Modular Forms Homework 1.
Due September 24, 2025.

1. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ and f be an elliptic function for Λ. Show that

$$\sum_{z \bmod \Lambda} \mathrm{ord}_z(f) = 0$$

and

$$\sum_{z \bmod \Lambda} \mathrm{ord}_z(f)\, z \equiv 0 \bmod \Lambda.$$

2. Show that for k even,

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where the Bernoulli numbers B_k are defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

3. Using the formula (1.46) from Iwaniec's book for $k = 2$, derive the following Fourier expansion of the Weierstrass function:

$$\mathfrak{p}(u; z) = \left(\frac{\pi}{\sin(\pi u)}\right)^2 - \frac{\pi^2}{3} + 16\pi^2 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \sin^2(\pi d u)\right) e(nz).$$

4. Show that $G_4\left(e^{\frac{2\pi i}{3}}\right) = 0$ and $G_6(i) = 0$.

5. Express $\begin{pmatrix} 70 & 213 \\ 23 & 70 \end{pmatrix}$ in terms of the generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$.

6. (a) Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.

(b) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Prove that γ has exactly one fixed point if $\mathbb{H}$ if $|a + d| < 2$ and no fixed point in $\mathbb{H}$ otherwise.

7. (a) Show that the stabilizer of i under the action of $\mathrm{SL}_2(\mathbb{R})$ is the special orthogonal group

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;:\; a, b \in \mathbb{R},\, a^2 + b^2 = 1 \right\}.$$

(b) Prove that the map

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) &\to \mathbb{H} \\ \gamma\mathrm{SO}_2(\mathbb{R}) &\mapsto \gamma i \end{aligned}$$

is a bijection.