Elliptic curves. Fall 2018

Assignment 2. Due Wednesday October 3.

Directions:

Undergraduate students answer 2 problems at their choice.
M.SC. students answer 3 problems at their choice.
Ph.D. students answer 4 problems at their choice.

1. Show that Propostion 1.2 and Theorem 2.3 (of Chapter II in Silverman) are true for $C = \mathbb{P}^1$ and $C_1 = C_2 = \mathbb{P}^1$ respectively.

2. Show that Theorem 2.3 (of Chapter II in Silverman) is true when C_1, C_2 are plane curves given by a single equation.
Hint: Use the resultant of homogeneous polynomials with respect to different variables.

3. Let $C : F(X, Y, Z) = 0$ be a plane curve given by a single equation. Show that a point P is smooth if and only if M_P is a principal ideal.

4. Let K be a field of characteristic different than 2. Let E/K be the curve with affine equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_1, a_2, a_3, a_4, a_6 \in K$.

(a) Show that E is isomorphic to a curve $y^2 = x^3 + px + q$, which is non-singular if and only if $4p^3 + 27q^2 \neq 0$. We suppose from now on that E is non-singular.

(b) Show that the rational map ϕ defined on E by $\phi(x, y) = (x, -y - a_1x - a_3)$ is an isomorphism.

(c) Let $f \in K(E)$ and $\phi^* f = f \circ \phi$. Let $P \in E(K)$, $Q = \phi(P)$ and t_Q be a uniformizer at Q. Show that $\phi^* t_Q$ is a uniformizer at P and $v_Q(f) = v_P(\phi^* f)$.

5. Let E be a curve as in **4.**. Let $f \in K(E)^*$. Show that $\deg(\mathrm{div}(f)) = 0$ by following the steps:

(a) Show that the result hold for $f(x) = (x - x_i)$, and then for any polynomial $a(x) \in K(E)$.

(b) Show that result hold for $f(x,y) = a(x) + yb(x)$. *Hint:* Use the map ϕ of **1.**

(c) Show that the result hold for a general $f \in K(E)$.

6. (Silverman II.2.2) Let $\phi : C_1 \to C_2$ be a non-constant map of smooth curves, $f \in \overline{K}(C_2)^*$, $P \in C_1$. Show that

$$\mathrm{ord}_P(\phi^* f) = e_\phi(P)\, \mathrm{ord}_{\phi(P)}(f).$$

7. (Silverman II.2.14) Find a smooth model for hyperelliptic curves. Do (a,b) only, and then describe explicitely the points at infinity.

8. (Silverman II.2.15))

9. (Silverman II.2.16)