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STATISTICS FOR ORDINARY ARTIN-SCHREIER COVERS AND OTHER p-RANK STRATA

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Abstract. We study the distribution of the number of points and of the zeroes of the zeta function in different p-rank strata of Artin-Schreier covers over $\mathbb{F}_q$ when q is fixed and the genus goes to infinity. The p-rank strata considered include the ordinary family, the whole family, and the family of covers with p-rank equal to $p-1$. While the zeta zeroes always approach the standard Gaussian distribution, the number of points over $\mathbb{F}_q$ has a distribution that varies with the specific family.

Contents

1. Introduction

Besides their central place in number theory, algebraic curves over finite fields also play a pivotal role in applications via such fields as cryptography and error-correcting codes. In both theory and applications, a key property of an algebraic curve over a finite field is its *zeta function*, which determines and is determined by the number of points on the curve over the finite extensions of the base field.

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These zeta functions exhibit a strong analogy with other zeta functions occurring in number theory, such as the Riemann zeta function, with the added benefit that the analogue of the Riemann hypothesis is known by results of Weil.

In addition to studying curves individually, it is also profitable to study curves in families and ask aggregate questions over families. Historically, this generally involved varying the finite field, as in the work of Deligne. More recently, a series of results have emerged in which the finite field is fixed and other geometric parameters are allowed to vary. Examples include the work Kurlberg and Rudnick [KR09] that studies the distribution of the number of points on hyperelliptic curves as the genus grows. Similar statistics for the number of points have been computed for cyclic ℓ-covers of the projective line [BDFL10b, BDFL11, Xio10a], plane curves [BDFL10a], complete intersections in projective spaces [BK12], general trigonal curves [Woo12], superelliptic curves [CWZ15], curves on Hirzebruch surfaces [EW15], and a subfamily of Artin-Schreier covers [Ent12].

A finer statistic for these curves is the distribution of the zeroes of the zeta function. (Note that the distribution of the points can be easily deduced from the distribution of the zeroes.) The problem of the distribution of the zeroes in the global and mesoscopic regimes was considered by Faifman and Rudnick [FR10] for hyperelliptic curves, while [Xio10b], [Xio15], and [BDFLS12] treated the cases of cyclic ℓ-covers, abelian covers of algebraic curves, and Artin-Schreier covers respectively. On the other hand, Entin [Ent12] used the distributions of the number of points of a subfamily of Artin-Schreier covers to obtain some partial results towards the pair correlation problem for the zeroes.

Artin-Schreier curves represent a special family because they cannot be uniformly obtained by base-changing a scheme defined over $\mathbb{Z}$. This is intimately related to the fact that their zeta function has an expression in terms of *additive* characters of $\mathbb{F}_p$, and not in terms of multiplicative characters, as is the case for the family of hyperelliptic curves and cyclic ℓ-covers. On the other hand, the factor corresponding to a fixed additive character has a nice description as an exponential sum (2.1), which allows one to do a fair number of concrete computations. For instance, they can sometimes be used to show that the Weil bound on the number of points is sharp (especially in the supersingular case [Gar05, GV92]).

The p-rank induces a stratification on the moduli space of Artin-Schreier covers of genus $\mathfrak{g}$. We would like to remark that this stratification is not specific to Artin-Schreier covers. Perhaps the best known example is the case of elliptic curves. The moduli space of elliptic curves has only two p-strata – p-rank 1 (ordinary) and p-rank 0 (supersingular) – and these two classes of elliptic curves behave fundamentally differently in many aspects. The ordinary stratum is Zariski dense in the moduli space, but there are only finitely many supersingular $\bar{\mathbb{F}}_q$-points in the moduli space of elliptic curves.

In the case of the Artin-Schreier covers, the picture is more complicated, as there are many intermediate strata besides the minimal p-rank and the maximal p-rank strata. But it is still the case that the p-rank 0 stratum, when non-empty, is the smallest stratum in the moduli space $\mathcal{AS}_{\mathfrak{g}}$ of Artin-Schreier covers of genus $\mathfrak{g}$. However, the p-rank 0 stratum appears if and only if $2\mathfrak{g}/(p-1) \not\equiv -1 \pmod p$. Moreover, the supersingular locus is usually strictly contained in this stratum, and it is not easy to locate the supersingular covers among those with p-rank 0. (See [Zhu].) On the other hand, the maximal p-rank stratum is irreducible in $\mathcal{AS}_{\mathfrak{g}}$, and in some sense, it is still the biggest stratum. As it is noted in [PZ12, Example

2.9], in the case of $p \geq 3$ that we are interested in, the ordinary locus is non-empty whenever $2\mathfrak{g}/(p-1)$ is even. Otherwise, we can still talk about the stratum of maximal p-rank, but that maximal rank will be strictly smaller than the genus (namely, equal to $\mathfrak{g} - \frac{p-1}{2}$), and there is no ordinary locus.

Fix a finite field $\mathbb{F}_q$ of odd characteristic p. An Artin-Schreier cover is an Artin-Schreier curve for which we fix an automorphism of order p and an isomorphism between the quotient and $\mathbb{P}^1$. Concretely, an $\mathbb{F}_q$-point of the moduli space of Artin-Schreier covers of genus $\mathfrak{g}$ consists of, up to $\mathbb{F}_q$-isomorphism, a curve of genus $\mathfrak{g}$ with affine model

$$C_f : y^p - y = f(x),$$

where $f(x) \in \mathbb{F}_q(x)$ is a rational function, together with the automorphism $y \mapsto y+1$.

The genus of C_f is given by

$$\mathfrak{g}(C_f) = \frac{p-1}{2}\left(-2 + \sum_{j=1}^{r+1}(d_j+1)\right) = \frac{p-1}{2}\left(r - 1 + \sum_{j=1}^{r+1} d_j\right),$$

where $r+1$ is the number of poles of $f(x)$ and d_j are their orders. (See [PZ12, Lemma 2.6].) The p-rank is the integer s such that the cardinality of $\mathrm{Jac}(C_f)[p](\bar{\mathbb{F}}_q)$ is p^s; by the Deuring-Shafarevich formula, we have $s = r(p-1)$. We will write $\mathcal{AS}_{\mathfrak{g},s}$ for the stratum with p-rank equal to s of the moduli space $\mathcal{AS}_{\mathfrak{g}}$. For example, $s=0$ corresponds to one pole, which can always be moved to infinity. This is the case where $f(x)$ is a polynomial that was considered in [Ent12, BDFLS12]. However, this case only corresponds to a piece, namely $\mathcal{AS}_{\mathfrak{g},0}$, of the whole moduli space $\mathcal{AS}_{\mathfrak{g}}$ of Artin-Schreier covers of genus $\mathfrak{g}$. The next case is $s = p-1$, which includes the case when $f(x)$ is a Laurent polynomial, but this is not the only way one may get this p-rank, as we explain in Section 5. For details on the moduli space of Artin-Schreier covers and the p-rank stratification, we refer the reader to [PZ12].

1.1. **Statement of results.**

The main object of this paper is the study of the distribution of the number of points and zeta zeroes for the ordinary locus $\mathcal{AS}_{\mathfrak{g},\mathfrak{g}}$ which only appears when $2\mathfrak{g}/(p-1)$ is even. In addition, we treat the cases of $\mathcal{AS}_{\mathfrak{g},p-1}$ of covers with p-rank equal to $p-1$ and the whole family $\mathcal{AS}_{\mathfrak{g}}$. More precisely, we have the following results.

Theorem 1.1. (1) *Assume that $2\mathfrak{g}/(p-1)$ is even. The average number of $\mathbb{F}_{q^k}$-points on an ordinary Artin-Schreier cover in $\mathcal{AS}_{\mathfrak{g},\mathfrak{g}}(\mathbb{F}_q)$ is*

$$\begin{cases} q^k + 1 + O\left(q^{(-1/2+\varepsilon)(1+\mathfrak{g}/(p-1))+2k}\right), & p \nmid k, \\ q^k + 1 + \frac{p-1}{1+q^{-1}-q^{-2}} + \sum_{u \mid \frac{k}{p}} \frac{p-1}{1+q^{-u}-q^{-2u}} \sum_{e \mid u} \mu(e) q^{u/e} & \\ \qquad + O\left(q^{(-1/2+\varepsilon)(1+\mathfrak{g}/(p-1))+2k}\right), & p \mid k. \end{cases}$$

(2) *The average number of* $\mathbb{F}_{q^k}$*-points on an Artin-Schreier cover in* $\mathcal{AS}_{\mathfrak{g}}(\mathbb{F}_q)$ *whose ramification divisor is supported at* $r+1$ *points and has degree* d *is*

$$\begin{cases} q^k + 1 + O\left(q^{(\varepsilon-1)d+2k}\right), & p \nmid k, \\ q^k + 1 + (p-1)q^{k/p} + \frac{p-1}{1+q^{-1}} \\ \quad -(p-1)\displaystyle\sum_{u \mid \frac{k}{p}} \frac{1}{1+q^u} \sum_{e \mid u} \mu(e) q^{u/e} + O\left(q^{(\varepsilon-1)d+2k}\right), & p \mid k. \end{cases}$$

(3) *The average number of* $\mathbb{F}_{q^k}$*-points on an Artin-Schreier cover in* $\mathcal{AS}_{\mathfrak{g},p-1}(\mathbb{F}_q)$ *is*

$$\begin{cases} q^k + 1, & p \nmid k, \\ q^k + 1 + (p-1)(q^{k/p} - 1), & p \mid k,\ k \text{ even}, \\ q^k + 1 + (p-1)q^{k/p}, & p \mid k,\ k \text{ odd}. \end{cases}$$

Remark 1.2. The results in the previous theorem are only meaningful in part (1) for $\mathfrak{g}$ sufficiently large with respect to k and in part (2) for d sufficiently large with respect to k.

By Weil's conjectures, the zeta function of C_f,

$$Z_{C_f}(u) = \exp\left(\sum_{k=1}^{\infty} N_k(C_f) \frac{u^k}{k}\right),$$

where $N_k(C_f)$ is the number of points on C_f defined over $\mathbb{F}_{q^k}$, can be written as

$$Z_{C_f}(u) = \frac{P_{C_f}(u)}{(1-u)(1-qu)},$$

where $P_{C_f}(u)$ is a polynomial of degree $2\mathfrak{g} = (p-1)(\Delta - 1)$ with $\Delta = r + \sum_{j=1}^{r+1} d_j$. Using Lemma 2.1 and the additive characters of $\mathbb{F}_p$ to write a formula for $N_k(C_f)$, it follows easily that

$$P_{C_f}(u) = \prod_{\psi} L(u, f, \psi), \tag{1.1}$$

where the product is taken over the *non-trivial* additive characters ψ of $\mathbb{F}_p$, and $L(u, f, \psi)$ are certain L-functions (given later by (2.1)). Understanding the distribution of the zeroes of $Z_{C_f}(u)$ amounts to understanding the distribution of the zeroes of each of the $L(u, f, \psi)$ as f runs in the relevant family of rational functions and the genus goes to infinity.

If we write

$$L(u, f, \psi) = \prod_{j=1}^{\Delta-1} (1 - \alpha_j(f, \psi) u),$$

we have that $\alpha_j(f, \psi) = \sqrt{q} e^{2\pi i \theta_j(f, \psi)}$ and $\theta_j(f, \psi) \in [-1/2, 1/2)$. We study the statistics of the set of angles $\{\theta_j(f, \psi)\}$ as f varies in the family. For an interval $\mathcal{I} \subset [-1/2, 1/2)$, let

$$N_{\mathcal{I}}(f, \psi) := \#\{1 \le j \le \Delta - 1 : \theta_j(f, \psi) \in \mathcal{I}\}$$

and

$$N_{\mathcal{I}}(C_f) := \sum_{j=1}^{p-1} N_{\mathcal{I}}(f, \psi^j).$$

We show that the number of zeroes with angle in a prescribed non-trivial subinterval $\mathcal{I}$ is asymptotic to $2\mathfrak{g}|\mathcal{I}|$, has variance asymptotic to $\frac{2(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|)$, and properly normalized has a Gaussian distribution.

Theorem 1.3. *Fix a finite field $\mathbb{F}_q$ of characteristic p. Let $\mathcal{AS}$ denote the family of Artin-Schreier covers, ordinary Artin-Schreier covers, or the p-rank $p-1$ Artin-Schreier covers. Then for any real numbers $a < b$ and $0 < |\mathcal{I}| < 1$ either fixed or $|\mathcal{I}| \to 0$ while $\mathfrak{g}|\mathcal{I}| \to \infty$,*

$$\lim_{\mathfrak{g}\to\infty} \mathrm{Prob}_{\mathcal{AS}(\mathbb{F}_q)}\left(a < \frac{N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}|}{\sqrt{\frac{2(p-1)}{\pi^2}\log\left(\mathfrak{g}|\mathcal{I}|\right)}} < b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}dx.$$

This result is analogous to what was obtained in [BDFLS12] for p-rank 0 Artin-Schreier covers and is compatible with the philosophy of Katz and Sarnak [KS99]. In fact, Katz [Kat87] shows that the monodromy of the L-functions defined in (2.1) is given by $\mathrm{SL}\left(2\mathfrak{g}/(p-1)\right)$ when the dimension of the moduli space is big enough. Since the dimension grows with the genus, this occurs when $\mathfrak{g}$ is big enough. In particular, [DS94] implies that the limiting distribution as $\mathfrak{g} \to \infty$ is Gaussian.

Remark 1.4. A similar result can be proved for $N_{\mathcal{I}}(f, \psi)$ with asymptotic mean and variance $(\Delta - 1)|\mathcal{I}|$ and $\frac{1}{\pi^2}\log\mathfrak{g}|\mathcal{I}|$ respectively with the additional restriction that the interval $\mathcal{I}$ is symmetric. In fact, under this condition, the $N_{\mathcal{I}}(f, \psi^j)$ for $j = 1, \ldots, (p-1)/2$ approach independently jointly normal distributions.

1.2. **About the results and their proofs.** While our work is inspired by the earlier work of Kurlberg and Rudnick [KR09] and Faifman and Rudnick [FR10] and resembles their work in the broad outlines, our techniques differ from theirs in several respects. Firstly, the zeta functions associated to the family of hyperelliptic curves studied by Rudnick et al. are expressed in terms of a real-valued multiplicative character of $\mathbb{F}_p$, whereas the zeta functions for the families of Artin-Schreier covers that we consider are expressed in terms of a complex-valued additive character of $\mathbb{F}_p$. This distinction necessitates using techniques developed in Entin [Ent12] and in [BDFLS12]. However, both of these papers only work with p-rank 0 Artin-Schreier covers. As remarked before, this stratum, when non-empty, is the smallest stratum in the moduli space, and therefore other bigger strata may better represent the behavior in the space of Artin-Schreier covers. In order to have results for all covers (and particularly the ordinary case) we need to combine the previous techniques with a careful use of the Tauberian Theorem in order to count the number of covers taking prescribed values. For example, counting the number of p-rank 0 Artin-Schreier covers of a given genus reduces to the counting of polynomials of fixed degree in $\mathbb{F}_q[X]$, while counting the number of ordinary Artin-Schreier covers amounts to the counting of pairs of homogeneous polynomials of fixed degree with various conditions (co-prime, square-free, and such), and requires some sieving (Proposition 3.6 and Corollary 3.7).

Secondly, our counting problem is in some sense more natural from a geometric perspective in that we are averaging over strata of the moduli space and therefore the results of [PZ12] play a role in our results. In the work of Rudnick et al. the statistics are computed for the family of hyperelliptic curves by running over all square-free polynomials of a fixed degree. This is not the same as running over the moduli space of hyperelliptic curves of a fixed genus, as not all points on the moduli space appear with the same multiplicity in this family.

We also mention that it is very interesting to contrast Theorem 1.1 to Theorem 1.3. In the first theorem the result is different for different families of Artin-Schreier covers, while the latter theorem has the same result for any of the families under consideration. Indeed, sets that describe different strata have distinct structures, and this phenomenon appears in the statistics for the number of points, but it does not appear in the statistics for the location of the zeroes of the zeta function.

1.3. **Outline of the article.** This article proceeds as follows. In the next section we review basic facts about Artin-Schreier theory and explicitly describe and set up notation for the various families we consider throughout the paper. In Sections 3 and 4 we use the Tauberian theorem to compute the expected number of $\mathbb{F}_{q^k}$-points on an Artin-Schreier cover defined over $\mathbb{F}_q$ for the ordinary locus and full space respectively, while the same problem for the prescribed factorization type is considered in Section 5. The results of these three sections combined are a generalization of Theorem 1.1 stated above. Along the way to proving this theorem we count the number of curves that take prescribed values. We will need these results in Section 8. In Section 6 we review some facts on Beurling-Selberg polynomials and approximate the characteristic function of $\mathcal{I}$ with a sum of these polynomials. In Section 7 we use the explicit formula as well as the results of the previous section to approximate $N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}|$ by a sum of characters of traces of a rational function evaluated at elements of $\mathbb{F}_{q^k}$. In Section 8 we combine results of the previous section to calculate the moments of the sum of characters from the previous section. Finally in Section 9 we complete the proof of Theorem 1.3 by proving that under suitable normalization $N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}|$ converges in mean square and hence in distribution to our approximating function.

2. Basic Artin-Schreier theory

Fix an odd prime p and let $\mathbb{F}_q$ be a finite field of characteristic p with q elements. We consider, up to $\mathbb{F}_q$-isomorphism, pairs of curves with affine model

$$C_f : y^p - y = f(x)$$

with $f(x)$ a rational function together with the automorphism $y \mapsto y + 1$.

For each integer $n \geq 1$, denote by $\mathrm{tr}_n : \mathbb{F}_{q^n} \to \mathbb{F}_p$ the absolute trace map (not the trace to $\mathbb{F}_q$).

