MATH699E(833C) Analytic Number Theory
Winter 2020
Solutions to Assignment 1

1. (a) Show that for any positive integer K

$$\mathrm{Li}(x) = x\sum_{k=1}^{K-1}\frac{(k-1)!}{\log^k x} + O_K\left(\frac{x}{\log^K x}\right).$$

Solution: We first show that

$$\mathrm{Li}(x) = x\sum_{k=1}^{K-1}\frac{(k-1)!}{\log^k x} + (K-1)!\int_2^x \frac{dt}{\log^K t} + O_K(1). \tag{1}$$

It is trivially true for $K=1$, and for $K=2$, integrating by part with $u = 1/\log t$ and $dv = dt$, we have that

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t}.$$

Then, by induction, and integrating by part with $u = 1/\log^K t$ and $dv = dt$, we have that

$$\begin{aligned}
\mathrm{Li}(x) &= x\sum_{k=1}^{K-1}\frac{(k-1)!}{\log^k x} + (K-1)!\int_2^x \frac{dt}{\log^K t} + O_K(1)\\
&= x\sum_{k=1}^{K-1}\frac{(k-1)!}{\log^k x} + \frac{(K-1)!x}{\log^K x} + \int_2^x \frac{K!dt}{\log^{K+1} t} + O_K(1)\\
&= x\sum_{k=1}^{K}\frac{(k-1)!}{\log^k x} + \int_2^x \frac{K!dt}{\log^{K+1} t} + O_K(1)
\end{aligned}$$

which shows (1). Finally, we have for any $0 < \theta < 1$,

$$\begin{aligned}
\int_2^x \frac{(K-1)!dt}{\log^K t} &= \int_2^{x^\theta} \frac{(K-1)!dt}{\log^K t} + \int_{x^\theta}^x \frac{(K-1)!dt}{\log^K x}\\
&\le (K-1)!x^\theta + \frac{(K-1)!x}{\theta^K \log^K x} \ll_K \frac{x}{\log^K x}.
\end{aligned}$$

(b)

2. (a) Let $\lambda(n)$ denote the Liouville's function given by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the total number of primes of n, counting multiplicities. Show that for $\mathrm{Re}(s) > 1$,

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

(b) Let $\tau(n)$ be the number of divisors of n. Show that for $\mathrm{Re}(s) > 1$,

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s}.$$

Solutions: (a) For $\Re(s) > 1$,

$$\begin{aligned} \frac{\zeta(2s)}{\zeta(s)} &= \frac{\prod_p (1-p^{-2s})^{-1}}{\prod_p (1-p^{-s})^{-1}} = \frac{\prod_p (1-p^{-s})^{-1}(1+p^{-s})^{-1}}{\prod_p (1-p^{-s})^{-1}} \\ &= \prod_p \left(1-(-p^{-2s})\right)^{-1} = \prod_p \left(1 - p^{-s} + p^{-2s} - p^{-3s} \dots\right) =: \sum_n \frac{a(n)}{n^s}. \end{aligned}$$

Let $n = p_1^{e_1} \dots p_s^{e_s}$. Then it is clear from above that

$$a(n) = (-1)^{e_1} \dots (-1)^{e_s} = (-1)^{\sum_{i=1}^{s} e_i} = (-1)^{\Omega(n)} = \lambda(n).$$

(b) We first remark that since $\tau(n)$ is multiplicative, so is $\tau(n^2)$. Then, for $\Re(s) > 1$,

$$\sum_n \frac{\tau(n)}{n^s} = \prod_p \sum_{k=0}^{\infty} \frac{\tau(p^{2k})}{p^{ks}} = \prod_p \sum_{k=0}^{\infty} \frac{2k+1}{p^{ks}}.$$

Note that

$$\frac{1}{(1-x)^2} = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{m=0}^{\infty} x^m\right) = \sum_{k=0}^{\infty} (k+1)\, x^k,$$

and then

$$\begin{aligned}\frac{1}{(1-x)^2}+\frac{x}{(1-x)^2} &= \sum_{k=0}^{\infty} k+1x^k+\sum_{k=0}^{\infty} k+1x^{k+1}\\ &= \sum_{k=0}^{\infty} k+1x^k+\sum_{k=1}^{\infty} kx^k=\sum_{k=0}^{\infty}(2k+1)\,x^k.\end{aligned}$$

Then,

$$\sum_{k=0}^{\infty}(2k+1)\,x^k=\frac{1+x}{(1-x)^2}=\frac{1-x^2}{(1-x)^3},$$

and using $x=p^{-s}$, we have

$$\prod_p\sum_{k=0}^{\infty}\frac{2k+1}{p^{ks}}=\prod_p\frac{1-p^{-2s}}{(1-p^{-s})^3}=\frac{\zeta^3(s)}{\zeta(2s)}.$$

3. (a) Show that

$$f(s)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}$$

converges for Re(s) >1 by showing that $A(x)=\sum_{n\le x}\tau(n)=O(x\log x)$.

