_n(\rho) + \mathrm{e}^{V_1(x)-V_1(\rho)}}{x-\rho}\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} + \\
&+\sum_{j=0}^{n-1}\underline{\phi}_j^{(0)}(x)\iint_{\varkappa}\phi_m(\eta)\mathrm{e}^{\rho\eta-V_1(\rho)}V_1'(\rho)\pi_j(\rho) = \\
=&(C) + \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)P_{jm} + \left[\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho)}{x-\rho}\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} + \\
&+\iint_{\varkappa}\phi_m(\eta)\mathrm{e}^{\rho\eta}V_1'(\rho)\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho) + \mathrm{e}^{V_1(x)-V_1(\rho)}}{x-\rho} = \\
=&\sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)P_{jm} + \left[\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho)}{x-\rho}\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} + \\
&+\iint_{\varkappa}\phi_m(\eta)\mathrm{e}^{\rho\eta}V_1'(\rho)\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho)}{x-\rho} + V_1'(x)\mathrm{e}^{V_1(x)}\iint_{\varkappa}\frac{\mathrm{e}^{-V_1(\xi)+\xi y}\phi_m(y)}{x-\xi} = \\
=&\sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)P_{jm} + \left[\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\Psi}_n(\rho)}{x-\rho}\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} + \iint_{\varkappa}\phi_m(\eta)\mathrm{e}^{\rho\eta}\frac{V_1'(\rho)-V_1'(x)}{x-\rho}\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho) \\
&+V_1'(x)\iint_{\varkappa}\phi_m(\eta)\mathrm{e}^{\rho\eta}\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho) + \mathrm{e}^{V_1(x)-V_1(\rho)}}{x-\rho} = \\
=&\sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)P_{jm} + \left[\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho)}{x-\rho}\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} - \underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\iint_{\varkappa}\widehat{\Psi}_n(\rho)\phi_m(\eta)\mathrm{e}^{\rho\eta}\frac{V_1'(\rho)-V_1'(x)}{\rho-x}
\end{aligned}
$$

$$
\begin{aligned}
&+ V_1'(x)\sum_{j=0}^{n-1}\underline{\phi}_j^{(0)}(x)\iint_{\varkappa}\phi_m(\eta)\mathrm{e}^{\rho\eta}\psi_j(\rho) = \\
&= \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(0)}(x)P_{jm} + \left[\frac{\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\rho)}{x-\rho}\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} - \underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\iint_{\varkappa}\widehat{\Psi}_n(\rho)\phi_m(\eta)\mathrm{e}^{\rho\eta}\frac{V_1'(\rho)-V_1'(x)}{\rho-x} \\
&+ V_1'(x)\underline{\phi}_{n-1}^{(0)}(x)\delta_{m,n-1}
\end{aligned}
\tag{1-12}
$$

We note that in this last expression we have $\partial_x\underline{\phi}_m^{(0)}(x)$ expressed purely in terms of $\underline{\phi}_\ell^{(0)}(x)$ for $\ell = n-1,\dots n+d_2$, the value $\ell = n+d_2$ entering only in the first expression. Given that $\underline{\phi}_n^{(0)}(x)$ satisfies the same multiplicative recurrence relations as the Fourier–Laplace transforms for $n\geqslant 1$, we can re-express $\underline{\phi}_{n+d_2}^{(0)}$ in terms of the elements of the window $\underline{\Phi}_n^{(0)}(x)$, obtaining the result.