Lemma 2.1. *For each $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^n})$, the number of points on the curve $C_f : y^p - y = f(x)$ in the fiber above α which are defined over $\mathbb{F}_{q^n}$ is given by*

$$\begin{cases} 1 & \text{if } f(\alpha) = \infty, \\ \\ p & \text{if } f(\alpha) \in \mathbb{F}_{q^n} \text{ with } \mathrm{tr}_n f(\alpha) = 0, \\ \\ 0 & \text{if } f(\alpha) \in \mathbb{F}_{q^n} \text{ with } \mathrm{tr}_n f(\alpha) \neq 0. \end{cases}$$

Proof. This is a simple application of Hilbert's Theorem 90. □

Let ψ_k, $k = 0, \ldots, p-1$, be the additive characters of $\mathbb{F}_p$ given by

$$\psi_k(a) = e^{2\pi i k a/p}, \quad k = 0, \ldots, p-1.$$

For each rational function $f \in \mathbb{F}_q(X)$ and non-trivial character ψ, we also define

$$S_n(f, \psi) = \sum_{\substack{x \in \mathbb{P}^1(\mathbb{F}_{q^n}) \\ f(x) \neq \infty}} \psi(\mathrm{tr}_n(f(x))).$$

Then, using the fact that for any $a \in \mathbb{F}_p$,

$$\sum_{k=0}^{p-1} \psi_k(a) = \begin{cases} p, & a = 0, \\ 0, & a \neq 0, \end{cases}$$

it is easy to check that

$$P_{C_f}(u) = \prod_{\psi \neq \psi_0} L(u, f, \psi)$$

where

$$L(u, f, \psi) = \exp\left(\sum_{n=1}^{\infty} S_n(f, \psi) \frac{u^n}{n}\right). \tag{2.1}$$

Recall that

$$L(u, f, \psi) = \prod_{j=1}^{\Delta-1} (1 - \sqrt{q} e^{2\pi i \theta_j(f,\psi)} u),$$

where $\theta_j(f, \psi) \in [-1/2, 1/2)$. For an interval $\mathcal{I} \subset [-1/2, 1/2)$, let

$$N_{\mathcal{I}}(f, \psi) := \#\{1 \leq j \leq \Delta - 1 : \theta_j(f, \psi) \in \mathcal{I}\}$$

and

$$N_{\mathcal{I}}(C_f) := \sum_{j=1}^{p-1} N_{\mathcal{I}}(f, \psi^j).$$

Let $\mathcal{S} = \mathbb{F}_q[X, Z]$ be the homogeneous coordinate ring of $\mathbb{P}^1$ and denote by $\mathcal{S}_d$ the $\mathbb{F}_q$-subspace of $\mathcal{S}$ of homogeneous polynomials of degree d. Notice that $\mathcal{S}_d$ contains the 0 polynomial and its size is exactly q^{d+1}.

Since each Artin-Schreier cover comes equipped with a prescribed map to $\mathbb{P}^1$, we can think of C_f as the cover given by

$$C_{g,h} : y^p - y = \frac{g(X, Z)}{h(X, Z)},$$

where the fraction on the right hand side is obtained by homogenizing $f(x)$ in the usual way.

Given $f \in \mathcal{S}_d$, we will denote by $f^*(X) \in \mathbb{F}_q[X]$ the non-homogeneous polynomial resulting from $f(X, Z)$ by setting $Z = 1$. We observe that f^* is polynomial of degree at most d. Similarly, let $f_*(Z) \in \mathbb{F}_q[Z]$ be the non-homogeneous polynomial resulting from $f(X, Z)$ by setting $X = 1$.

Given $\alpha = [\alpha_X : \alpha_Z] \in \mathbb{P}^1(\mathbb{F}_{q^k})$ and $h \in \mathcal{S}_d$ the value of $h(\alpha)$ can be zero or non-zero; but if it is non-zero, it is not well defined. When we want to discuss an actual non-zero value we will be talking about $h^*(\alpha) := h(\alpha_X/\alpha_Z, 1)$, which is defined for $\alpha \neq [1 : 0] = \infty$ and $h_*(\alpha) := h(1, \alpha_Z/\alpha_X)$, which is defined for $\alpha \neq [0 : 1] = 0$.

We recall that the rational function $\frac{g}{h}$ can be evaluated in $[\alpha_X : \alpha_Z]$ as long as $g, h \in \mathcal{S}_d$ and $(g(\alpha_X, \alpha_Z), h(\alpha_X, \alpha_Z)) \neq (0, 0)$.

Let $p_1, \dots, p_{r+1}$ be the set of poles of $f(x)$ and let d_j be the order of the pole p_j. By Artin-Schreier theory, we can assume that $p \nmid d_j$. Recall that the genus of C_f is given by

$$\mathfrak{g}(C_f) = \frac{p-1}{2}\left(-2 + \sum_{j=1}^{r+1}(d_j+1)\right) = \frac{p-1}{2}\left(r - 1 + \sum_{j=1}^{r+1} d_j\right). \tag{2.2}$$

We now proceed to explicitly describe the families to be considered. The ordinary case occurs when the p-rank is maximal, in other words, when r is maximal. For a given genus $\mathfrak{g}$, this happens when $d_i = 1$ in formula (2.2) and $2\mathfrak{g} = (p-1)2r$. (Notice once again that this imposes a restriction on the possible values for the genus, as $2\mathfrak{g}/(p-1)$ must be even.) Thus, $f(x)$ is a rational function with exactly $r+1$ simple poles. This corresponds to the fact that $g(X,Z)$ and $h(X,Z)$ are both homogeneous polynomials of degree $r+1$ with no common factors and $h(X,Z)$ is square-free.

We let

$$\mathcal{F}_d^{\mathrm{ord}} = \{(g(X,Z), h(X,Z)) : g(X,Z), h(X,Z) \in \mathcal{S}_d, h \text{ square-free}, (g,h) = 1\},$$

with the understanding that $d = r+1$.

As (g,h) range over $\mathcal{F}_d^{\mathrm{ord}}$, the cover $C_{g,h}$ ranges over each $\mathbb{F}_q$-point of $\mathcal{AS}_{\mathfrak{g},\mathfrak{g}}$ exactly $q-1$ times. Thus, our problem becomes the study of statistics for $C_{g,h}$ as (g,h) varies over $\mathcal{F}_d^{\mathrm{ord}}$ and d tends to infinity.

We will work with the full family of covers in $\mathcal{AS}_{\mathfrak{g}}$ as well. In this case we do not have the restriction of simple poles, but we still require $g(X,Z)$ and $h(X,Z)$ not to have common factors:

$$\mathcal{F}_d^{\mathrm{full}} = \{(g(X,Z), h(X,Z)) : g(X,Z), h(X,Z) \in \mathcal{S}_d, (g,h) = 1\}.$$

We will then study the statistics as d goes to infinity, which is the same as $\mathfrak{g}$ going to infinity provided that the number of poles $r+1$ remains bounded.

Finally, we will consider another family given as follows. We say that h has factorization type $v = (r_1^{d_{1,1}}, \dots, r_1^{d_{1,\ell_1}}, \dots, r_m^{d_{m,1}}, \dots, r_m^{d_{m,\ell_m}})$ if

$$h = P_{1,1}^{d_{1,1}} \cdots P_{1,\ell_1}^{d_{1,\ell_1}} \cdots P_{m,1}^{d_{m,1}} \cdots P_{m,\ell_m}^{d_{m,\ell_m}},$$

where the $P_{i,j}$ are distinct irreducible polynomials of degree r_i and $r_i \neq r_j$ if $i \neq j$. Thus the degree of h is given by $d = \sum_{i=1}^m r_i \sum_{j=1}^{\ell_i} d_{i,j}$.

Let

$$\mathcal{F}_d^{v} = \{(g(X,Z), h(X,Z)) : g(X,Z), h(X,Z) \in \mathcal{S}_d, (g,h) = 1, \\ h \text{ has factorization type } v\}.$$

In this case, formula (2.2) implies $2\mathfrak{g} = (p-1)\left(d - 2 + \sum_{i=1}^m \ell_i r_i\right)$. Here $\sum_{i=1}^m \ell_i r_i$ represents the number of poles and the p-rank is given by $(p-1)\left(\sum_{i=1}^m \ell_i r_i - 1\right)$. We will assume the parameters m, r_i's and ℓ_i's to be fixed. This implies that the covers considered are all in the same p-rank. However, in general, the set of the covers considered does not constitute the whole p-rank stratum. We will study the statistics as d goes to infinity, which is the same as $\mathfrak{g}$ going to infinity with a bound on the number of poles.

This family includes some important particular cases. Suppose that $v = (1^d)$. This corresponds to the case of only one pole of multiplicity d. This pole can always be moved to infinity (i.e., $h(X,Z)=Z^d$). After dehomogenizing with $Z=1$, this gives the family of p-rank 0 covers $\mathcal{AS}_{\mathfrak{g},0}$ indexed by polynomials of degree d:

$$\mathcal{F}_d^{\mathrm{rank}\,0}=\{g(x):\deg(g)=d\}.$$

The statistics for this family were studied in [Ent12, BDFLS12].

Another interesting case is $v=(1^{d_1},1^{d_2})$. In this case we have two poles defined over $\mathbb{F}_q$ that can always be moved to zero and infinity (i.e., $h(X,Z)=X^{d_1}Z^{d_2}$). After dehomogenizing with $Z=1$, this gives a piece of the family of p-rank $p-1$ covers $\mathcal{AS}_{\mathfrak{g},p-1}$ indexed by Laurent polynomials with bidegree (d_2,d_1):

$$\mathcal{F}_{d_1+d_2}^{\mathrm{rank}\,p-1}=\{g(x)/x^{d_1}:\deg(g)=d_2\}.$$

The other possibility within p-rank $p-1$ covers is having two poles defined over $\mathbb{F}_{q^2}\setminus\mathbb{F}_q$, corresponding to $v=(2^d)$. In terms of polynomials, we get, in this case,

$$\mathcal{F}_{2d}^{\mathrm{rank}\,p-1}=\{g(x)/h(x)^d:\deg(h)=2, h \text{ irreducible}, (g,h)=1\}.$$

We will show that the statistics for this family are very similar to the statistics for $\mathcal{AS}_{\mathfrak{g},0}$.

We will need to compute the number of elements in a family that take certain values at certain points. The following notation will be useful.

Definition 2.2. Let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\in\mathbb{P}^1(\mathbb{F}_{q^k})$. Let $\mathcal{F}_d$ be any of the families under consideration. We define

$$\mathcal{F}_d(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)=\left\{(g,h)\in\mathcal{F}_d:(\beta_{i,X}h-\beta_{i,Z}g)(\alpha_i)=0, 1\le i\le n\right\}.$$

We remark that when $\beta\neq\infty$ we identify $\beta=[\beta_X:\beta_Z]$ with $\frac{\beta_X}{\beta_Z}\in\mathbb{F}_{q^k}$, thus

$$(\beta_X h-\beta_Z g)(\alpha)=0\Longleftrightarrow\frac{g(\alpha)}{h(\alpha)}=\beta.$$

A particularly useful case is $\mathcal{F}_d(\alpha,\beta)$. We remark that the cardinality of this set does not depend on the value of β, provided that $\beta\neq\infty$, as we prove below.

Lemma 2.3. *Fix $\alpha\in\mathbb{P}^1(\mathbb{F}_{q^k})$ of degree u over $\mathbb{F}_q$. Let $\beta\in\mathbb{F}_{q^u}$. Let $\mathcal{F}_d$ be any of the families under consideration. Then*

$$|\mathcal{F}_d(\alpha,\beta)|=|\mathcal{F}_d(\alpha,0)|.$$

Proof. Recall that

$$\mathcal{F}_d(\alpha,\beta)=\{(g,h)\in\mathcal{F}_d:(\beta_X h-\beta_Z g)(\alpha)=0\}.$$

Now let $g'=\beta_X h-\beta_Z g$. Since $\beta_Z\neq 0$ we have that $(g,h)=1$ is equivalent to $(g',h)=1$. Then $(g,h)\in\mathcal{F}_d(\alpha,\beta)$ if and only if $(g',h)\in\mathcal{F}_d(\alpha,0)$. □

3. The ordinary case

In this section, we consider the family

$$\mathcal{F}_d^{\mathrm{ord}}=\left\{(g(X,Z),h(X,Z)):g(X,Z),h(X,Z)\in\mathcal{S}_d, h \text{ square-free}, (g,h)=1\right\}.$$

3.1. **Heuristics.** We want to calculate, for given $\alpha = [\alpha_X : \alpha_Z], \beta = [\beta_X : \beta_Z] \in \mathbb{P}^1(\mathbb{F}_{q^u})$ such that $\deg \alpha = u$, the probability that

$$(\beta_X h - \beta_Z g)(\alpha_X, \alpha_Z) = 0 \tag{3.1}$$

as $(g, h) \in \mathcal{F}_d^{\mathrm{ord}}$.

Locally at α this means that we want to look at pairs (g^*, h^*) such that $(m_\alpha^*)^2 \nmid h^*$ (where $m_\alpha^* \in \mathbb{F}_q[X]$ denotes the minimal polynomial of α over $\mathbb{F}_q$) and $(g^*(\alpha), h^*(\alpha)) \not\equiv (0,0) \pmod{(m_\alpha^*)^2}$.

Therefore

$$(g^*, h^*) \equiv (\gamma_1 + \delta_1 m_\alpha, \gamma_2 + \delta_2 m_\alpha) \pmod{(m_\alpha^*)^2},$$

with $\gamma_i, \delta_i \in \mathbb{F}_q[X]$, and if they are non-zero, $\deg \gamma_i, \deg \delta_i < u$. In addition, the conditions at α imply that $(\gamma_1, \gamma_2) \neq (0,0)$ and $(\gamma_2, \delta_2) \neq (0,0)$.

For each $\gamma_2 \neq 0$, there are q^u choices for each of the other parameters, thus $q^{3u}(q^u - 1)$ total possibilities. If $\gamma_2 = 0$, then there are $q^u - 1$ choices for each of γ_1 and δ_2, and q^u choices for δ_1, for a total of $q^u(q^u-1)^2$ possibilities.

For $(g^* \,(\mathrm{mod}(m_\alpha^*)^2),\ h^* \,(\mathrm{mod}(m_\alpha^*)^2))$, this yields a total of $q^u(q^u-1)(q^{2u}+q^u-1)$ possibilities.

Now if $\beta = [1 : 0] = \infty$, condition (3.1) reduces to $h^*(\alpha) = 0 \iff \gamma_2 = 0$. This leaves $q^u - 1$ choices for γ_1 and δ_2 respectively and q^u choices for δ_1. Thus the probability that $g/h \in \mathcal{F}_d^{\mathrm{ord}}$ takes the value ∞ at a given point α is

$$\frac{q^u(q^u-1)^2}{q^u(q^u-1)(q^{2u}+q^u-1)} = \frac{q^{-u}(1-q^{-u})}{1+q^{-u}-q^{-2u}}.$$

In all other cases, including $\beta = 0$, we must have $h^*(\alpha) \neq 0$. So there are $q^u - 1$ choices for γ_2. Once we know γ_2, equation (3.1) fixes $\gamma_1(\alpha)$ (and therefore γ_1, since its degree is less than u), and we have q^u choices for each of δ_1, δ_2. Thus the probability that $g/h \in \mathcal{F}_d^{\mathrm{ord}}$ takes the value $\beta \neq \infty$ at a given point α is

$$\frac{q^{2u}(q^u-1)}{q^u(q^u-1)(q^{2u}+q^u-1)} = \frac{q^{-u}}{1+q^{-u}-q^{-2u}}.$$

Then, the heuristic confirms the result of Proposition 3.10 and the expected number of points of Theorem 1.1 for the family $\mathcal{F}_d^{\mathrm{ord}}$.

3.2. **The number of covers with local conditions.** In this subsection, we are going to compute the proportion of polynomials with certain fixed values. We will obtain the size of the family and the expected number of points as corollaries.

Unless otherwise indicated, we fix $\alpha_1, \dots, \alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1, \dots, u_n$ over $\mathbb{F}_q$ and $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \leq i \leq n$ (i.e., none of the β_i's is ∞). Also, $\beta_1, \dots, \beta_\ell$ are not zero, and $\beta_{\ell+1} = \cdots = \beta_n = 0$. Finally, none of the α_i are Galois conjugate to each other; i.e., all the minimal polynomials m_{α_i} are distinct.

We start by making the following observation.

Remark 3.1. If $\alpha = [\alpha : 1] \in \mathbb{F}_{q^k}$ has degree u over $\mathbb{F}_q$, then the map $\mathcal{S}_d \to \mathbb{F}_{q^u}, h \mapsto h^*(\alpha)$ is a linear map of $\mathbb{F}_q$-vector spaces. The map is surjective as long as $d \geq u$, and in this case its kernel has dimension $d + 1 - u$. If $d < u$ the elements $1, \alpha, \alpha^2, \dots, \alpha^d$ are linearly independent over $\mathbb{F}_q$. Therefore the image has dimension $d+1$ and thus the kernel has dimension 0. In other words the map is injective and the preimage of any element is either empty or a point.

If $\alpha = [1:0] = \infty$, then it has degree 1 over $\mathbb{F}_q$, and a condition fixing a value for $h(\alpha)$ can be rewritten in terms of $h_*(1)$ such that it does become linear and the reasoning above applies.

Lemma 3.2. *Fix $\alpha_1, \dots, \alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1, \dots, u_n$ over $\mathbb{F}_q$ such that none of the α_i are conjugate to each other, and $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \le i \le n$ such that $\beta_1, \dots, \beta_\ell$ are not zero, and $\beta_{\ell+1} = \cdots = \beta_n = 0$. Fix $g \in \mathcal{S}_d$ such that $g(\alpha_i) = 0$ for $\ell+1 \le i \le n$, and $g(\alpha_i) \neq 0$ for $1 \le i \le \ell$. Then we have*

$$|\{h \in \mathcal{S}_d : (\beta_{i,X} h - \beta_{i,Z} g)(\alpha_i) = 0, 1 \le i \le n\}| = \begin{cases} q^{d+1-\sum_{i=1}^{\ell} u_i}, & d \ge \sum_{i=1}^{\ell} u_i, \\ \\ 0 \text{ or } 1, & \text{otherwise.} \end{cases}$$

Proof. For $\beta_i \neq 0$, the condition imposed over h is $h(\alpha_i) = \frac{g(\alpha_i)}{\beta_i}$, while there is no condition imposed if $\beta_i = 0$. By the Chinese Remainder Theorem, imposing all the conditions together for $\alpha_1, \dots, \alpha_\ell$ is the same as imposing a condition for h modulo the product $m_{\alpha_1} \cdots m_{\alpha_\ell}$. The result then follows from Remark 3.1. □

Let $D \in \mathcal{S}_d$. In all the following, the notation (D) means the ideal generated by the polynomial D.

Lemma 3.3. *Fix $\alpha_1, \dots, \alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1, \dots, u_n$ over $\mathbb{F}_q$ such that none of the α_i are conjugate to each other, $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \le i \le n$ such that $\beta_1, \dots, \beta_\ell$ are not zero, and $\beta_{\ell+1} = \cdots = \beta_n = 0$. Fix $g \in \mathcal{S}_d$ such that $g(\alpha_i) = 0$ for $\ell+1 \le i \le n$, and $g(\alpha_i) \neq 0$ for $1 \le i \le \ell$. Then we have for any $\varepsilon > 0$,*

$$\left| \left\{ h \in \mathcal{S}_d : (h,g) = 1,\ \frac{g(\alpha_i)}{h(\alpha_i)} = \beta_i, 1 \le i \le n \right\} \right| = q^{d+1-\sum_{i=1}^{\ell} u_i} \prod_{(P)|(g)} (1 - |P|^{-1}) + O\left(q^{\varepsilon d}\right).$$

If $g(\alpha_i) \neq 0$ for some $\ell+1 \le i \le n$ or $g(\alpha_i) = 0$ for some $1 \le i \le \ell$, then the above set is empty.

