Analytic Number Theory, Winter 2020

Assignment 3

Due Wednesday April 1

1. (a) Suppose that both f and $\hat{f}$ are in $L^1(\mathbb{R})$ and have bounded variation. Show that

$$\sum_{m\in\mathbb{Z}} f(vm+u) = \frac{1}{v}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{n}{v}\right) e\left(\frac{un}{v}\right),$$

and then by Fourier inversion

$$v\sum_{m\in\mathbb{Z}} \hat{f}(vm-u) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{v}\right) e\left(\frac{un}{v}\right),$$

(b) Use (a) to show that for a primitive character χ modulo q

$$\sum_{m\in\mathbb{Z}} f(m)\chi(m) = \frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{n}{q}\right)\overline{\chi}(n).$$

2. Let q and a be fixed relatively prime integers. Prove that

$$\sum_{\substack{n\le x \\ n\equiv a \pmod q}} \mu^2(n) \sim \frac{6}{\pi^2}\frac{c(q)}{\phi(q)}x$$

where $c(q) = \prod_{p|q} p/(p+1)$. What does it says about the distribution of square-free numbers that are relatively prime to q? [*Remark:* You can use the Tauberian Theorem if you want].

3. Let $a(n)$ be the number of non-isomorphic abelian groups of order n. The goal of this exercise is to show that

$$\sum_{n\le x} a(n) = Rx + O\left(x^{3/4}\right)$$

where

$$R \;=\; \zeta(2)\zeta(3)\zeta(4)\ldots \approx 2.29485$$

(a) Show that $a(n)$ is multiplicative, and that $a(p^\alpha)$ is the number of ways to separate a set of α elements in subsets, or equivalently the number of ways to write $a_1 + a_2 + \ldots + a_k = \alpha$ where the integers a_i are such that $1 \leq a_1 \leq a_2 \leq \ldots \leq a_k$.

(b) Show that for $\mathrm{Re}(s) > 0$,

$$\sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)}{p^{\alpha s}} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{p^{ks}}\right)^{-1},$$

and then

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)}{p^\alpha s} \\ &= \prod_{k=1}^{\infty} \zeta(ks), \end{aligned}$$

where the sum and the product are absolutely convergent for $\mathrm{Re}(s) > 1$. Then, $F(s)$ is analytic for $\mathrm{Re}(s) > 0$, except for simple poles at $s = 1, 1/2, 1/3, \ldots$.

(c) Let $F_1(s) = \zeta(s)\zeta(2s) = \sum_{n=1}^{\infty} b_n n^{-s}$. Show that

$$B(x) = \sum_{n \leq x} b_n = x\zeta(2) + O(x^{3/4}).$$

[*Remark:* This is an application of contour integration using Perron's formula. Use the convexity bound: $\zeta(\sigma + it) \ll t^{(1-\sigma)/2}$ for $0 < \sigma < 1$.]

(d) Let $F_2(s) = \prod_{k=3}^{\infty} \zeta(ks) = \sum_{n=1}^{\infty} c_n n^{-s}$. Show that $F_2(s)$ converges absolutely for $\mathrm{Re}(s) > 1/3$, and

$$\sum_{n \leq x} \frac{c_n}{n^\alpha} = F_2(\alpha) + O\left(x^{1/3-\alpha}\right)$$

for $\alpha > 1/3$.

(e) Conclude the desired estimate for $A(x) = \sum_{n \le x} a_n = \sum_{mn \le x} b_m c_n$.

4. (a) Show that for any integer $k \ge 1$ and any $c > 0$,

$$\frac{1}{2\pi i} \int_{(c)} \frac{x^s}{s^{k+1}}\, ds = \begin{cases} \log^k x/k! & \text{if } x \ge 1; \\ 0 & \text{if } 0 < x \le 1. \end{cases}$$

(b) Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series absolutely convergent in $\mathrm{Re}(s) > c - \epsilon$. Show that

$$\sum_{n \le x} a_n \left(\log \frac{x}{n} \right)^k = \frac{k!}{2\pi i} \int_{(c)} f(s) \frac{x^s}{s^{k+1}}\, ds.$$

5. Let χ be a real primitive character of conductor q. Show that if $L(s, \chi)$ has 2 real zeros $\beta_0 \le \beta_1 < 1$ (including the case of a double zero $\beta_0 = \beta_1$), then

$$\beta_0 < 1 - \frac{c}{\log q}$$

for some positive constant c.