The computation for the Fourier-Laplace transforms gives also the same differential equation, indeed

$$
\begin{aligned}
\partial_x\underline{\phi}_m^{(r)}(x) &= \int_{\Gamma_{y,r}}\mathrm{e}^{xy}\phi_m(y) = \sum_{j=0}^{n+d_2}\underline{\phi}_j^{(r)}(x)\iint_{\varkappa}\mathrm{e}^{\eta\rho}\eta\phi_m(\eta)\psi_j(\rho) = \\
&= \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(r)}(x)P_{jm} + \sum_{j=0}^{n-1}\underline{\phi}_j^{(r)}(x)\iint_{\varkappa}\mathrm{e}^{\eta\rho-V_1(\rho)}\phi_m(\eta)(-\partial_\rho+V_1'(\rho))\pi_j(\rho) + \\
&\quad + \sum_{j=0}^{n-1}\underline{\phi}_j^{(r)}(x)\left[\psi_j(\rho)\underline{\phi}_m^{(\varkappa)}(\rho)\right]_{\rho\in\partial_x\varkappa} = \\
&= \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(r)}(x)P_{jm} + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\iint_{\varkappa}\mathrm{e}^{\eta\rho}\phi_m(\eta)\frac{V_1'(\rho)}{x-\rho}\widehat{\Psi}_n(\rho) + \\
&\quad + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\left[\frac{\widehat{\Psi}_n(\rho)\underline{\phi}_m^{(\varkappa)}(\rho)}{x-\rho}\right]_{\rho\in\partial_x\varkappa} = \\
&= \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(r)}(x)P_{jm} + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\iint_{\varkappa}\widehat{\Psi}_n(\rho)\mathrm{e}^{\eta\rho}\phi_m(\eta)\frac{V_1'(\rho)-V_1'(x)}{x-\rho} + \\
&\quad + V_1'(x)\underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\iint_{\varkappa}\frac{\widehat{\Psi}_n(\rho)}{x-\rho}\mathrm{e}^{\eta\rho}\phi_m(\eta) + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\left[\frac{\widehat{\Psi}_n(\rho)\underline{\phi}_m^{(\varkappa)}(\rho)}{x-\rho}\right]_{\rho\in\partial_x\varkappa} = \\
&= \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(r)}(x)P_{jm} + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\iint_{\varkappa}\widehat{\Psi}_n(\rho)\mathrm{e}^{\eta\rho}\phi_m(\eta)\frac{V_1'(\rho)-V_1'(x)}{x-\rho} + \\
&\quad + V_1'(x)\sum_{j=0}^{n-1}\underline{\phi}_j^{(r)}(x)\iint_{\varkappa}\psi_j(\rho)\mathrm{e}^{\eta\rho}\phi_m(\eta) + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\left[\frac{\widehat{\Psi}_n(\rho)\underline{\phi}_m^{(\varkappa)}(\rho)}{x-\rho}\right]_{\rho\in\partial_x\varkappa} = \\
&= \sum_{j=n}^{n+d_2}\underline{\phi}_j^{(r)}(x)P_{jm} + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\iint_{\varkappa}\widehat{\Psi}_n(\rho)\mathrm{e}^{\eta\rho}\phi_m(\eta)\frac{V_1'(\rho)-V_1'(x)}{x-\rho} +
\end{aligned}
$$

$$+ V_1'(x)\delta_{m,n-1}\underline{\phi}_m^{(r)}(x) + \underline{\Phi}_n^{(r)}(x)\widehat{\mathbb{A}}(x)\left[\frac{\widehat{\Psi}_n(\rho)\underline{\phi}_m^{(\varkappa)}(\rho)}{x-\rho}\right]_{\rho\in\partial_x\varkappa} . \tag{1-13}$$

The coefficients of these expressions in terms of $\underline{\phi}_{n-1}, \dots \underline{\phi}_{n+d_2-1}$ are precisely the same as for the previous computation, hence completing the proof. **Q.E.D.**