Proof. If $g(\alpha_i) \neq 0$ for some $\ell+1 \le i \le n$ or $g(\alpha_i) = 0$ for some $1 \le i \le \ell$, then it is clear that the above set is empty. We then suppose $g(\alpha_i) = 0$ for $\ell+1 \le i \le n$, and $g(\alpha_i) \neq 0$ for $1 \le i \le \ell$.

By inclusion-exclusion and Lemma 3.2 we have

$$\begin{aligned} \left| \left\{ h \in \mathcal{S}_d : (h,g) = 1, \frac{g(\alpha_i)}{h(\alpha_i)} = \beta_i \right\} \right| &= \sum_{(D)|(g)} \mu(D) \sum_{\substack{h \in \mathcal{S}_d \\ D|h,\, \frac{g(\alpha_i)}{h(\alpha_i)} = \beta_i, 1 \le i \le \ell}} 1 \\ &= \sum_{\substack{(D)|(g) \\ \deg D \le d - \sum_{i=1}^{\ell} u_i}} \mu(D) q^{d+1-\deg D - \sum_{i=1}^{\ell} u_i} + \sum_{\substack{(D)|(g) \\ d - \sum_{i=1}^{\ell} u_\ell < \deg D \le d}} O(1) \\ &= q^{d+1-\sum_{i=1}^{\ell} u_i} \sum_{(D)|(g)} \mu(D) q^{-\deg D} + \sum_{\substack{(D)|(g) \\ d - \sum_{i=1}^{\ell} u_\ell < \deg D \le d}} O(1) \\ &= q^{d+1-\sum_{i=1}^{\ell} u_i} \prod_{(P)|(g)} (1 - |P|^{-1}) + O\left(q^{\varepsilon d}\right) \end{aligned}$$

where μ is the Möbius function. □

Definition 3.4. Let $g \in \mathcal{S}_d$. Set

$$A_d^g = \{h \in \mathcal{S}_d : h \text{ square free and } (h,g) = 1\}.$$

Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$. We define

$$A_d^g(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = \left\{h \in A_d^g : (\beta_{i,X} h - \beta_{i,Z} g)(\alpha_i) = 0, 1 \leq i \leq n\right\}.$$

Lemma 3.5. *Fix $\alpha_1, \dots, \alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1, \dots, u_n$ over $\mathbb{F}_q$ such that none of the α_i are conjugate to each other. Let $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \leq i \leq n$ such that $\beta_1, \dots, \beta_\ell$ are not zero, and $\beta_{\ell+1} = \cdots = \beta_n = 0$. Fix $g \in \mathcal{S}_d$ such that $g(\alpha_i) = 0$ for $\ell+1 \leq i \leq n$ and $g(\alpha_i) \neq 0$ for $1 \leq i \leq \ell$. Then*

$$\begin{aligned}&|A_d^g(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)| \\ &\quad = \frac{q^{d+1-\sum_{i=1}^{\ell} u_i}}{\zeta_q(2) \prod_{i=1}^{\ell}(1 - q^{-2u_i})} \prod_{(P)|(g)} (1 + |P|^{-1})^{-1} + O\left(q^{(1/2+\varepsilon)d}\right).\end{aligned}$$

If $g(\alpha_i) \neq 0$ for some $\ell+1 \leq i \leq n$, or $g(\alpha_i) = 0$ for some $1 \leq i \leq \ell$, then the above set is empty.

Proof. It is clear that $A_d^g(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ is empty if the condition on the values $g(\alpha_i)$ of the lemma are not satisfied, and we then suppose that $g(\alpha_i) = 0$ for $\ell+1 \leq i \leq n$, and $g(\alpha_i) \neq 0$ for $1 \leq i \leq \ell$.

By inclusion-exclusion,

$$\begin{aligned}&|A_d^g(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)| \\ &\quad = \sideset{}{'}\sum_{\substack{(D):(D,g)=1 \\ \deg(D) \leq d/2}} \mu(D) \left| \left\{ h_1 \in \mathcal{S}_{d-2\deg(D)} : (h_1, g) = 1, \frac{g(\alpha_i)}{h_1(\alpha_i)} = D^2(\alpha_i)\beta_i \right\} \right| \\ &\quad = q^{d+1-\sum_{i=1}^{\ell} u_i} \prod_{(P)|(g)} (1 - |P|^{-1}) \sideset{}{'}\sum_{\substack{(D):(D,g)=1 \\ \deg(D) \leq d/2}} \mu(D)|D|^{-2} + \sideset{}{'}\sum_{\substack{(D):(D,g)=1 \\ \deg D \leq d/2}} O\left(q^{\varepsilon d}\right)\end{aligned}$$

by Lemma 3.3, where we have written $\sideset{}{'}\sum_{(D)}$ for the sum over (monic) polynomials D such that $D(\alpha_i) \neq 0$ for $1 \leq i \leq \ell$.

But

$$\sideset{}{'}\sum_{(D):(D,g)=1} \mu(D)|D|^{-2s} = \prod_{\substack{(P):P \nmid g \\ P(\alpha_i) \neq 0, 1 \leq i \leq \ell}} (1 - |P|^{-2s}) = \prod_{(P):P \nmid g m_{\alpha_1} \dots m_{\alpha_\ell}} (1 - |P|^{-2s}),$$

where we made use of the fact that $(g, m_{\alpha_i}) = 1$ since $g(\alpha_i) \neq 0$. This can be rewritten as

$$\begin{aligned}&\frac{1}{\zeta_q(2s)} \prod_{(P)|(g m_{\alpha_1} \dots m_{\alpha_\ell})} (1 - |P|^{-2s})^{-1} \\ &\quad = \frac{1}{\zeta_q(2s) \prod_{i=1}^{\ell}(1 - q^{-2su_i})} \prod_{(P)|(g)} (1 - |P|^{-2s})^{-1}.\end{aligned}$$

Therefore

$$\sideset{}{'}\sum_{\substack{(D):(D,g)=1 \\ \deg(D) \leq d/2}} \mu(D)|D|^{-2} = \frac{1}{\zeta_q(2) \prod_{i=1}^{\ell}(1 - q^{-2u_i})} \prod_{(P)|(g)} (1 - |P|^{-2})^{-1} + O\left(q^{-d/2}\right)$$

and

$$\begin{aligned}&|A_d^g(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|\\&\quad=\frac{q^{d+1-\sum_{i=1}^{\ell}u_i}}{\zeta_q(2)\prod_{i=1}^{\ell}(1-q^{-2u_i})}\prod_{(P)|(g)}(1+|P|^{-1})^{-1}+O\left(q^{(1/2+\varepsilon)d}\right).\end{aligned}$$

□

Proposition 3.6. *Fix $\alpha_1,\dots,\alpha_n\in\mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1,\dots,u_n$ over $\mathbb{F}_q$ such that none of the α_i are conjugate to each other. Let $\beta_i\in\mathbb{F}_{q^{u_i}}$ for $1\le i\le n$. Then*

$$|\mathcal{F}_d^{\mathrm{ord}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|=\frac{H(1)q^{2d+2-\sum_{i=1}^{n}u_i}}{\zeta_q(2)^2\prod_{i=1}^{n}(1+q^{-u_i}-q^{-2u_i})}+O\left(q^{(3/2+\varepsilon)d}\right),$$

where

$$H(1)=\prod_{(P)}\left(1+\frac{1}{(|P|+1)(|P|^2-1)}\right).$$

Proof. Denote by m_{α_i} the homogenized minimal polynomial of α_i over $\mathbb{F}_q$. We have

$$|\mathcal{F}_d^{\mathrm{ord}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|=\sum_{g\in\mathcal{S}_d}|A_d^g(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|.$$

Assume without loss of generality that $\beta_1,\dots,\beta_\ell$ are not zero, and $\beta_{\ell+1}=\dots=\beta_n=0$. By Lemma 3.5, the above sum equals

$$\begin{aligned}&|\mathcal{F}_d^{\mathrm{ord}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|\\&=\sum_{\substack{g\in\mathcal{S}_d\\ g(\alpha_i)\neq0,1\le i\le\ell\\ g(\alpha_i)=0,\ell+1\le i\le n}}\left(\frac{q^{d+1-\sum_{i=1}^{\ell}u_i}}{\zeta_q(2)\prod_{i=1}^{\ell}(1-q^{-2u_i})}\prod_{(P)|(g)}(1+|P|^{-1})^{-1}+O\left(q^{(1/2+\varepsilon)d}\right)\right)\\&=\frac{q^{d+1-\sum_{i=1}^{n}u_i}}{\zeta_q(2)\prod_{i=1}^{\ell}(1-q^{-2u_i})}\sum_{\substack{g\in\mathcal{S}_d\\ g(\alpha_i)\neq0,1\le i\le\ell\\ g(\alpha_i)=0,\ell+1\le i\le n}}\prod_{(P)|(g)}(1+|P|^{-1})^{-1}+O\left(q^{(3/2+\varepsilon)d}\right).\end{aligned}$$

Set

$$b(g)=\prod_{(P)|(g)}(1+|P|^{-1})^{-1}$$

and

$$G(s)=\sum_{(g)\neq0}\frac{b(g)}{|g|^s}.$$

Since $b(g)$ is a multiplicative function, it follows that $G(s)$ has an Euler product of the form

$$\begin{aligned}G(s)&=\prod_{(P)}\left(\sum_{k=0}^{\infty}b(P^k)|P|^{-ks}\right)\\&=\prod_{(P)}\left(1+\frac{b(P)|P|^{-s}}{1-|P|^{-s}}\right)\\&=\prod_{(P)}\left(1+\frac{|P|^{-s}}{(1-|P|^{-s})(1+|P|^{-1})}\right).\end{aligned}$$

Thus

$$G(s) = \frac{\zeta_q(s)}{\zeta_q(2s)} H(s),$$

where

$$H(s) = \prod_{(P)} \left(1 - \frac{|P|^{-s}(1-|P|^{1-s}-|P|^{-s})}{(|P|+1)(1-|P|^{-2s})}\right),$$

which converges for $\mathrm{Re}(s) > 1/2$. In addition, $G(s)$ has a simple pole at $s=1$ with residue

$$\frac{H(1)}{\zeta_q(2)\log q} = \frac{1}{\zeta_q(2)\log q} \prod_{(P)} \left(1 + \frac{1}{(|P|+1)(|P|^2-1)}\right).$$

Define the additional Dirichlet series

$$\begin{aligned} G_1(s) \quad &= \quad \sum_{\substack{(m_{\alpha_i})\nmid(g), 1\le i\le \ell \\ (m_{\alpha_i})|(g), \ell+1\le i\le n}} \frac{b(g)}{|g|^s} = \prod_{(P)\neq(m_{\alpha_i}), 1\le i\le n} \left(1 + \frac{|P|^{-s}}{(1-|P|^{-s})(1+|P|^{-1})}\right) \\ &\quad \times \prod_{(P)=(m_{\alpha_i}), \ell+1\le i\le n} \left(\sum_{k=1}^{\infty} b(P^k)|P|^{-ks}\right) \\ &= \quad G(s) \prod_{i=1}^{n} \left(1 + \frac{q^{-u_i s}}{(1-q^{-u_i s})(1+q^{-u_i})}\right)^{-1} \prod_{i=\ell+1}^{n} \frac{q^{-u_i s}}{(1-q^{-u_i s})(1+q^{-u_i})} \\ &= \quad G(s) \prod_{i=1}^{\ell} \frac{(1-q^{-u_i s})(1+q^{-u_i})}{1+q^{-u_i}-q^{-u_i(s+1)}} \prod_{i=\ell+1}^{n} \frac{q^{-u_i s}}{1+q^{-u_i}-q^{-u_i(s+1)}}. \end{aligned}$$

Thus, $G_1(s)$ has a simple pole at $s=1$ with residue

$$\rho = \frac{H(1)}{\zeta_q(2)\log q} \prod_{i=1}^{\ell} \frac{1-q^{-2u_i}}{1+q^{-u_i}-q^{-2u_i}} \prod_{i=\ell+1}^{n} \frac{q^{-u_i}}{1+q^{-u_i}-q^{-2u_i}},$$

and

$$G_1(s) - \frac{\rho}{s-1}$$

is holomorphic for $\mathrm{Re}(s) > 1/2$. Then, using Theorem 17.1 of [Ros02], which is the function field version of the Wiener–Ikehara Tauberian Theorem, we get that

$$\begin{aligned} \sum_{\substack{(g), g\in\mathcal{S}_d \\ (m_{\alpha_i})\nmid(g), 1\le i\le \ell \\ (m_{\alpha_i})|(g), \ell+1\le i\le n}} b(g) &= \frac{H(1)q^{d+1}}{\zeta_q(2)} \prod_{i=1}^{\ell} \frac{1-q^{-2u_i}}{1+q^{-u_i}-q^{-2u_i}} \prod_{i=\ell+1}^{n} \frac{q^{-u_i}}{1+q^{-u_i}-q^{-2u_i}} \\ &\quad + O\left(q^{(1/2+\varepsilon)d}\right). \end{aligned}$$

Using the line above in the formula for $|\mathcal{F}_d^{\mathrm{ord}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|$, we get

$$\begin{aligned}
&|\mathcal{F}_d^{\mathrm{ord}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|\\
&= \frac{q^{d+1-\sum_{i=1}^{\ell} u_i}}{\zeta_q(2)\prod_{i=1}^{\ell}(1-q^{-2u_i})}\frac{H(1)q^{d+1}}{\zeta_q(2)}\prod_{i=1}^{\ell}\frac{1-q^{-2u_i}}{1+q^{-u_i}-q^{-2u_i}}\prod_{i=\ell+1}^{n}\frac{q^{-u_i}}{1+q^{-u_i}-q^{-2u_i}}\\
&\quad + O\left(q^{(3/2+\varepsilon)d}\right)\\
&= \frac{H(1)q^{2d+2-\sum_{i=1}^{n} u_i}}{\zeta_q(2)^2\prod_{i=1}^{n}(1+q^{-u_i}-q^{-2u_i})} + O\left(q^{(3/2+\varepsilon)d}\right).
\end{aligned}$$

□

The previous result may be used to obtain the number of covers in the whole ordinary family by specializing to $n=0$.

Corollary 3.7.

$$|\mathcal{F}_d^{\mathrm{ord}}| = \frac{H(1)q^{2d+2}}{\zeta_q(2)^2} + O\left(q^{(3/2+\varepsilon)d}\right).$$

By combining Proposition 3.6 and Corollary 3.7, we obtain the following result.

Proposition 3.8. *Fix $\alpha_1,\dots,\alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1,\dots,u_n$ over $\mathbb{F}_q$ such that none of the α_i are conjugate to each other. Let $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \le i \le n$. Then*

$$\begin{aligned}
\frac{|\mathcal{F}_d^{\mathrm{ord}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|}{|\mathcal{F}_d^{\mathrm{ord}}|} &= \frac{q^{-\sum_{i=1}^{n} u_i}}{\prod_{i=1}^{n}(1+q^{-u_i}-q^{-2u_i})} + O\left(q^{(-1/2+\varepsilon)d}\right)\\
&= q^{-\sum_{i=1}^{n} u_i}\left(1+O\left(\sum_{i=1}^{n} q^{-u_i}\right)\right) + O\left(q^{(-1/2+\varepsilon)d}\right).
\end{aligned}$$

We finish this section by computing the expected number of points in an ordinary Artin-Schreier cover. For this, we need to compute the case $n=1$, i.e., $|\mathcal{F}_d^{\mathrm{ord}}(\alpha,\beta)|$.

Corollary 3.9. *Fix $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degree u over $\mathbb{F}_q$. Let $\beta \in \mathbb{P}^1(\mathbb{F}_{q^u})$. Then*

$$|\mathcal{F}_d^{\mathrm{ord}}(\alpha,\beta)| = \begin{cases} \frac{H(1)q^{2d+2-u}(1-q^{-u})}{\zeta_q(2)^2(1+q^{-u}-q^{-2u})} + O\left(q^{(3/2+\varepsilon)d+u}\right), & \beta=\infty,\\ \\ \frac{H(1)q^{2d+2-u}}{\zeta_q(2)^2(1+q^{-u}-q^{-2u})} + O\left(q^{(3/2+\varepsilon)d}\right), & \beta \in \mathbb{F}_{q^u}. \end{cases}$$

Proof. The case of $\beta \in \mathbb{F}_{q^u}$ is a simple consequence of Proposition 3.6. For $\beta = [1:0]$, we have, by Lemma 2.3, that

$$\begin{aligned}
|\mathcal{F}_d^{\mathrm{ord}}(\alpha,\infty)| &= |\mathcal{F}_d^{\mathrm{ord}}| - \sum_{\beta\in\mathbb{F}_{q^u}} |\mathcal{F}_d^{\mathrm{ord}}(\alpha,\beta)|\\
&= |\mathcal{F}_d^{\mathrm{ord}}| - q^u|\mathcal{F}_d^{\mathrm{ord}}(\alpha,0)|\\
&= \frac{H(1)q^{2d+2-u}(1-q^{-u})}{\zeta_q(2)^2(1+q^{-u}-q^{-2u})} + O\left(q^{(3/2+\varepsilon)d+u}\right).
\end{aligned}$$

□

By combining Proposition 3.8 and Corollaries 3.7 and 3.9, we obtain the following result.

Proposition 3.10. *Fix $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ with degree u over $\mathbb{F}_q$. Let $\beta \in \mathbb{P}^1(\mathbb{F}_{q^u})$. Then*

$$\frac{|\mathcal{F}_d^{\mathrm{ord}}(\alpha,\beta)|}{|\mathcal{F}_d^{\mathrm{ord}}|} = \begin{cases} \frac{q^{-u}(1-q^{-u})}{1+q^{-u}-q^{-2u}} + O\left(q^{(-1/2+\varepsilon)d+u}\right), & \beta = \infty, \\ \\ \frac{q^{-u}}{1+q^{-u}-q^{-2u}} + O\left(q^{(-1/2+\varepsilon)d}\right), & \beta \in \mathbb{F}_{q^u}. \end{cases}$$

Lemma 3.11. *Fix $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degree u over $\mathbb{F}_q$. The expected number of $\mathbb{F}_{q^k}$-points in the fiber above α is*

$$\begin{cases} 1 + O\left(q^{(-1/2+\varepsilon)d+u}\right), & \text{if } p \nmid \frac{k}{u}, \\ \\ 1 + \frac{p-1}{1+q^{-u}-q^{-2u}} + O\left(q^{(-1/2+\varepsilon)d+u}\right), & \text{if } p \mid \frac{k}{u}. \end{cases}$$

Proof. By Lemma 2.1 and Proposition 3.10, the expected number of $\mathbb{F}_{q^k}$-points in the fiber above α is

$$\frac{q^{-u}(1-q^{-u})}{1+q^{-u}-q^{-2u}} + O\left(q^{(-1/2+\varepsilon)d+u}\right) \\ + \sum_{\beta\in\mathbb{F}_{q^u},\mathrm{tr}_k(\beta)=0} p\left(\frac{q^{-u}}{1+q^{-u}-q^{-2u}} + O\left(q^{(-1/2+\varepsilon)d}\right)\right).$$

If $p \nmid \frac{k}{u}$, then $\mathrm{tr}_k(\beta)=0$ iff $\mathrm{tr}_u(\beta)=0$ and there are $\frac{q^u}{p}$ points in $\mathbb{F}_{q^u}$ with that property.

