Modular Forms Homework 3.
Due Wednesday November 5.

1.

(a) Let χ be a Dirichlet character modulo q. Prove that

$$\sum_{n=0}^{q-1} \chi(n) = \begin{cases} \varphi(q) & \chi \text{ is principal} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let n be an integer. Prove that

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \varphi(q) & n \equiv 1 \bmod q \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all Dirichlet characters modulo q.

2. Let a, n be positive integers and let

$$c_q(n) := \sum_{\substack{a \bmod q \\ (a,q)=1}} e\left(\frac{an}{q}\right)$$

be the Ramanujan sum. Show that

$$c_q(n) = \mu\left(\frac{q}{(n,q)}\right) \varphi((n,q)).$$

3. Let χ be a *primitive* character of conductor q. Show that

$$\tau(m,\chi) = \overline{\chi}(m)\tau(1,\chi), \tag{0.1}$$

where $\tau(m,q)$ is the shifted Gauss sum

$$\tau(m,q) := \sum_{a \bmod q} \chi(a) e\left(\frac{am}{q}\right),$$

and in particular, $\tau(1,\chi) = \tau(\chi)$.
Remark: We showed in class that (0.1) is always true for *any character* χ when $(m,q) = 1$ by a simple change of variable in the sum. So, it remains to show that $\tau(m,\chi) = 0$ when $(m,q) > 1$, and χ is primitive.

Deduce that if χ is primitive, we have $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)q$ and $\tau(\chi)\overline{\tau(\chi)} = q$.

4. Let $k \in \mathbb{Z}_{>0}$, let $f \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})$ be an Hecke eigenform, normalized such that $a_1(f) = 1$, and let p be a prime number. Let $\alpha, \beta \in \mathbb{C}$ be the roots of the polynomial $t^2 - a_p(f)t + p^{k-1}$. You may use without proof the fact that $a_p(f)$ is real.

(a) Prove that for any $r \geq 0$, we have

$$a_{p^r}(f) = \sum_{j=0}^{r} \alpha^j \beta^{r-j}.$$

(b) Show that the following conditions are equivalent: (1) $|a_p(f)| \leq 2p^{(k-1)/2}$; (2) α and β are complex conjugates of absolute value $p^{(k-1)/2}$.