A.4 Proof of Thm. 4.4

Let $n - d_2 \leqslant m \leqslant n$ and let us compute

$$\begin{aligned}
\partial_x\widehat{\psi}_m(x) =& \mathrm{e}^{-V_1(x)}(\partial_x - V_1'(x))\widehat{\pi}_m(x) = -V_1'(x)\widehat{\psi}_m(x) + \sum_{j=0}^{n-1}\widehat{\psi}_j(x)\iint_\varkappa \widehat{\pi}_m'(\xi)\mathrm{e}^{\xi y - V_1(\xi)}\widehat{\phi}_j(y) = \\
=& -V_1'(x)\widehat{\psi}_m(x) + \sum_{j=0}^{n-1}\widehat{\psi}_j(x)\left[\widehat{\psi}_m(\xi)\widehat{\underline{\phi}}_j^{(\varkappa)}(\xi)\right]_{\xi\in\partial_x\varkappa} - \sum_{j=0}^{n-1}\widehat{\psi}_j(x)\iint_\varkappa \widehat{\psi}_m(\xi)\mathrm{e}^{\xi y} y\widehat{\phi}_j(y) = \\
=& -V_1'(x)\widehat{\psi}_m(x) + \left[\frac{\widehat{\psi}_m(\xi)\underline{\Phi}_n^{(\varkappa)}(\xi)}{\xi - x}\right]_{\xi\in\partial_x\varkappa}\widehat{\mathbb{A}}(x)\widehat{\Psi}_j(x) - \sum_{j=0}^{n-1}\widehat{\psi}_j(x)\iint_\varkappa \widehat{\psi}_m(\xi)\mathrm{e}^{\xi y}(y - V_1'(\rho))\widehat{\phi}_j(y) = \\
=& -V_1'(x)\widehat{\psi}_m(x) + \left[\frac{\widehat{\psi}_m(\xi)\underline{\Phi}_n^{(\varkappa)}(\xi)}{\xi - x}\right]_{\xi\in\partial_x\varkappa}\widehat{\mathbb{A}}(x)\widehat{\Psi}_j(x) + \sum_{j=0}^{n-1}\widehat{\psi}_j(x)\iint_\varkappa \widehat{\psi}_m(\xi)\mathrm{e}^{\xi y}V_1'(\rho)\widehat{\phi}_j(y) + \\
&- \sum_{j=\mathbf{m-1}}^{n-1}\widehat{\psi}_j(x)\iint_\varkappa \widehat{\psi}_m(\xi)\mathrm{e}^{\xi y}y\widehat{\phi}_j(y) = \\
=& -V_1'(x)\delta_{mn}\widehat{\psi}_n(x) + \left[\frac{\widehat{\psi}_m(\xi)\underline{\Phi}_n^{(\varkappa)}(\xi)}{\xi - x}\right]_{\xi\in\partial_x\varkappa}\widehat{\mathbb{A}}(x)\widehat{\Psi}_j(x) + \\
&+ \sum_{j=0}^{n-1}\widehat{\psi}_j(x)\iint_\varkappa \widehat{\psi}_m(\xi)\mathrm{e}^{\xi y}(V_1'(\rho) - V_1'(x))\widehat{\phi}_j(y) - \sum_{j=m-1}^{n-1}\widehat{\psi}_j(x)\widehat{P}_{mj} = \\
=& -V_1'(x)\delta_{mn}\widehat{\psi}_n(x) + \left[\frac{\widehat{\psi}_m(\xi)\underline{\Phi}_n^{(\varkappa)}(\xi)}{\xi - x}\right]_{\xi\in\partial_x\varkappa}\widehat{\mathbb{A}}(x)\widehat{\Psi}_j(x) + \\
&+ \iint_\varkappa \widehat{\psi}_m(\xi)\mathrm{e}^{\xi y}\frac{V_1'(\rho) - V_1'(x)}{\rho - x}\vec{\Phi}_n(y)\widehat{\mathbb{A}}(x)\widehat{\Psi}_j(x) - \sum_{j=m-1}^{n-1}\widehat{\psi}_j(x)\widehat{P}_{mj} \ .
\end{aligned} \tag{1-14}$$

The last term contains $\widehat{\psi}_{n-d_2-1}$ (for $m = n - d_2$) which is "outside" of the window of the quasipolynomials. Using the recurrence relations and re-expressing it in terms of elements in the window (using the ladder matrices) we obtain the formula.