If $p \mid \frac{k}{u}$, then $\mathrm{tr}_k(\beta) = \frac{k}{u}\,\mathrm{tr}_u(\beta) = 0$ for all $\beta \in \mathbb{F}_{q^u}$, and therefore the expected number of points in the fiber is

$$\frac{q^{-u}(1-q^{-u})}{1+q^{-u}-q^{-2u}} + O\left(q^{(-1/2+\varepsilon)d+u}\right) + \frac{p}{1+q^{-u}-q^{-2u}} + O\left(q^{(-1/2+\varepsilon)d+u}\right).$$

□

For our main result, we recall that an ordinary Artin-Schreier cover has $r+1$ simple poles. This corresponds to taking $d = r+1$. We are ready to prove the first part of Theorem 1.1.

Theorem 3.12. *The expected number of $\mathbb{F}_{q^k}$-points on an ordinary Artin-Schreier cover defined over $\mathbb{F}_q$ is*

$$\begin{cases} q^k + 1 + O\left(q^{(-1/2+\varepsilon)(r+1)+2k}\right), & p \nmid k, \\ \\ q^k + 1 + \frac{p-1}{1+q^{-1}-q^{-2}} + \sum_{u\mid\frac{k}{p}} \frac{p-1}{1+q^{-u}-q^{-2u}}\,\pi(u)u + O\left(q^{(-1/2+\varepsilon)(r+1)+2k}\right), & p \mid k, \end{cases}$$

where $\pi(u)$ is the number of monic irreducible polynomials in $\mathbb{F}_q[X]$ of degree u.

Proof. If $p \nmid k$, the result follows by adding the result of Lemma 3.11 over all $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$. If $p \mid k$ we still get the term q^k+1 and an additional term given by

$$\sum_{u\mid\frac{k}{p}} \sum_{\alpha,\deg\alpha=u} \frac{p-1}{1+q^{-u}-q^{-2u}} = \frac{p-1}{1+q^{-1}-q^{-2}} + \sum_{u\mid\frac{k}{p}} \frac{p-1}{1+q^{-u}-q^{-2u}}\pi(u)u,$$

where the first term on the right hand side accounts for the case $\alpha = \infty$. □

Remark 3.13. When $k=p$, we obtain

$$q^p+1+\frac{(p-1)(q+1)}{1+q^{-1}-q^{-2}}+O\left(q^{(-1/2+\varepsilon)(r+1)+2p}\right).$$

4. The full space

In this case, we consider the family

$$\mathcal{F}_d^{\text{full}}=\{(g(X,Z),h(X,Z)):g(X,Z),h(X,Z)\in\mathcal{S}_d,(g,h)=1\}.$$

Proposition 4.1. *Fix $\alpha_1,\dots,\alpha_n\in\mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1,\dots,u_n$ such that none of the α_i are conjugate to each other. Let $\beta_i\in\mathbb{F}_{q^{u_i}}$ for $1\le i\le n$. Then we have*

$$|\mathcal{F}_d^{\text{full}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|=\frac{q^{2d+2-\sum_{i=1}^n u_i}}{\zeta_q(2)\prod_{i=1}^n(1+q^{-u_i})}+O\left(q^{(1+\varepsilon)d}\right).$$

Proof. Assume without loss of generality that $\beta_1,\dots,\beta_\ell$ are not zero and $\beta_{\ell+1}=\dots=\beta_n=0$. We have, by Lemma 3.3, that

$$\begin{aligned}&|\mathcal{F}_d^{\text{full}}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|\\&\quad=\sum_{g\in\mathcal{S}_d}\left|\left\{h\in\mathcal{S}_d:(h,g)=1,\frac{g(\alpha_i)}{h(\alpha_i)}=\beta_i,1\le i\le n\right\}\right|\\&\quad=\sum_{g\in\mathcal{S}_d}q^{d+1-\sum_{i=1}^\ell u_i}\prod_{(P)|(g)}(1-|P|^{-1})+O\left(q^{(1+\varepsilon)d}\right).\end{aligned}$$

We set

$$b(g)=\prod_{(P)|(g)}(1-|P|^{-1})$$

and

$$G(s)=\sum_{(g)\neq 0}\frac{b(g)}{|g|^s}.$$

Since $b(g)$ is a multiplicative function, it follows that $G(s)$ has an Euler product of the form

$$\begin{aligned}G(s)&=\prod_{(P)}\left(\sum_{k=0}^\infty b(P^k)|P|^{-ks}\right)=\prod_{(P)}\left(1+\frac{b(P)|P|^{-s}}{1-|P|^{-s}}\right)\\&=\prod_{(P)}\left(1+\frac{(1-|P|^{-1})|P|^{-s}}{1-|P|^{-s}}\right)=\prod_{(P)}\left(\frac{1-|P|^{-1-s}}{1-|P|^{-s}}\right).\end{aligned}$$

Therefore

$$G(s)=\frac{\zeta_q(s)}{\zeta_q(1+s)}$$

is analytic for $\mathrm{Re}(s)>0$, except for a simple pole at $s=1$ with residue $\frac{1}{\zeta_q(2)\log q}$.

Now define the Dirichlet series

$$\begin{aligned} G_1(s) &= \sum_{\substack{(m_{\alpha_i}) \nmid (g), 1 \le i \le \ell \\ (m_{\alpha_i}) \mid (g), \ell+1 \le i \le n}} \frac{b(g)}{|g|^s} = \prod_{(P) \neq (m_{\alpha_i}), 1 \le i \le n} \left(\frac{1-|P|^{-1-s}}{1-|P|^{-s}} \right) \\ & \quad \times \prod_{(P)=(m_{\alpha_i}), \ell+1 \le i \le n} \left(\sum_{k=1}^{\infty} b(P^k)|P|^{-ks} \right) \\ &= G(s) \prod_{i=1}^{n} \left(\frac{1-q^{-u_i(1+s)}}{1-q^{-u_i s}} \right)^{-1} \prod_{i=\ell+1}^{n} \left(\frac{q^{-u_i s}(1-q^{-u_i})}{1-q^{-u_i s}} \right) \\ &= G(s) \prod_{i=1}^{\ell} \frac{1-q^{-u_i s}}{1-q^{-u_i(1+s)}} \prod_{i=\ell+1}^{n} \frac{q^{-u_i s}(1-q^{-u_i})}{1-q^{-u_i(1+s)}}. \end{aligned}$$

Thus $G_1(s)$ is analytic for $\mathrm{Re}(s) > 0$, except for a simple pole at $s = 1$ with residue

$$\frac{1}{\zeta_q(2) \log q} \prod_{i=1}^{\ell} \frac{1}{1+q^{-u_i}} \prod_{i=\ell+1}^{n} \frac{q^{-u_i}}{1+q^{-u_i}}.$$

Then, again using Theorem 17.1 of [Ros02], we get that

$$\begin{aligned} & |\mathcal{F}_d^{\text{full}}(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)| \\ & \quad = q^{d+1-\sum_{i=1}^{\ell} u_i} \sum_{\substack{(g), g \in \mathcal{S}_d \\ (m_{\alpha_i}) \nmid (g), 1 \le i \le \ell \\ (m_{\alpha_i}) \mid (g), \ell+1 \le i \le n}} b(g) + O\left(q^{(1+\varepsilon)d}\right) \\ & \quad = \frac{q^{2d+2-\sum_{i=1}^{\ell} u_i}}{\zeta_q(2)} \prod_{i=1}^{\ell} \left(\frac{1}{1+q^{-u_i}} \right) \prod_{i=\ell+1}^{n} \left(\frac{q^{-u_i}}{1+q^{-u_i}} \right) + O\left(q^{(1+\varepsilon)d}\right). \end{aligned}$$

□

We may now proceed to compute the number of covers in the whole family by setting $n = 0$ in the previous result.

Corollary 4.2.

$$|\mathcal{F}_d^{\text{full}}| = \frac{q^{2d+2}}{\zeta_q(2)} + O\left(q^{(1+\varepsilon)d}\right).$$

By combining Proposition 4.1 and Corollary 4.2, we obtain the following result.

Proposition 4.3. *Fix $\alpha_1, \dots, \alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1, \dots, u_n$ such that none of the α_i are conjugate to each other. Let $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \le i \le n$. Then we have*

$$\begin{aligned} \frac{|\mathcal{F}_d^{\text{full}}(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)|}{|\mathcal{F}_d^{\text{full}}|} &= \frac{q^{-\sum_{i=1}^{n} u_i}}{\prod_{i=1}^{n} (1+q^{-u_i})} + O\left(q^{(\varepsilon-1)d}\right) \\ &= q^{-\sum_{i=1}^{n} u_i} \left(1 + O\left(\sum_{i=1}^{n} q^{-u_i}\right)\right) + O\left(q^{(\varepsilon-1)d}\right). \end{aligned}$$

We finish the section by computing the expected number of points in the full Artin-Schreier family.

Corollary 4.4. *Fix $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degree u over $\mathbb{F}_q$. Let $\beta \in \mathbb{P}^1(\mathbb{F}_{q^u})$. Then*

$$|\mathcal{F}_d^{\text{full}}(\alpha,\beta)| = \frac{q^{2d+2-u}}{\zeta_q(2)(1+q^{-u})} + \begin{cases} O\left(q^{(\varepsilon+1)d+u}\right), & \beta = \infty, \\ \\ O\left(q^{(\varepsilon+1)d}\right), & \beta \in \mathbb{F}_{q^u}. \end{cases}$$

Proof. The case of $\beta \in \mathbb{F}_{q^u}$ easily follows from Proposition 4.1. For $\beta = [1:0]$, we have, by Lemma 2.3, that

$$\begin{aligned} |\mathcal{F}_d^{\text{full}}(\alpha,\infty)| &= |\mathcal{F}_d^{\text{full}}| - \sum_{\beta \in \mathbb{F}_{q^u}} |\mathcal{F}_d^{\text{full}}(\alpha,\beta)| \\ &= |\mathcal{F}_d^{\text{full}}| - q^u |\mathcal{F}_d^{\text{full}}(\alpha,0)| \\ &= \frac{q^{2d+2-u}}{\zeta_q(2)(1+q^{-u})} + O\left(q^{(\varepsilon+1)d+u}\right). \end{aligned}$$

□

We then obtain the following result.

Proposition 4.5. *Fix $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degree u over $\mathbb{F}_q$. Let $\beta \in \mathbb{P}^1(\mathbb{F}_{q^u})$. Then*

$$\frac{|\mathcal{F}_d^{\text{full}}(\alpha,\beta)|}{|\mathcal{F}_d^{\text{full}}|} = \frac{q^{-u}}{1+q^{-u}} + \begin{cases} O\left(q^{(\varepsilon-1)d+u}\right), & \beta = \infty, \\ \\ O\left(q^{(\varepsilon-1)d}\right), & \beta \in \mathbb{F}_{q^u}. \end{cases}$$

Lemma 4.6. *Fix $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degree u over $\mathbb{F}_q$. The expected number of $\mathbb{F}_{q^k}$-points in the fiber above α is*

$$\begin{cases} 1 + O\left(q^{(\varepsilon-1)d+u}\right), & \text{if } p \nmid \frac{k}{u}, \\ \\ 1 + \frac{p-1}{1+q^{-u}} + O\left(q^{(\varepsilon-1)d+u}\right), & \text{if } p \mid \frac{k}{u}. \end{cases}$$

Proof. By Lemma 2.1 and Proposition 4.5, we have

$$\frac{q^{-u}}{1+q^{-u}} + O\left(q^{(\varepsilon-1)d+u}\right) + \sum_{\beta \in \mathbb{F}_{q^u}, \operatorname{tr}_k(\beta)=0} p\left(\frac{q^{-u}}{1+q^{-u}} + O\left(q^{(\varepsilon-1)d}\right)\right).$$

If $p \nmid \frac{k}{u}$, then $\operatorname{tr}_k(\beta) = 0$ iff $\operatorname{tr}_u(\beta) = 0$, and there are $\frac{q^u}{p}$ points in $\mathbb{F}_{q^u}$ with that property.

If $p \mid \frac{k}{u}$, then $\operatorname{tr}_k(\beta) = \frac{k}{u}\operatorname{tr}_u(\beta) = 0$ for all $\beta \in \mathbb{F}_{q^u}$, and therefore the expected number of points in the fiber is

$$\frac{q^{-u}}{1+q^{-u}} + O\left(q^{(\varepsilon-1)d+u}\right) + \frac{p}{1+q^{-u}} + O\left(q^{(\varepsilon-1)d+u}\right).$$

□

We are ready to prove Theorem 1.1 (2).

Theorem 4.7. *The expected number of $\mathbb{F}_{q^k}$-points on an Artin-Schreier cover in $\mathcal{AS}_{\mathfrak{g}}$ defined over $\mathbb{F}_q$ is*

$$\begin{cases} q^k + 1 + O\left(q^{(\varepsilon-1)d+2k}\right), & p \nmid k, \\ \\ q^k + 1 + (p-1)q^{k/p} + \frac{p-1}{1+q^{-1}} - (p-1)\sum_{u \mid \frac{k}{p}} \frac{1}{1+q^u}\pi(u)u + O\left(q^{(\varepsilon-1)d+2k}\right), & p \mid k. \end{cases}$$

Proof. The result for $p \nmid k$ follows from Lemma 4.6. If $p \mid k$, we still get the term $q^k + 1$ and an additional term given by

$$\begin{aligned}\sum_{u \mid \frac{k}{p}} \sum_{\alpha, \deg \alpha = u} \frac{p-1}{1+q^{-u}} &= \frac{p-1}{1+q^{-1}} + (p-1) \sum_{u \mid \frac{k}{p}} \frac{q^u}{1+q^u} \pi(u) u \\ &= \frac{p-1}{1+q^{-1}} + (p-1) q^{k/p} - (p-1) \sum_{u \mid \frac{k}{p}} \frac{1}{1+q^u} \pi(u) u.\end{aligned}$$

□

Remark 4.8. When $k = p$, we obtain

$$q^p + 1 + (p-1)q + O\left(q^{(\varepsilon-1)d+2p}\right).$$

5. Prescribed factorization type

Recall that

$$\mathcal{F}_d^v = \{(g(X,Z), h(X,Z)) : g(X,Z), h(X,Z) \in \mathcal{S}_d, (g,h) = 1, \\ h \text{ has factorization type } v\},$$

where $v = (r_1^{d_{1,1}}, \dots, r_1^{d_{1,\ell_1}}, \dots, r_m^{d_{m,1}}, \dots, r_m^{d_{m,\ell_m}})$ and

$$h = P_{1,1}^{d_{1,1}} \cdots P_{1,\ell_1}^{d_{1,\ell_1}} \cdots P_{m,1}^{d_{m,1}} \cdots P_{m,\ell_m}^{d_{m,\ell_m}},$$

where the $P_{i,j}$ are distinct irreducible polynomials of degree r_i and $r_i \neq r_j$ if $i \neq j$. The degree of h is then given by $d = \sum_{i=1}^m r_i \sum_{j=1}^{\ell_i} d_{i,j}$.

We will first compute the expected number of points for this family. We need the following result.

Lemma 5.1. *Fix a polynomial $h \in \mathcal{S}_d$. Then, if $h \neq 0$,*

$$|\{g \in \mathcal{S}_d : (g,h) = 1\}| = q^{d+1} \prod_{(P)|(h)} (1 - |P|^{-1}).$$

We remark that this lemma follows directly from the proof of Lemma 3.3.

Proposition 5.2. *Fix $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees u over $\mathbb{F}_q$. Let $\beta \in \mathbb{P}^1(\mathbb{F}_{q^u})$. Then, if $u \leq d$,*

$$\frac{|\mathcal{F}_d^v(\alpha,\beta)|}{|\mathcal{F}_d^v|} = \begin{cases} q^{-u}, & \deg(\alpha) = u \neq r_i \forall i, \beta \neq \infty, \\ 0, & \deg(\alpha) = u \neq r_i \forall i, \beta = \infty, \\ \frac{q^{-r_{i_0}}(\pi(r_{i_0}) - \ell_{i_0})}{\pi(r_{i_0})}, & \deg(\alpha) = r_{i_0}, \beta \neq \infty, \\ \frac{\ell_{i_0}}{\pi(r_{i_0})}, & \deg(\alpha) = r_{i_0}, \beta = \infty. \end{cases}$$

If $u > d$ the above quotient is $O(q^{-d})$.

Proof. We first consider the size of the whole family. By Lemma 5.1 we have

$$\begin{aligned}|\mathcal{F}_d^v| &= \sum_{\deg P_{i,j}=r_i,\ \text{all different}} |\{g \in \mathcal{S}_d \,:\, (g,h)=1\}| \\ &= q^{d+1}\prod_{i=1}^{m}(1-q^{-r_i})^{\ell_i} \sum_{\deg P_{i,j}=r_i,\ \text{all different}} 1. \qquad (5.1)\end{aligned}$$

If $\deg(\alpha)=u\neq r_i$, and $\beta\in\mathbb{F}_{q^u}$, then by Lemma 2.3 it suffices to find $|\mathcal{F}_d^v(\alpha,\beta)|$ for $\beta=0$. If this is the case, then we need $g(\alpha)=0$ or $m_\alpha \mid g$.

$$\begin{aligned}|\mathcal{F}_d^v(\alpha,\beta)| &= \sum_{\deg P_{i,j}=r_i,\ \text{all different}} |\{g \in \mathcal{S}_d \,:\, (g,h)=1, m_\alpha \mid g\}| \\ &= q^{d+1-u}\prod_{i=1}^{m}(1-q^{-r_i})^{\ell_i} \sum_{\deg P_{i,j}=r_i,\ \text{all different}} 1.\end{aligned}$$

If $\deg(\alpha)=u\neq r_i$ and $\beta=\infty$, we get a contradiction and thus

$$|\mathcal{F}_d^v(\alpha,\infty)|=0.$$

Now assume that $\deg(\alpha)=u=r_{i_0}$, for some i_0, and that $\beta\in\mathbb{F}_{q^u}$. By Lemma 2.3 we can again assume that $\beta=0$. In this case we need to impose the condition that $h(\alpha)\neq 0$. Therefore,

$$|\mathcal{F}_d^v(\alpha,\beta)| = q^{d+1-r_{i_0}}\prod_{i=1}^{m}(1-q^{-r_i})^{\ell_i} \sum_{\deg P_{i,j}=r_i, P_{i_0,j}\neq m_\alpha,\ \text{all different}} 1.$$

Finally, if $\deg(\alpha)=r_{i_0}$ for some i_0 and $\beta=\infty$, we need that $h(\alpha)=0$ and $g(\alpha)\neq 0$.

$$|\mathcal{F}_d^v(\alpha,\infty)| = q^{d+1}\prod_{i=1}^{m}(1-q^{-r_i})^{\ell_i} \sum_{\deg P_{i,j}=r_i, \exists P_{i_0,j}=m_\alpha,\ \text{all different}} 1.$$

The result now follows from the identity

$$|\{\deg P_{i,j}=r_i,\ \text{all different}\}| = \prod_{i=1}^{m}\binom{\pi(r_i)}{\ell_i}.$$

□

We are now ready to prove the main result of this section.