(b) Show that if $|A(x)|\le Cx/\log^2 x$ for all $x\ge 2$, then $\sum_{n=1}^{\infty}a(n)/n$ converges.

(c) If $a(n)\ge 0$ for all n, and $A(x)\ge Cx/\log x$ for all $x\ge 2$, then $\sum_{n=1}^{\infty}a(n)/n$ diverges.

Solutions: We showed in class that

$$\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+O(x^{1/2})=O\left(x^{1+\epsilon}\right)$$

for any $\epsilon>0$, which implies that the Dirichelt series converges for $\Re(s)>1$ taking the Mellin transform as we saw in class.

(b) This can be proven by partial summation.

(c) This can be proven by partial summation.

4. (a) Assume that $\pi(x) \sim x/\log x$. Show that it implies that

$$\begin{aligned} \sum_{p \le x} \frac{\log p}{p} &\sim \log x \\ \sum_{p \le x} \frac{1}{p} &\sim \log\log x. \end{aligned}$$

[However, those assertions are weaker than the Prime Number Theorem and can be derived by elementary methods.]

(b)

Solutions: (a) We first show that $f(x) \sim g(x)$ implies that $\int_2^x f(t)\,dt \sim \int_2^x g(t)\,dt$ for positive integrable functions f, g such that for any $0 < \theta < 1$,

$$\int_2^{x^\theta} g(t)\,dt = \underline{o}\left(\int_2^x g(t)\,dt\right),$$

which is the case for most reasonable functions, and certainly for $g(x) = x/\log x$.

Since $f(x) \sim g(x)$, we have that $f(x) = g(x) + e(x)g(x)$ where $e(x) \to 0$ as $x \to \infty$. Then, for any $0 < \theta < 1$,

$$\begin{aligned} \int_2^x f(t)\,dt - \int_2^x g(t)\,dt &= \int_2^x e(t)g(t)\,dt \\ &= \int_2^{x^\theta} e(t)g(t)\,dt + \int_{x^\theta}^x e(t)g(t)\,dt \\ &\ll \int_2^{x^\theta} g(t)\,dt + \left(\sup_{x^\theta \le t \le x} |e(t)|\right) \int_{x^\theta}^x g(t)\,dt \\ &= \underline{o}\left(\int_2^x g(t)\,dt\right) \end{aligned}$$

since $\sup_{x^\theta \le t \le x} |e(t)| \to 0$ as $x \to \infty$. We use this result in the proofs of (a) and (b).

For the first sum, let

$$\begin{aligned} a_n &= \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases} \\ f(n) &= \frac{\log n}{n} \end{aligned}$$

Then,

$$\begin{aligned} \sum_{p \le x} \frac{\log p}{p} &= \pi(x)\frac{\log x}{x} - \int_1^x \pi(t)\left(\frac{1}{t^2} - \frac{\log t}{t^2}\right) dt \\ &\sim \int_2^x \frac{1}{t} - \frac{1}{t \log t}\, dt + O(1) \\ &= \log x - \log\log x + O(1) \sim \log x. \end{aligned}$$

Similarly, using a_n as above and $f(n) = 1/n$, we have

$$\begin{aligned} \sum_{p \le x} \frac{1}{p} &= \pi(x)\frac{1}{x} - \int_1^x \pi(t)\frac{-1}{t^2}\, dt \\ &\sim \int_2^x \frac{1}{t \log t}\, dt \sim \log\log x. \end{aligned}$$

(b) Similarly, we have using $a(n) = 1$ if $n = p$ and $n + 2 = q$ (where p and q are primes),

$$\sum_{p \le x, p+2 \text{ prime}} \frac{1}{p} \sim \mathfrak{S}\left(\frac{x}{\log^2 x}\frac{1}{x} + \int_1^x \frac{t}{\log^2 t}\frac{1}{t^2}\, dt\right) = O(1).$$

5. An integer n is power-full if $p \mid n \Rightarrow p^2 \mid n$. Let $\mathcal{F}$ be the set of power-full numbers.

(a) Show that for $\sigma > 1/2$,

$$\sum_{n \in \mathcal{F}} n^{-s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

(b) Show that any power-full number can be written as a^2b^3, and this representation is unique if b is square-free.