For completeness one should also consider the other columns of the fundamental system $\widehat{\mathbf{\Psi}}_n$ and show that they satisfy the same differential relation as the quasipolynomials. Let $n - d_2 \leqslant m \leqslant n$, then

$$\partial_x\widehat{\psi}_m^{(r)} = \frac{1}{2i\pi}\partial_x\int_{\widehat{\Gamma}_{y,r}}\iint_\varkappa \mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y - xs + \widehat{V}_2(s) - V_2(y)}\frac{\widehat{\psi}_m(\xi)}{x-\xi} =$$

$$=\frac{1}{2i\pi}\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}\widehat{\psi}_m(\xi)(\partial_x-s)\frac{\mathcal{B}_2(x;y,s)}{x-\xi}=$$
$$=\frac{1}{2i\pi}\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}\widehat{\psi}_m(\xi)\left[\frac{B_2(y)-B_2(s)}{(y-s)(x-\xi)}-s\frac{\mathcal{B}_2(x;y,s)}{(x-\xi)}-\mathcal{B}_2(x;y,s)\partial_\xi\frac{1}{x-\xi}\right]=$$
$$=-\overbrace{\frac{1}{2i\pi}\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\partial_\xi\left(\mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}\frac{\widehat{\psi}_m(\xi)}{x-\xi}\right)}^{=:(B)}+$$
$$+\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\frac{\widehat{\psi}_m(\xi)}{x-\xi}\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}\left[\frac{B_2(y)-B_2(s)}{y-s}+(y-s)\mathcal{B}_2(x;y,s)\right]+$$
$$+\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}\frac{(\partial_\xi-V_1'(\xi))\widehat{p}_m(\xi)}{x-\xi}=$$
$$=-(B)+\int_{\widehat{\Gamma}_{y,r}}\mathrm{e}^{\widehat{V}_2(s)-xs}\overbrace{\iint_{\varkappa}\widehat{\psi}_m(\xi)\mathrm{e}^{-V_2(y)+\xi y}B_2(y)}^{=\sqrt{\widehat{h}_0}\delta_{m0}\text{ because }\widehat{V}_2=V_2-\ln B_2}+$$
$$-\overbrace{\frac{1}{2i\pi}\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}\frac{V_1'(\xi)\widehat{\psi}_m(\xi)}{x-\xi}}^{=:(C)}+$$
$$+\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\mathcal{B}_2(x;y,s)\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}\frac{\widehat{p}_m'(\xi)}{x-\xi}=$$
$$=-(B)-(C)+\overbrace{\sqrt{\widehat{h}_0}\delta_{m0}\int_{\widehat{\Gamma}_{y,r}}\mathrm{e}^{\widehat{V}_2(s)-xs}}^{=:(D)}+\sum_{j=0}^{n-1}\widehat{\psi}_j^{(r)}(x)\iint_{\varkappa}\widehat{p}_m{}'(\xi)\widehat{\phi}_j(y)\mathrm{e}^{-V_1(\xi)+\xi y}=$$
$$=-(B)-(C)+(D)+\sum_{j=0}^{n-1}\widehat{\psi}_j^{(r)}(x)\left[\widehat{\psi}_m(\xi)\widehat{\underline{\phi}}_j^{(\varkappa)}(\xi)\right]_{\xi\in\partial_x\varkappa}+$$
$$+\sum_{j=0}^{n-1}\widehat{\psi}_j^{(r)}(x)\iint_{\varkappa}(V_1'(\xi)-V_1'(x))\widehat{\psi}_m(\xi)\widehat{\phi}_j(y)\mathrm{e}^{\xi y}+$$
$$+V_1'(x)(1-\delta_{m,n})\psi_m^{(r)}(x)-\sum_{j=0}^{n-1}\widehat{\psi}_j^{(r)}(x)\iint_{\varkappa}\widehat{\psi}_m(\xi)y\,\widehat{\phi}_j(y)\mathrm{e}^{\xi y}=$$
$$\overset{\text{[aux CDI]}}{=}-(B)-(C)+\frac{\widehat{\psi}_m(\xi)\widehat{\underline{\Phi}}_n^{(\varkappa)}(\xi)}{\xi-x}\bigg|_{\xi\in\partial_x\varkappa}\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(x)+$$
$$\overbrace{-\frac{1}{2i\pi}\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\partial_\xi\left(\frac{\widehat{\psi}_m(\xi)}{\xi-x}\mathcal{B}_2(x;y,s)\mathrm{e}^{y\xi-xs+\widehat{V}_2(s)-V_2(y)}\right)}^{=(B)}+$$
$$+\iint_{\varkappa}\frac{V_1'(\xi)-V_1'(x)}{\xi-x}\mathrm{e}^{\xi y}\widehat{\psi}_m(\xi)\vec{\Phi}_n(y)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n^{(r)}(x)+$$