Theorem 5.3. *The expected number of $\mathbb{F}_{q^k}$-points on an Artin-Schreier cover with poles given by the factorization type v defined over $\mathbb{F}_q$ is*

$$\begin{cases} q^k+1, & p\nmid k, \\ q^k+1+(p-1)q^{k/p}+(p-1)\left(1-\sum_{r_i\mid k}\ell_i r_i\right), & p\mid k. \end{cases}$$

Proof. We can assume that $p\nmid r_i$. This is because the $\mathbb{F}_q$-isomorphisms $(x,y)\mapsto(x,y+ax^k)$ allow us to eliminate all the terms in h such that x appears to a power multiple of p.

By Lemma 2.1, the final count becomes

$$\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^k})}\frac{|\mathcal{F}_d^v(\alpha,\infty)|}{|\mathcal{F}_d^v|}+\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^k})}\sum_{\beta\in\mathbb{F}_{q^{\deg(\alpha)}},\mathrm{tr}_k(\beta)=0}p\frac{|\mathcal{F}_d^v(\alpha,\beta)|}{|\mathcal{F}_d^v|}$$
$$=\sum_{r_i|k}\frac{\ell_i}{\pi(r_i)}\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^k}),\deg(\alpha)=r_i}1+\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^k})}\sum_{\beta\in\mathbb{F}_{q^{\deg(\alpha)}},\mathrm{tr}_k(\beta)=0}pq^{-\deg(\alpha)}$$
$$-\sum_{r_i|k}\frac{\ell_i}{\pi(r_i)}\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^k}),\deg(\alpha)=r_i}\sum_{\beta\in\mathbb{F}_{q^{r_i}},\mathrm{tr}_k(\beta)=0}pq^{-r_i}.$$

If $p\nmid k$, then $\mathrm{tr}_k(\beta)=0$ if and only if $\mathrm{tr}_u(\beta)=0$ and there are $\frac{q^u}{p}$ elements in $\mathbb{F}_{q^u}$ with that property. Thus we obtain q^k+1. If $p\mid k$, then since $p\nmid r_i$, if $r_i\mid k$, then $p\mid\frac{k}{r_i}$ and $\mathrm{tr}_k(\beta)=0$ for $\beta\in\mathbb{F}_{q^{r_i}}$. The final count then becomes

$$\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^k})}\sum_{\beta\in\mathbb{F}_{q^{\deg(\alpha)}},\mathrm{tr}_k(\beta)=0}pq^{-\deg(\alpha)}$$
$$+\sum_{r_i|k}\frac{\ell_i}{\pi(r_i)}\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^{r_i}}),\deg\alpha=r_i}\left(1-\sum_{\beta\in\mathbb{F}_{q^{r_i}},\mathrm{tr}_k(\beta)=0}pq^{-r_i}\right)$$
$$=q^k+1+(p-1)(q^{k/p}+1)-\sum_{r_i|k}\frac{\ell_i}{\pi(r_i)}\sum_{\alpha\in\mathbb{P}^1(\mathbb{F}_{q^{r_i}}),\deg\alpha=r_i}(p-1)$$
$$=q^k+1+(p-1)q^{k/p}+(p-1)\left(1-\sum_{r_i|k}\ell_ir_i\right).$$

□

Now suppose that we take the p-rank 0 family. We recall that this corresponds to $v=(1^d)$. A simple application of Theorem 5.3 yields the following.

Theorem 5.4. *The expected number of $\mathbb{F}_{q^k}$-points on a p-rank 0 Artin-Schreier cover in $\mathcal{AS}_{\mathfrak{g},0}$ defined over $\mathbb{F}_q$ is*

$$\begin{cases}q^k+1, & p\nmid k,\\ q^k+1+(p-1)q^{k/p}, & p\mid k.\end{cases}$$

This recovers the result from [Ent12].

Finally we consider the family of curves with p-rank equal to $p-1$. This means that we consider the case when $f(x)$ is a rational function with exactly two poles. If the poles happen to be at $\mathbb{F}_q$-rational points, we are in the case corresponding to $v=(1^{d_1},1^{d_2})$. Note that in this case we could use an automorphism of $\mathbb{P}^1(\mathbb{F}_q)$ to move the two poles to zero and infinity, and therefore this case corresponds to the case when $f(X)$ is a Laurent polynomial. Otherwise, the two poles have to be $\mathbb{F}_q$ Galois conjugate points in $\mathbb{F}_{q^2}$, and we find ourselves in the case of prescribed factorization $v=(2^d)$. The final answer for the whole p-rank equal to $p-1$ stratum is given by taking the average between these two cases. Again, by applying Theorem 5.3 we get the third part of Theorem 1.1.

Theorem 5.5. *The expected number of $\mathbb{F}_{q^k}$-points on a p-rank $p-1$ Artin-Schreier cover in $\mathcal{AS}_{\mathfrak{g},p-1}$ defined over $\mathbb{F}_q$ is*

$$\begin{cases} q^k+1, & p\nmid k,\\ q^k+1+(p-1)(q^{k/p}-1), & p\mid k,\ k\ \textit{even},\\ q^k+1+(p-1)q^{k/p}, & p\mid k,\ k\ \textit{odd}. \end{cases}$$

Proof. The different formulas occur when $p \mid k$. For k even we get that both $1 \mid k$ and $2 \mid k$, and therefore we always get $q^k+1+(p-1)(q^{k/p}-1)$ for $p \mid k$. When k is odd, the case $p \mid k$ will yield

$$q^k+1+(p-1)(q^{k/p}-1)$$

for $(1^{d_1},1^{d_2})$ and

$$q^k+1+(p-1)(q^{k/p}+1)$$

for (2^d).

Each case happens half of the time. To see this, notice that (2^d) corresponds to counting degree 2 irreducible monic polynomials over $\mathbb{F}_q$, while $(1^{d_1},1^{d_2})$ corresponds to counting degree 2 reducible monic polynomials with two different roots over $\mathbb{F}_q$. The number of degree 2 monic polynomials that are not squares is q^2-q, and exactly half of them are reducible. We take the average and obtain the final result. □

We now proceed to the case where we fix several values, which will be needed for the computation of the moments.

Proposition 5.6. *Let $\alpha_1,\dots,\alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1,\dots,u_n$ over $\mathbb{F}_q$ be such that none of the α_i are conjugate to each other. Let $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \le i \le n$. Then*

$$\frac{|\mathcal{F}_d^v(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|}{|\mathcal{F}_d^v|}=\prod_{i=1}^m(1-\tau(r_i,\ell_i;u_1,\dots,u_n))q^{-(u_1+\cdots+u_n)}+O(q^{(\varepsilon-1)d}),$$

where $0\le\tau(r_i,\ell_i;u_1,\dots,u_n)\le 1$ is a constant that depends on the number of u_j's that are equal to r_i and is equal to zero if $u_j\neq r_i$ for all j.

Proof. Without loss of generality we can assume that $\beta_1,\dots,\beta_\ell$ are not zero and that $\beta_{\ell+1}=\cdots=\beta_n=0$. We have that

(5.2)

$$\begin{aligned}&|\mathcal{F}_d^v(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|\\ &=\sum_{\substack{\deg P_{i,j}=r_i,\text{ all different}\\ P_{i,j}\neq m_\alpha}}\left|\left\{g_1\in\mathcal{S}_{d-\sum_{j=\ell+1}^n u_j}\ :\ (g_1,h)=1,\right.\right.\\ &\qquad\qquad\left.\left.\frac{g_1(\alpha_i)\prod_{j=\ell+1}^n m_{\alpha_j}(\alpha_i)}{h(\alpha_i)}=\beta_i,\ 1\le i\le\ell\right\}\right|.\end{aligned}$$

Notice that $\beta_i^{-1} \in \mathbb{F}_{q^{u_i}}^*$ for $1 \leq i \leq \ell$. By Lemma 3.3,

$$\begin{aligned}
&\left|\left\{ g_1 \in \mathcal{S}_{d-\sum_{j=\ell+1}^n u_j} \,:\, (g_1,h)=1, \frac{h(\alpha_i)}{g_1(\alpha_i)\prod_{j=\ell+1}^n m_{\alpha_j}(\alpha_i)} = \beta_i^{-1},\, 1 \leq i \leq \ell \right\}\right| \\
= \;& q^{d+1-\sum_{i=1}^n u_i} \prod_{(P)|(h)} (1-|P|^{-1}) + O\left(q^{\varepsilon d}\right) \\
= \;& q^{d+1-\sum_{i=1}^n u_i} \prod_{j=1}^m (1-q^{-r_j})^{\ell_j} + O\left(q^{\varepsilon d}\right).
\end{aligned}$$

On the other hand, $|\{\deg P_{i,j} = r_i, \text{all different}, P_{i,j} \neq m_\alpha\}|$ is a product of binomials of the form

$$\binom{\pi(r_i)-s_i}{\ell_i},$$

where s_i corresponds to the number of u_j's that equal the particular r_i.

This gives that

$$\frac{|\{\deg P_{i,j} = r_i, \text{all different}, P_{i,j} \neq m_\alpha\}|}{|\{\deg P_{i,j} = r_i, \text{all different}\}|}$$

is a product of terms of the form

$$\begin{aligned}
&(1-\tau(r_i,\ell_i;u_1,\dots,u_n)) \\
&\quad = \frac{\binom{\pi(r_i)-s_i}{\ell_i}}{\binom{\pi(r_i)}{\ell_i}} = \frac{(\pi(r_i)-\ell_i)(\pi(r_i)-\ell_i-1)\cdots(\pi(r_i)-\ell_i-s_i+1)}{\pi(r_i)(\pi(r_i)-1)\cdots(\pi(r_i)-s_i+1)}.
\end{aligned}$$

By dividing equation (5.2) by equation (5.1), we get

$$\frac{|\mathcal{F}_d^v(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)|}{|\mathcal{F}_d^v|} = q^{-\sum_{i=1}^n u_i} \prod_{i=1}^m (1-\tau(r_i,\ell_i;u_1,\dots,u_n)) + O(q^{(\varepsilon-1)d}),$$

where the constant satisfies the desired properties. □

6. Beurling–Selberg functions

In this section we start the development of the tools needed to prove Theorem 1.3. By the functional equation, the conjugate of a root of $Z_{C_f}(u)$ is also a root, so we can restrict to considering symmetric intervals. Let $0 < \beta < 1$ and set $\mathcal{I} = [-\beta/2, \beta/2] \subset [-1/2, 1/2)$. Our goal is to estimate the quantity

$$N_{\mathcal{I}}(f,\psi) := \#\left\{ 1 \leq j \leq \frac{2\mathfrak{g}}{p-1} \,:\, \theta_j(f,\psi) \in \mathcal{I} \right\} = \sum_{j=1}^{2\mathfrak{g}/(p-1)} \chi_{\mathcal{I}}(\theta_j(f,\psi)),$$

where $\chi_{\mathcal{I}}$ denotes the characteristic function of $\mathcal{I}$. We are going to approximate $\chi_{\mathcal{I}}$ with Beurling–Selberg polynomials $I_K^{\pm}$.

In what follows, we use the standard notation $e(x) := e^{2\pi i x}$. Let K be a positive integer, and let $h(\theta) = \sum_{|k|\leq K} a_k e(k\theta)$ be a trigonometric polynomial. Then, the coefficients a_k are given by the Fourier transform

$$a_k = \widehat{h}(k) = \int_{-1/2}^{1/2} h(\theta) e(-k\theta) d\theta.$$

Here is a list of a series of useful properties of the Beurling–Selberg polynomials (see [Mon94, Ch. 1.2] that will be used in this paper.

(a) The $I_K^{\pm}$ are trigonometric polynomials of degree $\leq K$, i.e.,

$$I_K^{\pm}(x) = \sum_{|k|\leq K} \widehat{I}_K^{\pm}(k)e(kx).$$

(b) The Beurling–Selberg polynomials yield upper and lower bounds for the characteristic function:

$$I_K^- \leq \chi_{\mathcal{I}} \leq I_K^+.$$

(c) The integral of Beurling–Selberg polynomials approximates the length of the interval:

$$\int_{-1/2}^{1/2} I_K^{\pm}(x)dx = \int_{-1/2}^{1/2} \chi_{\mathcal{I}}(x)dx \pm \frac{1}{K+1} = |\mathcal{I}| \pm \frac{1}{K+1}.$$

(d) The $I_K^{\pm}$ are even (because the interval $\mathcal{I}$ is symmetric about the origin). Therefore the Fourier coefficients are also even, i.e., $\widehat{I}_K^{\pm}(-k) = \widehat{I}_K^{\pm}(k)$ for $|k| \leq K$.

(e) The non-zero Fourier coefficients of the Beurling–Selberg polynomials approximate those of the characteristic function:

$$|\widehat{I}_K^{\pm}(k) - \widehat{\chi}_{\mathcal{I}}(k)| \leq \frac{1}{K+1} \quad \Longrightarrow \quad \widehat{I}_K^{\pm}(k) = \frac{\sin(\pi k|\mathcal{I}|)}{\pi k} + O\left(\frac{1}{K+1}\right), \quad k \geq 1.$$

Therefore we obtain the following bound:

$$|\widehat{I}_K^{\pm}(k)| \leq \frac{1}{K+1} + \min\left\{|\mathcal{I}|, \frac{\pi}{|k|}\right\}, \quad 0 < |k| \leq K.$$

We now list some results that will be useful in future sections.

Proposition 6.1 ([FR10, Proposition 4.1])**.** *For $K \geq 1$ such that $K|\mathcal{I}| > 1$, we have*

$$\begin{aligned}
\sum_{k\geq 1} \widehat{I}_K^{\pm}(2k) &= O(1),\\
\sum_{k\geq 1} \widehat{I}_K^{\pm}(k)^2 k &= \frac{1}{2\pi^2}\log(K|\mathcal{I}|) + O(1),\\
\sum_{k\geq 1} \widehat{I}_K^{+}(k)\widehat{I}_K^{-}(k) k &= \frac{1}{2\pi^2}\log(K|\mathcal{I}|) + O(1).
\end{aligned}$$

We remark that for a given K the above sums are actually finite, since the Beurling–Selberg polynomials $I_K^{\pm}$ have degree at most K. We will also need the following estimates.

Proposition 6.2 ([BDFLS12, Proposition 5.2])**.** *For $\alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_r > 0$ and $\beta_1, \ldots, \beta_r \in \mathbb{R}$, we have*

$$\sum_{k_1,\ldots,k_r\geq 1} \widehat{I}_K^{\pm}(k_1)^{\alpha_1} \ldots \widehat{I}_K^{\pm}(k_r)^{\alpha_r} k_1^{\beta_1} \ldots k_r^{\beta_r} q^{-\gamma_1 k_1 - \cdots - \gamma_r k_r} = O(1).$$

For $\alpha_1, \alpha_2, \gamma > 0$ and $\beta \in \mathbb{R}$,

$$\sum_{k \geq 1} \widehat{I_K^{\pm}}(k)^{\alpha_1} \widehat{I_K^{\pm}}(2k)^{\alpha_2} k^{\beta} q^{-\gamma k} = O(1).$$

7. Set-up for the distribution of the zeroes

We state here an explicit formula that will be used to relate $L(u, f, \psi)$ to the Beurling–Selberg polynomials. Recall that $2\mathfrak{g} = (p-1)(\Delta - 1)$.

Lemma 7.1 ([BDFLS12, Lemma 3.1]). *Let $h(\theta) = \sum_{|k| \leq K} \widehat{h}(k) e(k\theta)$ be a trigonometric polynomial. Let $\theta_j(f, \psi)$ be the eigenangles of the L-function $L(u, f, \psi)$. Then we have*

$$\sum_{j=1}^{\Delta-1} h(\theta_j(f, \psi)) = (\Delta - 1)\widehat{h}(0) - \sum_{k=1}^{K} \frac{\widehat{h}(k) S_k(f, \psi) + \widehat{h}(-k) S_k(f, \overline{\psi})}{q^{k/2}}, \tag{7.1}$$

where

$$S_k(f, \psi) = \sum_{\substack{x \in \mathbb{P}^1(\mathbb{F}_{q^k}) \\ f(x) \neq \infty}} \psi(\operatorname{tr}_k(f(x))).$$

We use the Beurling–Selberg approximation of the characteristic function of the interval $\mathcal{I}$ to rewrite $N_{\mathcal{I}}(f, \psi)$ and $N_{\mathcal{I}}(C_f)$ where f varies over one of the families $\mathcal{F}_d$. By property (b) of the Beurling–Selberg polynomials, we have

$$\sum_{j=1}^{\Delta-1} I_K^{-}(\theta_j(f, \psi)) \leq N_{\mathcal{I}}(f, \psi) \leq \sum_{j=1}^{\Delta-1} I_K^{+}(\theta_j(f, \psi)),$$

and using the explicit formula of Lemma 7.1 and property (c), we have

$$\sum_{j=1}^{\Delta-1} I_K^{\pm}(\theta_j(f, \psi)) = (\Delta - 1)|\mathcal{I}| - S^{\pm}(K, f, \psi) \pm \frac{\Delta - 1}{K+1},$$

where

$$S^{\pm}(K, f, \psi) := \sum_{k=1}^{K} \frac{\widehat{I_K^{\pm}}(k) S_k(f, \psi) + \widehat{I_K^{\pm}}(-k) S_k(f, \bar{\psi})}{q^{k/2}}. \tag{7.2}$$

This gives

$$-S^{-}(K, f, \psi) - \frac{\Delta - 1}{K+1} \leq N_{\mathcal{I}}(f, \psi) - (\Delta - 1)|\mathcal{I}| \leq -S^{+}(K, f, \psi) + \frac{\Delta - 1}{K+1}, \tag{7.3}$$

and

$$-\sum_{h=1}^{p-1} S^{-}(K, f, \psi^h) - \frac{2\mathfrak{g}}{K+1} \leq N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}| \leq -\sum_{h=1}^{p-1} S^{+}(K, f, \psi^h) + \frac{2\mathfrak{g}}{K+1}. \tag{7.4}$$

In the next section we are going to compute the moments

$$\frac{1}{|\mathcal{F}_d|} \sum_{f \in \mathcal{F}_d} S^{\pm}(K, f, \psi^h)^n \quad \text{and} \quad \frac{1}{|\mathcal{F}_d|} \sum_{f \in \mathcal{F}_d} S^{\pm}(K, C_f)^n,$$

where

$$S^{\pm}(K,C_f)^n = \sum_{h_1,\dots,h_n=1}^{p-1} S^{\pm}(K,f,\psi^{h_1})\dots S^{\pm}(K,f,\psi^{h_n}). \tag{7.5}$$

We will show that they approach the Gaussian moments when properly normalized. We will then use this result to show that

$$\frac{N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}|}{\sqrt{\frac{2(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|)}}$$

converges to a normal distribution as $\mathfrak{g} \to \infty$ since it converges in mean square to

$$\frac{S^{\pm}(K,C_f)}{\sqrt{\frac{2(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|)}}.$$

8. Moments

Our goal is to compute the moments of $S^{\pm}(K,C_f)$ when f varies in any of the families of curves $\mathcal{F}_d^{\text{ord}}$, $\mathcal{F}_d^{\text{full}}$, and $\mathcal{F}_d^{v}$.

