MATH699E(494O) Analytic Number Theory
Winter 2020

Assignment 1

Due January 31, 2020

1. (a) Show that for any positive integer K

$$\mathrm{li}(x) = x\sum_{k=1}^{K-1} \frac{(k-1)!}{\log^k x} + O_K\left(\frac{x}{\log^K x}\right).$$

(b) Let $c, \varepsilon > 0$ and A be any positive integer. Show that there exists $x_0, x_1 > 0$ such that

$$\exp\left(c\sqrt{\log x}\right) \leq x^{\varepsilon} \qquad \text{for all } x > x_0$$

$$(\log x)^A \leq \exp\left(c\sqrt{\log x}\right) \qquad \text{for all } x > x_1$$

2. (a) Let $\lambda(n)$ denote the Liouville's function given by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the total number of primes of n, counting multiplicities. Show that for $\mathrm{Re}(s) > 1$,

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

(b) Let $\tau(n)$ be the number of divisors of n. Show that for $\mathrm{Re}(s) > 1$,

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s}.$$

3. (a) Show that

$$f(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

converges for Re(s) > 1 by showing that $A(x) = \sum_{n\leq x} \tau(n) = O(x\log x)$.

(b) Show that if $|A(x)| \leq Cx/\log^2 x$ for all $x \geq 2$, then $\sum_{n=1}^{\infty} a(n)/n$ converges.

(c) If $a(n) \geq 0$ for all n, and $A(x) \geq Cx/\log x$ for all $x \geq 2$, then $\sum_{n=1}^{\infty} a(n)/n$ diverges.

4. (a) Assume that $\pi(x) \sim x/\log x$. Show that it implies that

$$\sum_{p \leq x} \frac{\log p}{p} \sim \log x$$
$$\sum_{p \leq x} \frac{1}{p} \sim \log\log x.$$

[However, those assertions are weaker than the Prime Number Theorem and can be derived by elementary methods.]

(b) Assume the Twin Prime Conjecture under the form

$$\#\{p \leq x \;:\; p+2 \text{ is prime}\} \sim \mathfrak{S}\frac{x}{\log^2 x}.$$

Show that this implies that the sum $\displaystyle\sum_p \frac{1}{p}$, where p runs over the set of primes p such that $p+2$ is prime, converges. [But this can be proven without any hypothesis using the Brun's sieve.]

5. An integer n is power-full if $p \mid n \Rightarrow p^2 \mid n$. Let $\mathcal{F}$ be the set of power-full numbers.

(a) Show that for $\sigma > 1/2$,

$$\sum_{n \in \mathcal{F}} n^{-s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

(b) Show that any power-full number can be written as $a^2 b^3$, and this representation is unique if b is square-free.

(c) Show that

$$\begin{aligned}
\sum_{a^2b^3 \le x} 1 &= \zeta(3/2)x^{1/2} + O(x^{1/3}) \\
\sum_{\substack{n \le x \\ n \in \mathcal{F}}} 1 &= \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3})
\end{aligned}$$