MATH699E(833C) Analytic Number Theory
Winter 2020

Assignment 2

Due Friday February 21

1. An integer n is power-full if $p \mid n \Rightarrow p^2 \mid n$. Let $\mathcal{F}$ be the set of power-full numbers. Show that

$$\begin{aligned} \sum_{a^2b^3 \le x} 1 &= \zeta(3/2)x^{1/2} + \zeta(2/3)x^{1/3} + O(x^{1/5}) \\ \sum_{\substack{n \le x \\ n \in \mathcal{F}}} 1 &= \frac{\zeta(3/2)}{\zeta(3)}x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)}x^{1/3} + O(x^{1/5}) \end{aligned}$$

2. Let $\tau(n)$ be the number of divisors of n, and let χ be a Dirichlet character modulo q. Show that for $\sigma > 1$,

(a)
$$\sum_{n=1}^{\infty} (-1)^{n-1}\chi(n)n^{-s} = \left(1 - \chi(2)2^{1-s}\right) L(s,\chi);$$

(b)
$$\sum_{n=1}^{\infty} \frac{\tau(n)^2\chi(n)}{n^s} = \frac{L(s,\chi)^4}{L(2s,\chi^2)}.$$

3. Let $f_1(n)$ and $f_2(n)$ be totally multiplicative functions and suppose that $|f_i(n)| \le 1$ for all n.

(a) Show that if $\sigma > 1$, then

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} f_1(d) \right) \left(\sum_{d|n} f_2(d) \right) n^{-s}$$
$$= \frac{\zeta(s) \left(\sum_{n=1}^{\infty} f_1(n)n^{-s} \right) \left(\sum_{n=1}^{\infty} f_2(n)n^{-s} \right) \left(\sum_{n=1}^{\infty} f_1(n)f_2(n)n^{-s} \right)}{\left(\sum_{n=1}^{\infty} f_1(n)f_2(n)n^{-2s} \right)}$$

$$= \frac{\prod_p \left(1 - \frac{f_1(p)f_2(p)}{p^{2s}}\right)}{\prod_p \left(1 - \frac{1}{p^s}\right) \prod_p \left(1 - \frac{f_1(p)}{p^s}\right) \prod_p \left(1 - \frac{f_2(p)}{p^s}\right) \prod_p \left(1 - \frac{f_1(p)f_2(p)}{p^s}\right)}$$

(b) By considering

$$F(s) = \sum_{n=1}^{\infty} \left| \sum_{d|n} \chi(d) d^{-iu} \right|^2 n^{-s},$$

show that $L(1+iu,\chi) \neq 0$.

4. (a) Show that for arbitrary real or complex numbers $c_1, \ldots, c_q$,

$$\sum_{\chi} \left| \sum_{n=1}^{q} c_n \chi(n) \right|^2 = \phi(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2$$

where the sum over χ runs over all Dirichlet characters modulo q.

(b) Show that for arbitrary real or complex numbers c_χ,

$$\sum_{n=1}^{q} \left| \sum_{\chi} c_\chi \chi(n) \right|^2 = \phi(q) \sum_{\chi} |c_\chi|^2$$

where the sum over χ runs over all Dirichlet characters modulo q.

5. Let $S(T)$ be the set of primitive quadratic characters with odd conductor smaller or equal to T. Let n be a positive integer. Show that

$$\#S(T) = \frac{2}{3\zeta(2)} T + O\left(\sqrt{T}\right)$$

and that

$$\frac{1}{\#S(T)} \sum_{\chi \in S(T)} \chi(n) \sim \begin{cases} \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1} & \text{if } n \text{ is an odd square} \\ \frac{3}{2} \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1} & \text{if } n \text{ is an even square} \end{cases}$$

$$\sum_{\chi \in S(T)} \chi(n) = \underline{o}\left(\#S(T)\right) \quad \text{when } n \text{ is not a square.}$$

Hint: You will find the following formula

$$\sum_{k^2|d} \mu(k) = \begin{cases} 1 & \text{if } d \text{ is square-free} \\ 0 & \text{otherwise} \end{cases}$$

very useful. Prove it first.