Definition 8.1. Let

$$E_{\mathcal{F}_d}(u) = \begin{cases} (1+q^{-u}-q^{-2u})^{-1}, & \mathcal{F}_d = \mathcal{F}_d^{\text{ord}}, \\ (1+q^{-u})^{-1}, & \mathcal{F}_d = \mathcal{F}_d^{\text{full}}, \\ \dfrac{\pi(r_i)-\ell_i}{\pi(r_i)}, & \mathcal{F}_d = \mathcal{F}_d^{v} \text{ and } u = r_i \text{ for some } i, \\ 1, & \mathcal{F}_d = \mathcal{F}_d^{v} \text{ and } u \neq r_i \text{ for any } i. \end{cases}$$

More generally, we have

$$E_{\mathcal{F}_d}(u_1,\dots,u_n) = \begin{cases} \displaystyle\prod_{i=1}^{n} E_{\mathcal{F}_d}(u_i), & \mathcal{F}_d = \mathcal{F}_d^{\text{ord}}, \mathcal{F}_d^{\text{full}}, \\ \displaystyle\prod_{i=1}^{m}(1-\tau(r_i,\ell_i;u_1,\dots,u_n)), & \mathcal{F}_d = \mathcal{F}_d^{v}, \end{cases}$$

where $\tau(r_i,\ell_i;u_1,\dots,u_n)$ is as defined in Proposition 5.6.

Remark 8.2. Let $\mathcal{F}_d$ be any one of the families considered. Then

$$E_{\mathcal{F}_d}(u) = 1 + O\left(uq^{-u}\right).$$

The estimate can be improved to $E_{\mathcal{F}_d}(u) = 1 + O\left(q^{-u}\right)$ for $\mathcal{F}_d^{\text{ord}}$ and $\mathcal{F}_d^{\text{full}}$. In the case of $\mathcal{F}_d^{v}$, we are assuming that the ℓ_i are fixed constants and using the estimate $\pi(m) = \frac{q^m}{m} + O\left(\frac{q^{m/2}}{m}\right)$ (see [Ros02, Theorem 2.2]).

In addition, we have that

$$E_{\mathcal{F}_d}(u_1,\dots,u_n) \ll 1.$$

From now on we will use the notation $\alpha_1 \sim \alpha_2$ to indicate that α_1 and α_2 are Galois conjugate, and $\alpha_1 \not\sim \alpha_2$ for the opposite statement.

Then, for all the families under consideration we have the following result.

Lemma 8.3. *Let $\alpha \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degree u over $\mathbb{F}_q$. Let $\beta \in \mathbb{F}_{q^u}$. Let $\mathcal{F}_d$ be any of the families under consideration. Then*

$$\frac{|\mathcal{F}_d(\alpha,\beta)|}{|\mathcal{F}_d|} = \frac{|\mathcal{F}_d(\alpha,0)|}{|\mathcal{F}_d|} = \frac{E_{\mathcal{F}_d}(u)}{q^u} + O\left(q^{-d/2}\right). \tag{8.1}$$

Let $\alpha_1, \alpha_2 \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees u_1, u_2 respectively over $\mathbb{F}_q$. Let $\beta_1 \in \mathbb{F}_{q^{u_1}}$, $\beta_2 \in \mathbb{F}_{q^{u_2}}$. Let $\mathcal{F}_d$ be any of the families under consideration. Then, if $\alpha_1 \not\sim \alpha_2$,

$$\frac{|\mathcal{F}_d(\alpha_1,\alpha_2,\beta_1,\beta_2)|}{|\mathcal{F}_d|} = \frac{E_{\mathcal{F}_d}(u_1,u_2)}{q^{u_1+u_2}} + O\left(q^{-d/2}\right), \tag{8.2}$$

where $E_{\mathcal{F}_d}(u_1,u_2)$ does not depend on the values of β_1, β_2.

If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ by the same automorphism, then

$$\frac{|\mathcal{F}_d(\alpha_1,\alpha_2,\beta_1,\beta_2)|}{|\mathcal{F}_d|} = \frac{|\mathcal{F}_d(\alpha_1,\beta_1)|}{|\mathcal{F}_d|} = \frac{E_{\mathcal{F}_d}(u_1)}{q^{u_1}} + O\left(q^{-d/2}\right). \tag{8.3}$$

Otherwise, we get zero.

Let $\alpha_1, \dots, \alpha_n \in \mathbb{P}^1(\mathbb{F}_{q^k})$ of degrees $u_1, \dots, u_n$ over $\mathbb{F}_q$ and let $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \le i \le n$.

If none of the α_i are conjugate to each other, then

$$\frac{|\mathcal{F}_d(\alpha_1,\dots,\alpha_n,\beta_1,\dots\beta_n)|}{|\mathcal{F}_d|} = \frac{E_{\mathcal{F}_d}(u_1,\dots,u_n)}{q^{u_1+\cdots+u_n}} + O(q^{-d/2}), \tag{8.4}$$

where $E_{\mathcal{F}_d}(u_1,\dots,u_n)$ does not depend on the values of $\beta_1,\dots,\beta_n$.

If some of the α_i's are conjugate to others, then we get zero, unless the corresponding β_i's are conjugate by the same automorphisms, and in that case we get formula (8.4), *where the u_i's correspond to the degrees for each of the* different *conjugacy classes of the α_i's.*

Proof. This follows from Propositions 3.8, 3.10, 4.3, 4.5, 5.2, and 5.6. □

We recall that for a family $\mathcal{F}$, a function G depending on f, and a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, we have the notation

$$\langle G(f)\rangle_{\mathcal{F}} := \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} G(f),$$
$$\langle G(f)\rangle_{\mathcal{F},\boldsymbol{\alpha}} := \frac{1}{|\mathcal{F}|}\sum_{\substack{f\in\mathcal{F}\\ f(\alpha_i)\neq\infty, 1\le i\le n}} G(f).$$

The main idea in the computations of moments is that if we sum the value of a non-trivial additive character ψ evaluated at a linear combination of the traces $\mathrm{tr}_{u_i}(\beta_i)$ over all $\beta_i \in \mathbb{F}_{q^{u_i}}$ for $1 \le i \le s$, then the sum will be 0 unless each coefficient is divisible by p.

Lemma 8.4. *Let $m_1, \dots, m_s \in \mathbb{Z}$, and ψ a non-trivial additive character of $\mathbb{F}_p$. Then*

$$\sum_{\beta_i \in \mathbb{F}_{q^{u_i}},\ 1\le i\le s} \psi(m_1\,\mathrm{tr}_{u_1}(\beta_1) + \cdots + m_s\,\mathrm{tr}_{u_s}(\beta_s)) = \begin{cases} q^{u_1+\cdots+u_s}, & p \mid m_i \textit{ for } 1 \le i \le s, \\ 0, & \textit{otherwise.}\end{cases}$$

8.1. First moment.

Lemma 8.5. *Let h be an integer such that $p \nmid h$, $e \in \{-1,1\}$, and $k > 0$. Let $\alpha \in \mathbb{F}_{q^k}$ of degree u over $\mathbb{F}_q$. Let $\mathcal{F}_d$ be any of the families under consideration. We have*

$$\langle \psi(eh \operatorname{tr}_k f(\alpha)) \rangle_{\mathcal{F}_{d,\alpha}} = \begin{cases} E_{\mathcal{F}_d}(u) + O\left(q^{u-d/2}\right), & p \mid \frac{k}{u}, \\ \\ O\left(q^{u-d/2}\right), & \textit{otherwise.} \end{cases}$$

Proof. By reversing the order of summation, we obtain

$$\langle \psi(eh \operatorname{tr}_k f(\alpha)) \rangle_{\mathcal{F}_{d,\alpha}} = \sum_{\beta \in \mathbb{F}_{q^u}} \psi(eh \operatorname{tr}_k(\beta)) \frac{|\mathcal{F}_d(\alpha,\beta)|}{|\mathcal{F}_d|}.$$

We now apply Lemma 8.3 in order to obtain

$$\frac{E_{\mathcal{F}_d}(u)}{q^u} \sum_{\beta \in \mathbb{F}_{q^u}} \psi\left(\frac{ehk}{u} \operatorname{tr}_u(\beta)\right) + O\left(q^{u-d/2}\right).$$

Lemma 8.4 implies that the main term is zero unless $p \mid \frac{k}{u}$. This completes the proof of the statement. □

For positive integers k, h with $p \nmid h$ and $e \in \{-1,1\}$, set

$$\begin{aligned} M_{1,d}^{k,e,h} &:= \left\langle q^{-k/2} \sum_{\substack{\alpha \in \mathbb{F}_{q^k} \\ f(\alpha) \neq \infty}} \psi(eh \operatorname{tr}_k f(\alpha)) \right\rangle_{\mathcal{F}_d} \\ &= q^{-k/2} \sum_{\alpha \in \mathbb{F}_{q^k}} \langle \psi(eh \operatorname{tr}_k f(\alpha)) \rangle_{\mathcal{F}_{d,\alpha}}. \end{aligned}$$

Lemma 8.5 has the following consequence.

Theorem 8.6. *Let h be an integer such that $p \nmid h$ and let $\mathcal{F}_d$ be any of the families under consideration. Then*

$$\begin{aligned} M_{1,d}^{k,e,h} &= e_{p,k}\left(E_{\mathcal{F}_d}(k/p)\, q^{-(1/2-1/p)k} + O\left(q^{-(1/2-1/2p)k}\right)\right) + O\left(q^{3k/2-d/2}\right) \\ &= O\left(q^{-(1/2-1/p)k} + q^{3k/2-d/2}\right), \end{aligned}$$

where

$$e_{p,k} = \begin{cases} 0, & p \nmid k, \\ 1, & p \mid k. \end{cases}$$

Proof. By Lemma 8.5, we have that

$$\begin{aligned} M_{1,d}^{k,e,h} &= q^{-k/2} \sum_{\substack{u,pu \mid k \\ \alpha \in \mathbb{F}_{q^k}, \deg(\alpha)=u}} E_{\mathcal{F}_d}(u) + q^{-k/2} \sum_{\alpha \in \mathbb{F}_{q^k}} O(q^{\deg(\alpha)-d/2}) \\ &= \frac{e_{p,k}}{q^{k/2}} \sum_{m,pm \mid k} E_{\mathcal{F}_d}(m)\pi(m)m + O\left(q^{3k/2-d/2}\right). \end{aligned}$$

Finally, if $p \mid k$, the estimates from Remark 8.2 yield

$$\sum_{m,pm \mid k} E_{\mathcal{F}_d}(m)\pi(m)m = E_{\mathcal{F}_d}(k/p)\, q^{k/p} + O\left(q^{k/2p}\right).$$ □

Notice that changing h allows us to vary the character from ψ to ψ^h. This will be useful later.

Theorem 8.7. *Let h be an integer such that $p \nmid h$ and let $\mathcal{F}_d$ be any of the families under consideration. Then for any K with* $\max\{1, 1/|\mathcal{I}|\} < K < d/3$,

$$\left\langle S^{\pm}(K, f, \psi^h)\right\rangle_{\mathcal{F}_d} = O(1).$$

Proof. We have that

$$\begin{aligned}\left\langle S^{\pm}(K, f, \psi^h)\right\rangle_{\mathcal{F}_d} &= \sum_{k=1}^{K} \frac{\widehat{I}_K^{\pm}(k)\left\langle S_k(f, \psi^h)\right\rangle_{\mathcal{F}_d} + \widehat{I}_K^{\pm}(-k)\left\langle S_k(f, \bar{\psi}^h)\right\rangle_{\mathcal{F}_d}}{q^{k/2}} \\ &= \sum_{k=1}^{K} \widehat{I}_K^{\pm}(k) M_{1,d}^{k,1,h} + \widehat{I}_K^{\pm}(-k) M_{1,d}^{k,-1,h} \\ &= \sum_{k=1}^{K} \widehat{I}_K^{\pm}(k) O\left(q^{-(1/2-1/p)k} + q^{3k/2-d/2}\right),\end{aligned}$$

and the result follows from Proposition 6.2. □

Theorem 8.8. *Let $\mathcal{F}_d$ be any of the families under consideration. Then,*

$$\begin{aligned}\langle N_{\mathcal{I}}(f, \psi)\rangle_{\mathcal{F}_d} = \frac{1}{|\mathcal{F}_d|}\sum_{f\in\mathcal{F}_d} N_{\mathcal{I}}(f, \psi) &= (\Delta - 1)|\mathcal{I}| + O(1), \\ \langle N_{\mathcal{I}}(C_f)\rangle_{\mathcal{F}_d} = \frac{1}{|\mathcal{F}_d|}\sum_{f\in\mathcal{F}_d} N_{\mathcal{I}}(C_f) &= 2\mathfrak{g}|\mathcal{I}| + O(1).\end{aligned}$$

Proof. This follows from Theorem 8.7 and equations (7.3) and (7.4) using $K = \varepsilon d$ for any $0 < \varepsilon < 1/3$. □

8.2. Second moment.

Lemma 8.9. *Let h_1, h_2 be integers such that $p \nmid h_1h_2$, $e_1, e_2 \in \{-1, 1\}$ and $k_1, k_2 > 0$. Let $\alpha_1 \in \mathbb{F}_{q^{k_1}}$, $\alpha_2 \in \mathbb{F}_{q^{k_2}}$ of degrees u_1, u_2 respectively over $\mathbb{F}_q$. For any of the families under consideration, we have*

$$\begin{aligned}&\langle \psi(e_1h_1 \operatorname{tr}_{k_1} f(\alpha_1) + e_2h_2 \operatorname{tr}_{k_2} f(\alpha_2))\rangle_{\mathcal{F}_{d},(\alpha_1,\alpha_2)} \\ &\quad= \begin{cases} E_{\mathcal{F}_d}(u_1) + O\left(q^{u_1-d/2}\right), & \alpha_1 \sim \alpha_2,\ p \mid \dfrac{e_1h_1k_1 + e_2h_2k_2}{u_1}, \\ O\left(1 + q^{u_1+u_2-d/2}\right), & \alpha_1 \not\sim \alpha_2,\ p \mid \left(\frac{k_1}{u_1}, \frac{k_2}{u_2}\right), \\ O\left(q^{u_1+u_2-d/2}\right), & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. Reversing the order of summation, we write

$$\begin{aligned}&\langle \psi(e_1h_1 \operatorname{tr}_{k_1} f(\alpha_1) + e_2h_2 \operatorname{tr}_{k_2} f(\alpha_2))\rangle_{\mathcal{F}_{d},(\alpha_1,\alpha_2)} \\ (8.5)\quad &= \sum_{\beta_1\in\mathbb{F}_{q^{u_1}},\beta_2\in\mathbb{F}_{q^{u_2}}} \psi(e_1h_1 \operatorname{tr}_{k_1} \beta_1 + e_2h_2 \operatorname{tr}_{k_2} \beta_2)\frac{|\mathcal{F}_d(\alpha_1, \alpha_2, \beta_1, \beta_2)|}{|\mathcal{F}_d|}.\end{aligned}$$

Assume that $\alpha_1 \not\sim \alpha_2$. By Lemma 8.3 we can write (8.5) as

$$\frac{E_{\mathcal{F}_d}(u_1, u_2)}{q^{u_1+u_2}} \sum_{\beta_1\in\mathbb{F}_{q^{u_1}},\beta_2\in\mathbb{F}_{q^{u_2}}} \psi\left(\frac{e_1h_1k_1}{u_1} \operatorname{tr}_{u_1} \beta_1 + \frac{e_2h_2k_2}{u_2} \operatorname{tr}_{u_2} \beta_2\right) + O\left(q^{u_1+u_2-d/2}\right).$$

Then Lemma 8.4 implies that the sum is zero unless $p \mid \frac{k_1}{u_1}$ and $p \mid \frac{k_2}{u_2}$.

Now assume that $\alpha_1 \sim \alpha_2$. Then $f(\alpha_1) \sim f(\alpha_2)$ and $\operatorname{tr}_{u_1} f(\alpha_1) = \operatorname{tr}_{u_1} f(\alpha_2)$. By Lemma 8.3 we can write (8.5) as

$$\frac{E_{\mathcal{F}_d}(u_1)}{q^{u_1}} \sum_{\beta_1 \in \mathbb{F}_{q^{u_1}}} \psi\left(\frac{e_1h_1k_1 + e_2h_2k_2}{u_1} \operatorname{tr}_{u_1} \beta_1\right) + O\left(q^{u_1-d/2}\right).$$

Then Lemma 8.4 implies that the sum is zero unless $p \mid \frac{e_1h_1k_1+e_2h_2k_2}{u_1}$. □

Lemma 8.10. *Let h_1, h_2 be integers such that $p \nmid h_1h_2$, $e_1, e_2 \in \{-1, 1\}$ and $k_1, k_2 > 0$, $k_1 \geq k_2$. Let $\mathcal{F}_d$ be any of the families under consideration. Then,*

$$\sum_{\substack{m \mid (k_1,k_2) \\ mp \nmid k_1,k_2 \\ mp \mid (e_1h_1k_1+e_2h_2k_2)}} E_{\mathcal{F}_d}(m)\pi(m)m^2$$
$$= \begin{cases} E_{\mathcal{F}_d}(k_1)k_1q^{k_1} + O\left(k_1q^{k_1/2}\right), & k_1 = k_2, p \mid (e_1h_1 + e_2h_2), \\ 0, & k_1 = k_2, p \nmid (e_1h_1 + e_2h_2), \\ O\left(k_1q^{k_1/2}\right), & k_1 = 2k_2, \\ O\left(k_1q^{k_1/3}\right), & k_1 \neq k_2, 2k_2. \end{cases}$$

Proof. For the first case when $k_1 = k_2$, the conditions on the summation indices become $m \mid k_1$, $mp \nmid k_1$, and $mp \mid (e_1h_1 + e_2h_2)k_1$, a contradiction unless $p \mid (e_1h_1 + e_2h_2)$. In this case, one gets

$$\sum_{\substack{m \mid k_1 \\ mp \nmid k_1}} E_{\mathcal{F}_d}(m)\pi(m)m^2 = E_{\mathcal{F}_d}(k_1)k_1q^{k_1} + O\left(k_1q^{k_1/2}\right),$$

where we have used the estimates for $\pi(m)$ and $E_{\mathcal{F}_d}(m)$ discussed in Remark 8.2.