$$
\begin{aligned}
&-\frac{1}{2i\pi}\int_{\widehat{\Gamma}_{y,r}}\iint_{\varkappa}\Big(\overset{[\text{gives }(C)]}{\overset{\uparrow}{V_1'(\xi)}}-V_1'(x)\Big)\frac{\widehat{\psi}_m(\xi)}{\xi-x}\mathrm{e}^{\xi y-xs+\widehat{V}_2(s)-V_2(y)}+\\
&+V_1'(x)(1-\delta_{m,n})\psi_m^{(r)}(x)-\sum_{j=m-1}^{n-1}\widehat{P}_{mj}\widehat{\psi}_j^{(r)}(x)=\\
=&\frac{\widehat{\psi}_m(\xi)\widehat{\underline{\Phi}}_n^{(\varkappa)}(\xi)}{\xi-x}\bigg|_{\xi\in\partial_x\varkappa}\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(x)+\iint_{\varkappa}\frac{V_1'(\xi)-V_1'(x)}{\xi-x}\mathrm{e}^{\xi y}\widehat{\psi}_m(\xi)\vec{\Phi}_n(y)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n^{(r)}(x)\\
&-\delta_{mn}V_1'(x)\psi_m^{(r)}(x)-\sum_{j=0}^{n-1}\widehat{\psi}_j^{(r)}(x)\iint_{\varkappa}\widehat{\psi}_m(\xi)y\,\widehat{\phi}_j(y)\mathrm{e}^{\xi y}
\end{aligned}
\tag{1-15}
$$

This is the same expression as for the quasipolynomials: since the auxiliary wave functions $\widehat{\psi}_j^{(r)}(x)$ satisfy the same multiplicative recurrence relation (for n large enough) as the quasipolynomials, re-expressing $\widehat{\psi}_{n+1}^{(r)}(x)$ in terms of the elements of the window yields the same differential equation. **Q.E.D.**

A.5 Proof of Thm. 5.1

For brevity we denote $\widehat{\mathbb{A}}_n(x)$ simply by $\widehat{\mathbb{A}}(x)$ during this proof. Since the rows (columns) of $\underline{\mathbf{\Phi}_n}$ ($\widehat{\mathbf{\Psi}}_n$) are of two types, we need to carry out four types of computations

$$
\begin{array}{ll}
(\mathbf{a})=\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n^{(0)}(x)\ , & (\mathbf{b})=\underline{\Phi}_n^{(0)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n^{(j)}(x)\ ,\ j=1\ldots d_2\\
(\mathbf{c})=\underline{\Phi}_n^{(j)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n^{(0)}(x)\ ,\ j=1\ldots d_2 & (\mathbf{d})=\underline{\Phi}_n^{(\ell)}(x)\widehat{\mathbb{A}}(x)\widehat{\Psi}_n^{(m)}(x)\ ,\ \ell,m=1\ldots d_2
\end{array}
\tag{1-16}
$$

