Modular Forms Homework 2.
Due October 15, 2025.

1.

(a) Prove that

$$\sigma_9(n) = \frac{21}{11}\sigma_5(n) - \frac{10}{11}\sigma_3(n) + \frac{5040}{11}\sum_{j=1}^{n-1}\sigma_3(j)\sigma_5(n-j) \text{ for all } n \in \mathbb{Z}_{>0}.$$

(b) Find similar expressions for σ_{13} in terms of σ_3 and σ_9, and in terms of σ_5 and σ_7.

2.

(a) Find rational numbers λ and μ such that

$$\Delta = \lambda E_4^3 + \mu E_{12}.$$

(b) Let $\tau(n)$ be the nth coefficient of the Fourier expansion of Δ at infinity. Show that $\tau(n) \in \mathbb{Z}$, and prove Ramanujan's congruence:

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691.$$

3.

(a) Define

$$\begin{aligned} G_2(z) &:= 2\zeta(2) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2} \\ E_2(z) &:= -\frac{1}{8\pi^2}G_2(z). \end{aligned}$$

Show that

$$\begin{aligned} G_2(z) &= 2\zeta(2) - 8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)q^n \\ E_2(z) &= -\frac{B_2}{4} + \sum_{n=1}^{\infty}\sigma_1(n)q^n. \end{aligned}$$

(b) Use the fact that

$$\begin{aligned} \sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) &= 0 \\ \sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) &= -\frac{2\pi i}{z} \end{aligned}$$

to show that

$$z^{-2}G_2(-1/z) = G_2(z) - \frac{2\pi i}{z} \quad \text{and} \quad z^{-2}E_2(-1/z) = E_2(z) - \frac{1}{4\pi i z},$$

and then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$E_2(\gamma z) = (cz+d)^2 \left(E_2(z) - \frac{1}{4\pi i} \frac{c}{cz+d} \right).$$

(c) Show that if f is a modular form of weight k for Γ, then $\partial f := \frac{1}{2\pi i} f' + 2kE_2 f$ is a modular form of weight $k+2$.

(d) Prove that $\frac{7}{10}E_6 = \frac{1}{2\pi i}E_4' + 8E_2E_4$ and $\frac{10}{21}E_8 = \frac{1}{2\pi i}E_6' + 12E_2E_6$.

(e) Prove that $\frac{5}{6}E_4 = \frac{1}{2\pi i}E_2' + 2E_2^2$.

(f) Conclude that the ring $\mathbb{C}[E_2, E_4, E_6]$ is closed under derivation.

4. Let $\mathcal{L}_(N)$ be the set of pairs (Λ, P) where Λ is a lattice in $\mathbb{C}$ and P is a point of order N in the (additive) group $\mathbb{C}/\Lambda$.

(a) Show that the relation on $\mathcal{L}_1(N)$ given by:
$(\Lambda, P) \sim (\Lambda', P')$ iff there exists $\alpha \in \mathbb{C}^*$ such that for any $\omega \in \mathbb{C}$ with $\omega + \Lambda = P$ in $\mathbb{C}/\Lambda$, we have $\alpha\Lambda = \Lambda'$ and $\alpha\omega + \Lambda' = P'$ in $\mathbb{C}/\Lambda'$
is an equivalence relation.

(b) Prove that there is a bijection between the equivalence classes in $\mathcal{L}_1(N)$ and $\Gamma_1(N)\backslash\mathbb{H}$.

5. Let $T_n = T_n^{k,\chi}$ be the nth Hecke operator. Show (6.24) In Iwaniec's book, i.e. show that

$$T_{mn} = \sum_{d|(m,n)} \mu(d)\chi(d)d^{k-1} T_{\frac{m}{d}} T_{\frac{n}{d}}.$$

6. Let $T_n = T_n^{k,\chi}$ be the nth Hecke operator. Show (6.27) In Iwaniec's book, i.e. show the formal relation

$$\sum_{v=0}^{\infty} T_{p^v} X^v = \left(1 - T_p X + \chi(p)p^{k-1}X^2\right)^{-1}.$$