On the other hand, when $k_1 = 2k_2$, one gets

$$\sum_{\substack{m \mid k_2 \\ mp \nmid k_2 \\ mp \mid (2e_1h_1+e_2h_2)k_2}} E_{\mathcal{F}_d}(m)\pi(m)m^2 = O\left(k_1q^{k_1/2}\right).$$

Finally, if $k_1 > k_2$ but $k_1 \neq 2k_2$, we have $(k_1, k_2) \leq k_1/3$ and

$$\sum_{\substack{m \mid (k_1,k_2) \\ mp \nmid k_1,k_2 \\ mp \mid (e_1h_1k_1+e_2h_2k_2)}} E_{\mathcal{F}_d}(m)\pi(m)m^2 = O\left(k_1q^{k_1/3}\right).$$

This completes the proof. □

For positive integers k_1, k_2, h_1, h_2 with $p \nmid h_1h_2$ and $e_1, e_2 \in \{-1, 1\}$, let

$$M_{2,d}^{(k_1,k_2),(e_1,e_2),(h_1,h_2)}$$
$$:= \left\langle q^{-(k_1+k_2)/2} \sum_{\substack{\alpha_1 \in \mathbb{F}_{q^{k_1}}, \alpha_2 \in \mathbb{F}_{q^{k_2}} \\ f(\alpha_1) \neq \infty, f(\alpha_2) \neq \infty}} \psi(e_1h_1 \operatorname{tr}_{k_1} f(\alpha_1) + e_2h_2 \operatorname{tr}_{k_2} f(\alpha_2)) \right\rangle_{\mathcal{F}_d}$$
$$= q^{-(k_1+k_2)/2} \sum_{\substack{\alpha_1 \in \mathbb{F}_{q^{k_1}} \\ \alpha_2 \in \mathbb{F}_{q^{k_2}}}} \langle \psi(e_1h_1 \operatorname{tr}_{k_1} f(\alpha_1) + e_2h_2 \operatorname{tr}_{k_2} f(\alpha_2)) \rangle_{\mathcal{F}_{d,(\alpha_1,\alpha_2)}}.$$

Using Lemma 8.10, we can prove the following analogue of Theorem 8 in [Ent12].

Theorem 8.11. *Let $0 < h_1, h_2 \leq (p-1)/2$, $e_1, e_2 \in \{-1, 1\}$, $k_1 \geq k_2 > 0$, and let $\mathcal{F}_d$ be any of the families under consideration. Then*

$$\begin{aligned}
&M_{2,d}^{(k_1,k_2),(e_1,e_2),(h_1,h_2)}\\
&\quad = \begin{cases} \delta_{k_1,k_2}\left(E_{\mathcal{F}_d}(k_1)k_1 + O\left(k_1 q^{-k_1/2} + k_1 q^{(k_1-d/2)}\right)\right), & e_1 = -e_2,\ h_1 = h_2,\\ 0, & \textit{otherwise,} \end{cases}\\
&\qquad + \delta_{k_1,2k_2} O\left(k_1 q^{-k_2/2} + k_1 q^{k_2/2-d/2}\right)\\
&\qquad + O\left(k_1 q^{-k_2/2-k_1/6} + k_1 q^{k_1/6-k_2/2-d/2}\right)\\
&\qquad + O\left(q^{(1/p-1/2)(k_1+k_2)} + q^{3(k_1+k_2)/2-d/2}\right)
\end{aligned}$$

where

$$\delta_{k_1,k_2} = \begin{cases} 1, & k_1 = k_2,\\ 0, & k_1 \neq k_2. \end{cases}$$

Proof. From Lemma 8.9, we have

$$\begin{aligned}
&M_{2,d}^{(k_1,k_2),(e_1,e_2),(h_1,h_2)}\\
&\quad = \frac{e_{p,e_1h_1k_1+e_2h_2k_2}}{q^{(k_1+k_2)/2}} \sum_{\substack{m\mid(k_1,k_2)\\ mp\nmid k_1,k_2\\ mp\mid(e_1h_1k_1+e_2h_2k_2)}} \pi(m)m^2\left(E_{\mathcal{F}_d}(m) + O(q^{m-d/2})\right)\\
&\qquad + O\left(\frac{e_{p,k_1}e_{p,k_2}}{q^{(k_1+k_2)/2}} \sum_{\substack{\deg\alpha_1=u_1,\deg\alpha_2=u_2\\ p\mid\frac{k_1}{u_1},p\mid\frac{k_2}{u_2}}} \left(1 + q^{u_1+u_2-d/2}\right)\right)\\
&\qquad + O\left(\frac{1}{q^{(k_1+k_2)/2}} \sum_{\substack{\deg\alpha_1=u_1,\deg\alpha_2=u_2\\ u_1\mid k_1,u_2\mid k_2}} q^{u_1+u_2-d/2}\right).
\end{aligned}$$

It is easy to see that the last two terms are

$$O\left(q^{(1/p-1/2)(k_1+k_2)} + q^{3(k_1+k_2)/2-d/2}\right).$$

For the first term, we use Lemma 8.10. As a final observation, the condition $p \mid e_1h_1 + e_2h_2$ translates into $h_1 = h_2$ and $e_1 = -e_2$ because of the restriction on the possible values for h_1, h_2. This concludes the proof of the theorem. □

Using Lemma 8.10, we can prove the following result which will also be used in the general moments.

Proposition 8.12. *Let h_1, h_2 be integers such that $p \nmid h_1 h_2$, $e_1, e_2 \in \{-1, 1\}$ and $k_1, k_2 > 0$. Let $\mathcal{F}_d$ be any of the families under consideration. Then,*

$$\sum_{k_1,k_2=1}^{K} \widehat{I}_K^{\pm}(e_1 k_1)\widehat{I}_K^{\pm}(e_2 k_2) q^{-(k_1+k_2)/2} \sum_{\substack{m|(k_1,k_2)\\ mp\nmid k_1,k_2\\ mp|(e_1h_1k_1+e_2h_2k_2)}} E_{\mathcal{F}_d}(m)\pi(m)m^2$$

$$= \begin{cases} \dfrac{1}{2\pi^2}\log(K|\mathcal{I}|) + O(1), & p \mid (e_1h_1 + e_2h_2), \\ \\ O(1), & \textit{otherwise.} \end{cases}$$

Proof. Using Lemma 8.10, we see that the sum is

$$e_{p,e_1h_1+e_2h_2} \sum_{k_1=1}^{K} \widehat{I}_K^{\pm}(k_1)\widehat{I}_K^{\pm}(-k_1)\left(E_{\mathcal{F}_d}(k_1)k_1 + O\left(k_1 q^{-k_1/2}\right)\right)$$

$$+O\left(\sum_{k_1=1}^{K} k_1 q^{-k_1/4} + \sum_{k_1,k_2=1}^{K} k_1 q^{-k_1/6} q^{-k_2/2}\right)$$

$$= e_{p,e_1h_1+e_2h_2} \sum_{k_1=1}^{K} \widehat{I}_K^{\pm}(k_1)\widehat{I}_K^{\pm}(-k_1)E_{\mathcal{F}_d}(k_1)k_1 + O(1).$$

Now the estimates from Remark 8.2 and Proposition 6.1 yield

$$\begin{aligned}\sum_{k_1=1}^{K} \widehat{I}_K^{\pm}(k_1)\widehat{I}_K^{\pm}(-k_1)E_{\mathcal{F}_d}(k_1)k_1 &= \sum_{k_1=1}^{K} \widehat{I}_K^{\pm}(k_1)\widehat{I}_K^{\pm}(-k_1)k_1 + O\left(\sum_{k_1=1}^{K} k_1^2 q^{-k_1}\right) \\ &= \frac{1}{2\pi^2}\log(K|\mathcal{I}|) + O(1),\end{aligned}$$

which finishes the proof of the statement. □

Finally, we are able to compute the covariances.

Theorem 8.13. *Let $0 < h_1, h_2 \leq (p-1)/2$, and let $\mathcal{F}_d$ be any of the families under consideration. Then for any K with $1/|\mathcal{I}| < K < d/6$,*

$$\left\langle S^{\pm}(K,f,\psi^{h_1})S^{\pm}(K,f,\psi^{h_2})\right\rangle_{\mathcal{F}_d} = \left\langle S^{\pm}(K,f,\psi^{h_1})S^{\mp}(K,f,\psi^{h_2})\right\rangle_{\mathcal{F}_d}$$

$$= \begin{cases} \dfrac{1}{\pi^2}\log(K|\mathcal{I}|) + O(1), & h_1 = h_2, \\ \\ O(1), & h_1 \neq h_2. \end{cases}$$

Proof. By definition,

$$\begin{aligned}\left\langle S^{\pm}(K,f,\psi^{h_1})S^{\pm}(K,f,\psi^{h_2})\right\rangle_{\mathcal{F}_d} = \sum_{k_1,k_2=1}^{K} &\widehat{I}_K^{\pm}(k_1)\widehat{I}_K^{\pm}(k_2)M_{2,d}^{(k_1,k_2),(1,1),(h_1,h_2)} \\ &+ \widehat{I}_K^{\pm}(k_1)\widehat{I}_K^{\pm}(-k_2)M_{2,d}^{(k_1,k_2),(1,-1),(h_1,h_2)} \\ &+ \widehat{I}_K^{\pm}(-k_1)\widehat{I}_K^{\pm}(k_2)M_{2,d}^{(k_1,k_2),(-1,1),(h_1,h_2)} \\ &+ \widehat{I}_K^{\pm}(-k_1)\widehat{I}_K^{\pm}(-k_2)M_{2,d}^{(k_1,k_2),(-1,-1),(h_1,h_2)}.\end{aligned}$$

Using Theorem 8.11 to replace the terms above, we first remark that the contribution of the last two error terms from Theorem 8.11 to the sum is

$$\ll \sum_{k_1,k_2=1}^{K} k_1 q^{-k_2/2-k_1/6} + k_1 q^{k_1/6-k_2/2-d/2} + q^{(1/p-1/2)(k_1+k_2)} + q^{3(k_1+k_2)/2-d/2} \ll 1$$

provided that $d > 6K$.

Similarly, the contribution of the error terms for $k_1 = k_2$ and $k_1 = 2k_1$ is bounded by

$$\ll \sum_{k=1}^{K} kq^{-k/2} + kq^{k-d/2} \ll 1$$

provided that $d > 2K$. Finally, the main term comes from summing $E_{\mathcal{F}_d}(k_1)k_1$ when $k_1 = k_2$, and this occurs only when $h_1 = h_2$ and $\{e_1, e_2\} = \{1, -1\}$. Proceeding as in the proof of Proposition 8.12, we then get that

$$\begin{aligned} \left\langle S^{\pm}(K, f, \psi^{h_1})^2 \right\rangle_{\mathcal{F}_d} &= 2 \sum_{k_1=1}^{K} \widehat{I}_K^{\pm}(k_1) \widehat{I}_K^{\pm}(-k_1) k_1 E_{\mathcal{F}_d}(k_1) + O(1) \\ &= \frac{1}{\pi^2} \log(K|\mathcal{I}|) + O(1). \end{aligned}$$

The proof for $\left\langle S^{\pm}(K, f, \psi^{h_1}) S^{\mp}(K, f, \psi^{h_2}) \right\rangle_{\mathcal{F}_d}$ follows exactly along the same lines. □

Corollary 8.14. *For any K with $1/|\mathcal{I}| < K < d/6$,*

$$\left\langle S^{\pm}(K, C_f)^2 \right\rangle_{\mathcal{F}_d} = \left\langle S^{+}(K, C_f) S^{-}(K, C_f) \right\rangle_{\mathcal{F}_d} = \frac{2(p-1)}{\pi^2} \log(K|\mathcal{I}|) + O(1).$$

Proof. First we note that

$$\left\langle S^{\pm}(K, C_f)^2 \right\rangle_{\mathcal{F}_d} = \sum_{h_1,h_2=1}^{p-1} \left\langle S^{\pm}(K, f, \psi^{h_1}) S^{\pm}(K, f, \psi^{h_2}) \right\rangle_{\mathcal{F}_d}.$$

Notice that by Theorem 8.13, the mixed average contributes $\frac{1}{\pi^2} \log(K|\mathcal{I}|) + O(1)$ for each term where $h_1 = h_2$ or $h_1 = p - h_2$. The proof for $\langle S^{+}(K, C_f) S^{-}(K, C_f) \rangle_{\mathcal{F}_d}$ is identical. □

8.3. **General moments.** Let $n, k_1, \dots, k_n$ be positive integers, let $e_1, \dots, e_n$ take values ± 1 and let $h_1, \dots, h_n$ be integers such that $p \nmid h_i$, $1 \le i \le n$. Let $\mathbf{k} = (k_1, \dots, k_n)$, $\mathbf{e} = (e_1, \dots, e_n)$, and $\mathbf{h} = (h_1, \dots, h_n)$. Let $\alpha_i \in \mathbb{F}_{q^{k_i}}$, $1 \le i \le n$, and let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Let $\mathcal{F}_d$ be any of the families under consideration. Then, we define

$$\begin{aligned} m_n^{\mathbf{k},\mathbf{e},\mathbf{h}}(\boldsymbol{\alpha}) &= \langle \psi(e_1 h_1 \operatorname{tr}_{k_1} f(\alpha_1) + \dots + e_n h_n \operatorname{tr}_{k_n} f(\alpha_n)) \rangle_{\mathcal{F}_{d,\boldsymbol{\alpha}}} \\ &= \frac{1}{|\mathcal{F}_d|} \sum_{\substack{f \in \mathcal{F}_d \\ f(\alpha_i) \neq \infty, 1 \le i \le n}} \psi(e_1 h_1 \operatorname{tr}_{k_1} f(\alpha_1) + \dots + e_n h_n \operatorname{tr}_{k_n} f(\alpha_n)) \end{aligned}$$

and

$$M_n^{\mathbf{k},\mathbf{e},\mathbf{h}} = \sum_{\substack{\alpha_i \in \mathbb{F}_{q^{k_i}} \\ i=1,\dots,n}} q^{-(k_1+\dots+k_n)/2} m_n^{\mathbf{k},\mathbf{e},\mathbf{h}}(\boldsymbol{\alpha}).$$

Lemma 8.15. *Let $\mathcal{F}_d$ be any of the families under consideration. Let $C_1, \dots, C_s$ be the distinct conjugacy classes of the $\alpha_1, \dots, \alpha_n$. Let u_i be the degree of the elements of C_i. For $i = 1, \dots, s$, let*

$$\eta_i = \frac{1}{u_i} \sum_{\alpha_j \in C_i} e_j h_j k_j.$$

Then

$$m_n^{\mathbf{k},\mathbf{e},\mathbf{h}}(\boldsymbol{\alpha}) = \begin{cases} E_{\mathcal{F}_d}(u_1, \dots, u_s) + O\left(q^{u_1+\cdots+u_s-d/2}\right), & \text{if } p \mid \eta_i \text{ for } 1 \le i \le s, \\ \\ O\left(q^{u_1+\cdots+u_s-d/2}\right), & \text{otherwise.} \end{cases}$$

Proof. Renumbering, suppose that $\alpha_i \in C_i$ for $1 \le i \le s$. Since $\operatorname{tr}_{k_i} f(\alpha_i) = \frac{k_i}{u_i} \operatorname{tr}_{u_i} f(\alpha_i)$ for $i = 1, \dots, s$, by the definition of η_i, we have that

$$\begin{aligned} m_n^{\mathbf{k},\mathbf{e},\mathbf{h}}(\boldsymbol{\alpha}) &= \frac{1}{|\mathcal{F}_d|} \sum_{\substack{f \in \mathcal{F}_d \\ f(\alpha_i) \neq \infty, 1 \le i \le n}} \psi\left(e_1 h_1 \operatorname{tr}_{k_1} f(\alpha_1) + \cdots + e_n h_n \operatorname{tr}_{k_n} f(\alpha_n)\right) \\ &= \frac{1}{|\mathcal{F}_d|} \sum_{\substack{f \in \mathcal{F}_d \\ f(\alpha_i) \neq \infty, 1 \le i \le n}} \psi\left(\eta_1 \operatorname{tr}_{u_1} f(\alpha_1) + \cdots + \eta_s \operatorname{tr}_{u_s} f(\alpha_s)\right) \\ &= \sum_{\beta_i \in \mathbb{F}_{q^{u_i}},\, 1 \le i \le s} \psi\left(\eta_1 \operatorname{tr}_{u_1} \beta_1 + \cdots + \eta_s \operatorname{tr}_{u_s} \beta_s\right) \frac{|\mathcal{F}_d(\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_s)|}{|\mathcal{F}_d|} \\ &= \frac{E_{\mathcal{F}_d}(u_1, \dots, u_s)}{q^{u_1+\cdots+u_s}} \sum_{\beta_i \in \mathbb{F}_{q^{u_i}},\, 1 \le i \le s} \psi\left(\eta_1 \operatorname{tr}_{u_1} \beta_1 + \cdots + \eta_s \operatorname{tr}_{u_s} \beta_s\right) \\ &\quad + O\left(q^{u_1+\cdots+u_s-d/2}\right) \end{aligned}$$

by Lemma 8.3. The result now follows from Lemma 8.4. □

Lemma 8.16. *The quantity $M_n^{\mathbf{k},\mathbf{e},\mathbf{h}}$ is bounded by a sum of terms*

$$q^{-(k_1+\cdots+k_n)/2} T(k_1, \dots, k_n),$$

where each $T(k_1, \dots, k_n)$ is a product of elementary terms of the type

$$\sum_{\substack{m \mid (j_1, \dots, j_r) \\ mp \mid \sum_{i=1}^r e_i h_i j_i}} \pi(m) m^r$$

such that the indices $j_1, \dots, j_r$ of the elementary terms appearing in each $T(k_1, \dots, k_n)$ are in bijection with $k_1, \dots, k_n$.