It follows trivially from (5-3) that $(\mathbf{c})=0$ (set $x=x'$ in the LHS). For $(\mathbf{a})$ we have

$$
\begin{aligned}
(\mathbf{a})\ &=\ \mathrm{e}^{V_1(x)}\iint_{\varkappa}\frac{\Phi_n(\xi)}{x-\zeta}\mathrm{e}^{-V_1(\zeta)+\xi\zeta}\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(x)= \qquad (1\text{-}17)\\
&=\mathrm{e}^{V_1(x)}\iint_{\varkappa}\mathrm{d}\zeta\,\mathrm{e}^{-V_1(\zeta)+\xi\zeta}\sum_{j=0}^{n-1}\widehat{\phi}_j(\xi)\widehat{\psi}_j(x)=\mathrm{e}^{V_1(x)}\iint_{\varkappa}\mathrm{d}\zeta\,\mathrm{e}^{-V_1(\zeta)+\xi\zeta}\widehat{\phi}_0(\xi)\widehat{\psi}_0(x)=1
\end{aligned}
$$

where we have used that $\widehat{\phi}_j(\zeta)$, $j\geqslant 1$ are orthogonal to $p(\xi)\equiv 1$. Note also that we had to use the CDI in the form (5-2). Then we have to compute for $1\leqslant\ell,m\leqslant d_2$ (we suppress explicit reference to the variables of integration because there is no possibility of ambiguity)

$$
(\mathbf{d})\ =\ \frac{1}{2i\pi}\underline{\Phi}_n^{(\ell)}(x)\iint_{\varkappa}\int_{s\in\widehat{\Gamma}_m}\mathcal{B}_2(x;\eta,s)\frac{\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\xi)}{x-\xi}\mathrm{e}^{\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)}= \tag{1-18}
$$

$$
=\ \frac{1}{2i\pi}\sum_{j=0}^{n-1}\underline{\phi}_j^{(\ell)}(x)\iint_{\varkappa}\int_{s\in\widehat{\Gamma}_m}\mathcal{B}_2(x;\eta,s)\psi_j(\xi)\mathrm{e}^{\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)}= \tag{1-19}
$$

$$= \frac{1}{2i\pi}\sum_{j=0}^{n-1}\int_{\widehat{\Gamma}_m}\mathrm{d}s\int_{\Gamma_\ell}\mathrm{d}y\,\phi_j(y)\mathrm{e}^{xy}\iint_{\varkappa}\mathcal{B}_2(x;\eta,s)\psi_j(\xi)\mathrm{e}^{\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)} \stackrel{\star}{=} \tag{1-20}$$

$$= \frac{1}{2i\pi}\int_{\widehat{\Gamma}_m}\mathrm{d}s\int_{\Gamma_\ell}\mathrm{d}y\,\mathcal{B}_2(x;y,s)\mathrm{e}^{x(y-s)-V_2(y)+\widehat{V}_2(s)} = \tag{1-21}$$

$$= \mathcal{B}_2(\Gamma_{y,\ell},\widehat{\Gamma}_{y,m}) = \delta_{\ell m}\ , \tag{1-22}$$

where in the step marked with a star we have used that for the polynomial of η $P(\eta) := \mathcal{B}_2(x;\eta,s)$ is reproduced by the kernel

$$P(y) = \sum_{j=0}^{n-1}s_j(y)\iint_{\varkappa}\mathrm{d}\eta\mathrm{d}\xi\psi_j(\xi)\mathrm{e}^{-V_2(\eta)+\xi\eta}P(\eta) \tag{1-23}$$

provided that $n-1 \geqslant \deg P = d_2-1$. Note also that in this latter computation we are forced to use the other form of the CDI (5-3). Finally we need to compute $(\mathbf{b})$, which involves quintuple integrals