(c) Show that

$$\begin{aligned}\sum_{a^2b^3\le x} 1 &= \zeta(3/2)x^{1/2}+O(x^{1/3})\\ \sum_{\substack{n\le x\\ n\in\mathcal{F}}} 1 &= \frac{\zeta(3/2)}{\zeta(3)}x^{1/2}+O(x^{1/3})\end{aligned}$$

Solutions: (a) Let

$$\mathcal{F}(x)=\sum_{\substack{n\le x\\ n\in\mathcal{F}}} 1.$$

Then, using (c), we have that $\mathcal{F}(x)\sim\frac{\zeta(3/2)}{\zeta(3)}x^{1/2}$ and

$$f(s)=\sum_{n\in\mathcal{F}} n^{-s}$$

is analytic for $\mathrm{Re}(s)>1/2$ and we have the Euler product

$$\begin{aligned}\sum_{n\in\mathcal{F}} n^{-s} &= \prod_p\left(1+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\dots\right)\\ &= \prod_p\left(\left(1-\frac{1}{p^s}\right)^{-1}-\frac{1}{p^s}\right)\\ &= \prod_p\frac{p^{2s}-p^s+1}{p^s(p^s-1)}.\end{aligned}$$

We also compute for $\mathrm{Re}(s)>1/2$

$$\begin{aligned}\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} &= \prod_p\left(\frac{p^{2s}}{p^{2s}-1}\right)\left(\frac{p^{3s}}{p^{3s}-1}\right)\left(\frac{p^{6s}-1}{p^{6s}}\right)\\ &= \prod_p\left(\frac{p^{6s}-1}{p^s(p^{2s}-1)(p^{3s}-1)}\right)\\ &= \prod_p\frac{p^{2s}-p^s+1}{p^s(p^s-1)},\end{aligned}$$

where the last equality follows from the identities

$$\frac{x^6-1}{(x^2-1)(x^3-1)} = \frac{x^3+1}{x^2-1} = \frac{x^2-x+1}{x-1}.$$

(b) Let n be a power-full number, i.e.

$$n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} p_{k+1}^{e_{k+1}} \dots p_\ell^{e_\ell}$$

with $e_1, \dots, e_k$ odd (and then ≥ 3) and $e_{k+1}, \dots, e_\ell$ even. Then $n = a^2 b^3$ where $a = p_1^{(e_1-3)/2} \dots p_k^{(e_k-3)/2} p_{k+1}^{e_{k+1}/2} \dots p_\ell^{e_\ell/2}$ and $b = p_1 \dots p_k$. This representation is unique, as the primes dividing b will have odd order in $n = a^2 b^3$ since b is square-free, and every prime which appears with odd order in $n = a^2 b^3$ must divide b.

(c) We have that

$$\begin{aligned}
\sum_{a^2 b^3 \leq x} 1 &= \sum_{b^3 \leq x} \sum_{a^2 \leq x/b^3} 1 = \sum_{b^3 \leq x} \left[\left(\frac{x}{b^3} \right)^{1/2} \right] \\
&= \sum_{b^3 \leq x} \left(\left(\frac{x}{b^3} \right)^{1/2} + O(1) \right) = x^{1/2} \sum_{b^3 \leq x} \frac{1}{b^{3/2}} + O\left(\sum_{b^3 \leq x} 1 \right) \\
&= x^{1/2} \sum_{b=1}^{\infty} \frac{1}{b^{3/2}} - x^{1/2} \sum_{b^3 > x} \frac{1}{b^{3/2}} + O(x^{1/3}) \\
&= \zeta(3/2) x^{1/2} + O\left(x^{1/2} \int_{x^{1/3}}^{\infty} \frac{dt}{t^{3/2}} \right) + O(x^{1/3}) \\
&= \zeta(3/2) x^{1/2} + O\left(x^{1/2} x^{-1/6} \right) + O(x^{1/3}) \\
&= \zeta(3/2) x^{1/2} + O(x^{1/3})
\end{aligned}$$

Similarly,

$$\sum_{\substack{n \leq x \\ n \in \mathcal{F}}} 1 = \sum_{a^2 b^3 \leq x} \mu^2(b)$$

$$
\begin{aligned}
&= \sum_{b^3 \le x} \mu^2(b) \sum_{a^2 \le x/b^3} 1 = \sum_{b^3 \le x} \mu^2(b) \left[\left(\frac{x}{b^3} \right)^{1/2} \right] \\
&= \sum_{b^3 \le x} \mu^2(b) \left(\frac{x}{b^3} \right)^{1/2} + O(1) = x^{1/2} \sum_{b^3 \le x} \frac{\mu^2(b)}{b^{3/2}} + O\left(\sum_{b^3 \le x} 1 \right) \\
&= x^{1/2} \sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{3/2}} - x^{1/2} \sum_{b^3 > x} \frac{\mu^2(b)}{b^{3/2}} + O(x^{1/3}) \\
&= \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O\left(x^{1/2} \int_{x^{1/3}}^{\infty} \frac{dt}{t^{3/2}} \right) + O(x^{1/3}) \\
&= \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3})
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{b=1}^{\infty} \frac{\mu^2(b)}{b^{3/2}} &= \prod_p \left(1 + \frac{1}{p^{3/2}} \right) \\
\frac{\zeta(3/2)}{\zeta(3)} &= \prod_p \frac{1 - \frac{1}{p^3}}{1 - \frac{1}{p^{3/2}}} = \prod_p \left(1 + \frac{1}{p^{3/2}} \right).
\end{aligned}
$$