MATH203. Winter 2019

Solutions to the midterm

1. (a) We have

$$\begin{aligned} 2\log_2(x) = 2 + \log_2(x+3) &\iff \log_2(x^2) = 2 + \log_2(x+3) \\ &\iff 2^{\log_2(x^2)} = 2^{2+\log_2(x+3)} \\ &\iff x^2 = 2^2 2^{\log_2(x+3)} = 4(x+3) \\ &\iff x^2 - 4x - 12 = (x-6)(x+2) = 0. \end{aligned}$$

Then, the only solution to the original equation is $x = 6$ as x must be positive to write $2\log_2(x) = \log(x^2)$ as we did.

(b) We have that

$$(f \circ g)(x) = \log_2(3 - (4x^2 - 1)) = \log_2(4 - 4x^2) = \log_2(4(1 - x^2)),$$

which is only defined only when

$$1 - x^2 > 0 \iff x^2 < 1 \iff x \in (-1, 1).$$

The range of $f \circ g$ is $(-\infty, \log_2(4)] = (-\infty, 2]$.

(c) The function $f(x) = 3^{x^4-1}$ is not 1-to-1, since $f(x) = f(-x)$ for all values of x. Then, it is not invertible.

The function $g(x) = 3^{5x-1}$ is 1-to-1, and we can find the inverse. We have

$$y = 3^{5x-1} \iff \log_3 y = 5x - 1 \iff x = \frac{\log_3 y + 1}{5}.$$

Changing variables, the inverse is $g^{-1}(x) = \dfrac{\log_3 x + 1}{5}$.

2. (a) For the horizontal asymptotes, we compute the limits at infinity.

$$\begin{aligned} \lim_{x\to\pm\infty} \frac{|x|\sqrt{9x^4 + 6x^2 + 2}}{(2x+3)(x+1)^2} &= \lim_{x\to\pm\infty} \frac{|x|^3\sqrt{9 + 6x^{-2} + 2x^{-4}}}{x^3(2 + 3x^{-1})(1 + x^{-1})^2} \\ &= \begin{cases} \frac{3}{2} & x \to \infty \\ -\frac{3}{2} & x \to -\infty, \end{cases} \end{aligned}$$

and the lines $x = 3/2$ and $x = -3/2$ are horizontal asymptotes.

(b) For the vertical asymptote, we are looking for points where the function does not exist (tends to infinity). We have

$$\begin{aligned}\lim_{x\to -1} \frac{|x|\sqrt{9x^4+6x^2+2}}{(2x+3)(x+1)^2} &= \infty \\ \lim_{x\to -3/2} \frac{|x|\sqrt{9x^4+6x^2+2}}{(2x+3)(x+1)^2} &= \infty,\end{aligned}$$

and the lines $y=-1$ and $y=-3/2$ are vertical asymptotes.

3. (a) We compute

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2-2x}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2-2x}-x\right)\frac{\left(\sqrt{x^2-2x}+x\right)}{\left(\sqrt{x^2-2x}+x\right)} \\ &= \lim_{x\to\infty}\frac{(x^2-2x-x^2)}{\left(\sqrt{x^2-2x}+x\right)} = \lim_{x\to\infty}\frac{-2x}{\left(|x|\sqrt{1-2/x}+x\right)} \\ &= \lim_{x\to\infty}\frac{-2}{\left(\sqrt{1-2/x}+1\right)} = \frac{-2}{2} = -1.\end{aligned}$$

(b) We have

$$\lim_{x\to 0}\frac{\sqrt{2x^2+5x^4}}{x} = \lim_{x\to 0}\frac{|x|\sqrt{2+5x^2}}{x},$$

and then

$$\lim_{x\to 0^+}\frac{\sqrt{2x^2+5x^4}}{x} = \sqrt{2}, \quad \lim_{x\to 0^-}\frac{\sqrt{2x^2+5x^4}}{x} = -\sqrt{2},$$

so the limit does not exist.

4. (a) With the definition of the derivative,

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{\frac{2}{x+h-3}-\frac{2}{x-3}}{h} = \lim_{h\to 0}\frac{2(x-3)-2(x+h-3)}{(x-3)(x+h-3)h} \\ &= \lim_{h\to 0}\frac{-2h}{(x-3)(x+h-3)h} = \frac{-2}{(x-3)^2}\end{aligned}$$

(b) We have that $f'(1)=-1/2$, and the equation of the tangent line is $y=-\frac{1}{2}x+b$ with $-1=-\frac{1}{2}\cdot 1+b \Rightarrow b=-\frac{1}{2}$, and the equation of the tangent line is

$$y=-\frac{1}{2}x-\frac{1}{2}.$$

5. We have $f(x) = e^{bx}(e^{bx} + e^{3-bx}) = e^{2bx} + e^3$. Then,

$$f'(x) = 2be^{2bx}, \quad f''(x) = (2b)^2 e^{2bx}, \quad f'''(x) = (2b)^3 e^{2bx},$$

and $f'''(0) = (2b)^3 e^0 = 8b^3$.

6. (a) We can write

$$f(x) = \frac{2x^3 + x - 10}{x\sqrt{x}} = x^{-3/2}\left(2x^3 + x - 10\right) = 2x^{3/2} + x^{-1/2} - 10x^{-3/2},$$

and then

$$f'(x) = 3x^{1/2} - \frac{1}{2}x^{-3/2} + 15x^{-5/2}.$$

(b) We compute

$$f'(x) = e^x + ex^{e-1} - e.$$

(c) We use the quotient rule to compute

$$f'(x) = \frac{3\sec^2(3x)(1+x^2) - 2x\tan(3x)}{(1+x^2)^2}$$

(d) We use the chain rule to compute

$$\begin{aligned} f'(x) &= 2\cos\left(\sin(3x) + x^3e^x\right) \cdot \frac{d}{dx}\left(\cos\left(\sin(3x) + x^3e^x\right)\right) \\ &= -2\cos\left(\sin(3x) + x^3e^x\right)\ \sin\left(\sin(3x) + x^3e^x\right) \cdot \frac{d}{dx}\left(\sin(3x) + x^3e^x\right) \\ &= -2\cos\left(\sin(3x) + x^3e^x\right)\ \sin\left(\sin(3x) + x^3e^x\right)\left(3\cos(3x) + 3x^2e^x + x^3e^x\right) \end{aligned}$$

BONUS First, since $f(x)$ is differentiable, it has to be continuous. It is clearly continuous at all $x \neq 1$ and at $x = 1$, we need to have $1 + a = a + b$, and then $b = 1$. For the differentiability, the function is clearly differentiable at all $x \neq 1$ and for $x = 1$, we need to have $1 = 2a$ equating the derivatives at $x = 1$. Then, $a = 1/2$.