$$\begin{aligned}
(\mathbf{b}) = {}& \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\iint_{\varkappa}\frac{\Phi_n(\rho)}{x-\zeta}\mathrm{e}^{-V_1(\zeta)+\rho\zeta}\iint_{\varkappa}\int_{s\in\widehat{\Gamma}_m}\mathcal{B}_2(x;\eta,s)\frac{\widehat{\mathbb{A}}(x)\widehat{\Psi}_n(\xi)}{x-\xi}\mathrm{e}^{\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)} = \\
= {}& \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\iint_{\varkappa}\frac{\Phi_n(\rho)}{x-\zeta}\mathrm{e}^{-V_1(\zeta)+\rho\zeta}\iint_{\varkappa}\int_{s\in\widehat{\Gamma}_m}\mathcal{B}_2(x;\eta,s)\frac{\widehat{\mathbb{A}}(\zeta)\widehat{\Psi}_n(\xi)}{x-\xi}\mathrm{e}^{\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)} + \\
& + \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\underbrace{\iint_{\varkappa}\Phi_n(\rho)[\widehat{L},p_n]\mathrm{e}^{-V_1(\zeta)+\rho\zeta}}_{=0 \text{ if } n\geqslant q_1}\iint_{\varkappa}\int_{s\in\widehat{\Gamma}_m}\mathcal{B}_2(x;\eta,s)\frac{\widehat{\Psi}_n(\xi)}{x-\xi}\mathrm{e}^{\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)} =
\end{aligned} \tag{1-24}$$

$$\begin{aligned}
= {}& \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\sum_{j=0}^{n-1}\iint_{\varkappa}\iint_{\varkappa}\int_{s\in\check{\Gamma}_m}\mathrm{d}s\,\mathrm{e}^{-V_1(\zeta)+\zeta\rho+\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)}\phi_j(\rho)\psi_j(\xi)\mathcal{B}_2(x;\eta,s)\frac{\zeta-\xi}{(x-\zeta)(x-\xi)} = \\
= {}& \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\sum_{j=0}^{n-1}\int\!\mathrm{d}y\iint_{\varkappa}\iint_{\varkappa}\int_{s\in\widehat{\Gamma}_m}\mathrm{d}s\,\mathrm{e}^{-V_1(\zeta)+\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)+\zeta\rho}\phi_j(\rho)\psi_j(\xi)\mathcal{B}_2(x;\eta,s)\left(\frac{1}{x-\zeta}-\frac{1}{x-\xi}\right) = \\
= {}& \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\sum_{j=0}^{n-1}\int\!\mathrm{d}y\iiiint_{\check{\Gamma}}\mathrm{d}s\,\mathrm{e}^{-V_1(\zeta)+\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)+\zeta\rho}\phi_j(\rho)\psi_j(\xi)\mathcal{B}_2(x;\eta,s)\frac{1}{x-\zeta} + \\
& - \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\iiint_{\check{\Gamma}}\mathrm{d}s\,\mathcal{B}_2(x;\eta,s)\frac{1}{x-\xi}\mathrm{e}^{\xi\eta-xs-V_2(\eta)+\widehat{V}_2(s)-V_1(\xi)} = \\
= {}& \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\int\!\mathrm{d}y\iint_{\check{\Gamma}}\mathrm{d}s\,\mathrm{e}^{-V_1(\zeta)-V_2(\rho)+\zeta\rho-xs+\widehat{V}_2(s)}\mathcal{B}_2(x;\rho,s)\frac{1}{x-\zeta} + \\
& - \frac{\mathrm{e}^{V_1(x)}}{2i\pi}\iiint_{\check{\Gamma}}\mathrm{d}s\,\mathrm{e}^{-V_1(\xi)-V_2(\eta)+\xi\eta-xs+\widehat{V}_2(s)}\mathcal{B}_2(x;\eta,s)\frac{1}{x-\xi} \equiv 0
\end{aligned} \tag{1-25}$$

Once more, we are forced to use the CDI in the form (5-3). This concludes the proof. **Q.E.D.**

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