For $n = 2\ell$ even, let $N_n^{\mathbf{k},\mathbf{e},\mathbf{h}}$ be the sum of all possible terms

$$q^{-(k_1+\cdots+k_n)/2} T(k_1, \dots, k_n),$$

where the $T(k_1, \dots, k_n)$ are made exclusively of the following nested sums:

(8.6)
$$\sum_{\substack{m_1 \mid (j_1, j_{\ell+1}) \\ m_1 p \mid e_1 h_1 j_{\ell+1} + e_{\ell+1} h_{\ell+1} j_{\ell+1}}} \pi(m_1) m_1^2 \cdots \sum_{\substack{m_\ell \mid (j_\ell, j_{2\ell}) \\ m_\ell p \mid e_\ell h_\ell j_{2\ell} + e_{2\ell} h_{2\ell} j_{2\ell}}} \pi(m_\ell) m_\ell^2 E_{\mathcal{F}_d}(m_1, \dots, m_\ell).$$

If $n = 2\ell + 1$ is odd, let $N_n^{\mathbf{k},\mathbf{e},\mathbf{h}}$ be the sum of all possible terms $q^{-(k_1+\cdots+k_n)/2}$ $T(k_1, \dots, k_n)$, where $T(k_1, \dots, k_n)$ are made exclusively of the following nested sums:

$$\sum_{\substack{m_1|(j_1,j_{\ell+1}) \\ m_1p|e_1h_1j_{\ell+1}+e_{\ell+1}h_{\ell+1}j_{\ell+1}}} \pi(m_1)m_1^2 \cdots \sum_{\substack{m_\ell|(j_\ell,j_{2\ell}) \\ m_\ell p|e_\ell h_\ell j_{2\ell}+e_{2\ell}h_{2\ell}j_{2\ell}}} \pi(m_\ell)m_\ell^2$$
$$\times \sum_{\substack{m_{\ell+1}|j_{2\ell+1} \\ m_{\ell+1}p|e_{2\ell+1}h_{2\ell+1}j_{2\ell+1}}} \pi(m_{\ell+1})m_{\ell+1}E_{\mathcal{F}_d}(m_1, \dots, m_\ell, m_{\ell+1}).$$

Let $L_n^{\mathbf{k},\mathbf{e},\mathbf{h}}$ be the sum of all the other terms $q^{-(k_1+\cdots+k_n)/2}T(k_1, \dots, k_n)$ as defined above. Then,

$$M_n^{\mathbf{k},\mathbf{e},\mathbf{h}} = N_{n,d}^{\mathbf{k},\mathbf{e},\mathbf{h}} + O\left(L_n^{\mathbf{k},\mathbf{e},\mathbf{h}}\right) + O\left(q^{3(k_1+\cdots+k_n)/2-d/2}\right).$$

Proof. Using Lemma 8.15, we first write

$$M_n^{\mathbf{k},\mathbf{e},\mathbf{h}} = q^{-(k_1+\cdots+k_n)/2} \sum_{\substack{\alpha_i \in \mathbb{F}_{q^{k_i}},\ i=1,\dots,n \\ (\alpha_1,\dots,\alpha_n)\in\mathcal{A}}} E_{\mathcal{F}_d}(u_1, \dots, u_s) + O\left(q^{3(k_1+\cdots+k_n)/2-d/2}\right),$$

where the set $\mathcal{A}$ of admissible $(\alpha_1, \dots, \alpha_n)$ are those where $p \mid \eta_i$, $i = 1, \dots, s$. To count the number of admissible $(\alpha_1, \dots, \alpha_n)$, we first fix a partition of $\{1, \dots, n\}$ in s classes $C_1, \dots, C_s$. Let $k(C_w)$ be the gcd of the k_i such that $i \in C_w$ and let $\delta(C_w) = \sum_{i\in C_w} e_ih_ik_i$. Then, for any such partition, the number of $(\alpha_1, \dots, \alpha_n) \in \mathbb{F}_{q^{k_1}} \times \cdots \times \mathbb{F}_{q^{k_n}}$ such that α_i and α_j are conjugate when i, j are in the same class C_w and which are counted in $\mathcal{A}$ is bounded by

$$\prod_{i=1}^{s} \sum_{\substack{m|k(C_i) \\ mp|\delta(C_i)}} \pi(m)m^{|C_i|}, \tag{8.7}$$

where we have used the fact that the number of $(\alpha_1, \dots, \alpha_t) \in \mathbb{F}_{q^{k_1}} \times \cdots \times \mathbb{F}_{q^{k_t}}$ which are conjugate over $\mathbb{F}_q$ is given by

$$\sum_{m|(k_1,\dots,k_t)} \pi(m)m^t.$$

Since $E_{\mathcal{F}}(u_1, \dots, u_s) \ll 1$ by Remark 8.2, we get the first result of the statement by summing (8.7) over all partitions of $\{1, \dots, n\}$ in s classes $C_1, \dots C_s$.

Suppose that $n = 2\ell$ is even. Then, using inclusion-exclusion, the number of $(\alpha_1, \dots, \alpha_n) \in \mathbb{F}_{q^{k_1}} \times \cdots \times \mathbb{F}_{q^{k_n}}$ such that α_i and α_j are conjugate, if and only if $i \equiv j (\mathrm{mod} \ell)$, can be written as

$$\left(\sum_{\substack{m_1|(k_1,k_{\ell+1}) \\ m_1p|e_1h_1k_1+e_{\ell+1}h_{\ell+1}k_{\ell+1}}} \pi(m_1)m_1^2 \cdots \sum_{\substack{m_\ell|(k_\ell,k_{2\ell}) \\ m_\ell p|e_\ell h_\ell k_\ell+e_{2\ell}h_{2\ell}k_{e\ell}}} \pi(m_\ell)m_\ell^2 E_{\mathcal{F}_d}(m_1, \dots, m_\ell) \right.$$
$$\left. + S(k_1, \dots, k_n), \right.$$

where $S(k_1, \dots, k_n)$ is a sum of terms in $L_n^{\mathbf{k},\mathbf{e},\mathbf{h}}$. (We have to do inclusion-exclusion to remove the cases where conjugate values of α belong to two different classes C_w.)

The case of $n = 2\ell + 1$ follows similarly, taking into account that one has to multiply by the factor $q^{-k_n/2} \sum_{\substack{m|k_n \\ mp|ek_n}} \pi(m)m$. □

Theorem 8.17. *Let $\mathcal{F}_d$ be any of the families under consideration. For any K with $1/|\mathcal{I}| < K < d/n$,*

$$\left\langle S^{\pm}(K,f,\psi)^n \right\rangle_{\mathcal{F}_d} = \begin{cases} \frac{(2\ell)!}{\ell!(2\pi^2)^\ell} \log^\ell(K|\mathcal{I}|) \left(1 + O\left(\log^{-1}(K|\mathcal{I}|)\right)\right), & n = 2\ell, \\ \\ O\left(\log^\ell(K|\mathcal{I}|)\right), & n = 2\ell+1. \end{cases}$$

More generally, let $0 < h_1, \dots, h_n \leq (p-1)/2$. Then for any K with $1/|\mathcal{I}| < K < d/n$,

$$\begin{aligned} &\left\langle S^{\pm}(K,f,\psi^{h_1}) \dots S^{\pm}(K,f,\psi^{h_n}) \right\rangle_{\mathcal{F}_d} \\ &= \begin{cases} \frac{\Theta(h_1,\dots,h_n)}{(2\pi^2)^\ell} \log^\ell(K|\mathcal{I}|) \left(1 + O\left(\log^{-1}(K|\mathcal{I}|)\right)\right), & n = 2\ell, \\ \\ O\left(\log^\ell(K|\mathcal{I}|)\right), & n = 2\ell+1. \end{cases} \end{aligned}$$

The constant $\Theta(h_1, \dots, h_n)$ is given by

$$\begin{aligned} \#\{(e_1,\dots,e_n) \in \{-1,1\}, \sigma \in \mathbb{S}_n \ : \ & e_1 h_{\sigma(1)} + e_2 h_{\sigma(2)} \\ & \equiv \dots \equiv e_{2\ell-1} h_{\sigma(2\ell-1)} + e_{2\ell} h_{\sigma(2\ell)} \equiv 0 \pmod p\}, \end{aligned}$$

where $\mathbb{S}_n$ denotes the permutations of the set of n elements.

Proof. We have that

$$\left\langle S^{\pm}(K,f,\psi^{h_1}) \dots S^{\pm}(K,f,\psi^{h_n}) \right\rangle_{\mathcal{F}_d} = \sum_{\substack{k_1,\dots,k_n=1 \\ e_1,\dots,e_n=\pm1}}^{K} I_K^{\pm}(e_1k_1) \dots I_K^{\pm}(e_nk_n) M_n^{\mathbf{k},\mathbf{e},\mathbf{h}},$$

and we use Lemma 8.16 to replace $M_n^{\mathbf{k},\mathbf{e},\mathbf{h}}$ in the sum. The error term satisfies

$$\sum_{\substack{k_1,\dots,k_n=1 \\ e_1,\dots,e_n=\pm1}}^{K} I_K^{\pm}(e_1k_1) \dots I_K^{\pm}(e_nk_n) O\left(q^{3(k_1+\dots+k_n)/2-d/2}\right) \ll \left(\sum_{k=1}^{K} q^{3k/2-d/2n}\right)^n \ll 1$$

when $d > 3nK$.

For the main term, we have to consider the sum of the terms $T(k_1, \dots, k_n)$ from Lemma 8.16. For each fixed $T(k_1, \dots, k_n)$, we write the sum over $k_1, \dots, k_n$ as s nested sums $\Sigma_1 \dots \Sigma_s E_{\mathcal{F}_d}(m_1, \dots, m_s)$ where Σ_w is a sum over the k_i such that $i \in C_w$, and $|E_{\mathcal{F}_d}(m_1, \dots, m_s)| \ll 1$. If $|C_w| = 1$, then we have a sum

$$\sum_{k=1}^{K} \widehat{I}_K^{\pm}(k) q^{-k/2} \sum_{\substack{m|k \\ mp|e_k}} \pi(m)m \ll 1, \tag{8.8}$$

because of Theorem 8.7. For $r = |C_w| \geq 2$, we have a sum of the type

$$\sum_{k_1,\dots,k_r=1}^{K} \widehat{I}_K^{\pm}(e_1k_1) \dots \widehat{I}_K^{\pm}(e_rk_r) q^{-(k_1+\dots+k_r)/2} \sum_{\substack{m|(k_1,\dots,k_r) \\ mp|\sum_{i=1}^r e_ih_ik_i}} \pi(m)m^r.$$

When $r = |C_w| > 2$, we will show in Lemma 8.18 that the contribution from the terms of the sum over $k_1, \dots, k_r$ is bounded. Assuming this result, we have by Lemma 8.16 that the leading term in $S^{\pm}(K, f, \psi)^n$ will come from the contributions $N_{n,d}^{\mathbf{k},\mathbf{e},\mathbf{h}}$.

If $n = 2\ell$, the leading terms are of the form

$$\sum_{k_1,\dots,k_r=1}^{K} \widehat{I}_K^{\pm}(e_1k_1)\dots\widehat{I}_K^{\pm}(e_rk_r)q^{-(k_1+\dots+k_r)/2}$$
$$\times \sum_{\substack{m_1|(k_1,k_{\ell+1})\\ m_1p|e_1h_1k_{\ell+1}+e_{\ell+1}h_{\ell+1}k_{\ell+1}}} \pi(m_1)m_1^2 \dots \sum_{\substack{m_\ell|(k_\ell,k_{2\ell})\\ m_\ell p|e_\ell h_\ell k_{2\ell}+e_{2\ell}h_{2\ell}k_{2\ell}}} \pi(m_\ell)m_\ell^2 E_{\mathcal{F}_d}(m_1,\dots,m_\ell).$$

By Definition 8.1 and Remark 8.2 combined with Proposition 8.12, for $\mathcal{F}_d = \mathcal{F}_d^{\mathrm{ord}}, \mathcal{F}_d^{\mathrm{full}}$ the above sum gives

$$\left(\frac{1}{2\pi^2}\log\left(K|\mathcal{I}|\right)\right)^{\ell}.$$

For $\mathcal{F}_d^v$, we have that $E_{\mathcal{F}_d}(m_1,\dots,m_\ell) = 1$ unless some of the m_j's equal some of the r_i's. Since the r_i's are fixed constants, this simply introduces an error term of the form $O\left(\log^{\ell-1}\left(K|\mathcal{I}|\right)\right)$ which does not change the final result.

If $n = 2\ell + 1$, the leading terms are of the form

$$O\left(\log^{\ell}\left(K|\mathcal{I}|\right)\right).$$

The final coefficient is obtained by counting the number of ways to choose the ℓ coefficients k_i's with positive sign ($e_i = 1$) and to pair them with those with negative sign ($e_j = -1$). □

Lemma 8.18. *Let $r > 2$. Then*

$$S := \sum_{k_1,\dots,k_r=1}^{K} \widehat{I}_K^{\pm}(k_1)\dots\widehat{I}_K^{\pm}(k_r)q^{-(k_1+\dots+k_r)/2} \sum_{\substack{m|(k_1,\dots,k_r)\\ mp\nmid(k_1,\dots,k_r)}} \pi(m)m^r = O(1).$$

Proof. Suppose that $k_1 \geq \dots \geq k_r$. We use repeatedly the estimates from Remark 8.2. If $k_1 = k_r$, we have

$$\sum_{\substack{m|(k_1,\dots,k_r)\\ mp\nmid(k_1,\dots,k_r)}} \pi(m)m^r = O\left(k_1^{r-1}q^{k_1}\right).$$

If $k_1 = 2k_r$ and all the other k_i are equal to k_1 or k_r, we have

$$\sum_{\substack{m|(k_1,\dots,k_r)\\ mp\nmid(k_1,\dots,k_r)}} \pi(m)m^r = O\left(k_1^{r-1}q^{k_1/2}\right).$$

In all the other cases, the estimate is

$$\sum_{\substack{m|(k_1,\dots,k_r)\\ mp\nmid(k_1,\dots,k_r)}} \pi(m)m^r = O\left(k_1^{r-1}q^{k_1/3}\right).$$

Putting things together, we get

$$\begin{aligned} S &\ll \sum_{k=1}^{K} \widehat{I}_K^{\pm}(k)^r k^{r-1} q^{-(r-2)k/2} + \sum_{\ell=1}^{r-1}\sum_{k=1}^{K} \widehat{I}_K^{\pm}(2k)^{\ell}\widehat{I}_K^{\pm}(k)^{r-\ell}k^{r-1}q^{(1-r/2-\ell/2)k} \\ &\quad + \sum_{k_1,\dots,k_r=1}^{K} \widehat{I}_K^{\pm}(k_1)\dots\widehat{I}_K^{\pm}(k_r)k_1^{r-1}q^{-k_1/6-(k_2+\dots+k_r)/2} \\ &\ll 1 \end{aligned}$$

by Proposition 6.2. □

Remark 8.19. We note that if $n = 2\ell$,

$$\sum_{h_1,\dots,h_n=1}^{(p-1)/2} \Theta(h_1,\dots,h_n) = \frac{(p-1)^{\ell}(2\ell)!}{2^{\ell}\ell!}. \tag{8.9}$$

There are $\frac{(2\ell)!}{\ell!2^{\ell}}$ ways of choosing unordered pairs of the form $\{e_i, e_j\}$. Inside each pair, exactly one of $\{e_i, e_j\}$ is positive and the other is negative, so there is a total of 2^{ℓ} choices for the signs. Finally, for each pair there are $(p-1)/2$ possible values for h_i which automatically determine the value of h_j.

Remark 8.20. By Theorem 8.17, the moments are given by sums of products of covariances. Thus, they are the same as the moments of a multivariate normal distribution. Moreover, the generating function of the moments converges due to (8.9). Therefore, our random variables are jointly normal. Since the variables are uncorrelated (cf. Theorem 8.13), it follows that our random variables (for $h = 1,\dots,\frac{p-1}{2}$) are independent.

Recall that

$$S^{\pm}(K, C_f) = \sum_{j=1}^{p-1} S^{\pm}(K, f, \psi^j).$$

Theorem 8.21. *Assume that $K = \mathfrak{g}/\log\log(\mathfrak{g}|\mathcal{I}|)$, $\mathfrak{g} \to \infty$ and either $|\mathcal{I}|$ is fixed or $|\mathcal{I}| \to 0$ while $\mathfrak{g}|\mathcal{I}| \to \infty$. Then*

$$\frac{S^{\pm}(K, C_f)}{\sqrt{\frac{2(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|)}}$$

has a standard Gaussian limiting distribution when $\mathfrak{g} \to \infty$.

Proof. First we compute the moments and then we normalize them.

With our choice of K we have

$$\frac{\log(K|\mathcal{I}|)}{\log(\mathfrak{g}|\mathcal{I}|)} = 1 - \frac{\log\log\log(\mathfrak{g}|\mathcal{I}|)}{\log(\mathfrak{g}|\mathcal{I}|)} \to 1 \text{ as } \mathfrak{g} \to \infty.$$

Because of this, $\log(K|\mathcal{I}|)$ can be replaced by $\log(\mathfrak{g}|\mathcal{I}|)$ in our formulas.

Recall that $S^{\pm}(K,f,\psi^h)=S^{\pm}(K,f,\psi^{p-h})$; then

$$\begin{aligned} S^{\pm}(K,C_f)^n &= \left(2\sum_{h=1}^{(p-1)/2} S^{\pm}(K,f,\psi^h)\right)^n \\ &= 2^n \sum_{h_1,\dots,h_n=1}^{(p-1)/2} S^{\pm}(K,f,\psi^{h_1})\dots S^{\pm}(K,f,\psi^{h_n}). \end{aligned}$$

Therefore, the moment is given by

$$\left\langle S^{\pm}(K,C_f)^n\right\rangle_{\mathcal{F}_d} \quad = \quad 2^n \sum_{h_1,\dots,h_n=1}^{(p-1)/2} \langle S^{\pm}(K,f,\psi^{h_1})\dots S^{\pm}(K,f,\psi^{h_n})\rangle_{\mathcal{F}_d}.$$

First assume that $n=2\ell$. By Theorem 8.17, this is asymptotic to

$$\frac{2^n}{(2\pi^2)^\ell}\log^\ell(\mathfrak{g}|\mathcal{I}|)\sum_{h_1,\dots,h_n=1}^{(p-1)/2}\Theta(h_1,\dots,h_n).$$

Finally we use equation (8.9) to conclude that when $n=2\ell$,

$$\left\langle S^{\pm}(K,C_f)^n\right\rangle_{\mathcal{F}_d} \sim \frac{2^n(p-1)^\ell(2\ell)!}{2^\ell\ell!(2\pi^2)^\ell}\log^\ell(\mathfrak{g}|\mathcal{I}|)=\frac{(2\ell)!}{\ell!\pi^{2\ell}}(p-1)^\ell\log^\ell(\mathfrak{g}|\mathcal{I}|).$$

In particular, the variance is asymptotic to $\frac{2(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|)$.

Now assume that n is odd, $n=2\ell+1$. Theorem 8.17 yields

$$\left\langle S^{\pm}(K,C_f)^n\right\rangle_{\mathcal{F}_d} \quad = \quad O\left(\log^\ell(\mathfrak{g}|\mathcal{I}|)\right).$$

Hence the normalized moment converges to

$$\lim_{\mathfrak{g}\to\infty}\frac{\left\langle S^{\pm}(K,C_f)^{2\ell}\right\rangle}{\left(\sqrt{\frac{2(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|)}\right)^{2\ell}}=\frac{(2\ell)!}{\ell!2^\ell},$$

for $n=2\ell$, and to zero for n odd. Hence, we have obtained the moments of the standard Gaussian distribution. □

9. The distribution of zeroes

We prove in this section that

$$\frac{N_{\mathcal{I}}(C_f)-2\mathfrak{g}|\mathcal{I}|}{\sqrt{(2(p-1)/\pi^2)\log(\mathfrak{g}|\mathcal{I}|)|}}$$

converges in mean square to

$$\frac{S^{\pm}(K,C_f)}{\sqrt{(2(p-1)/\pi^2)\log(\mathfrak{g}|\mathcal{I}|)}}.$$

Then, using Theorem 8.21, we get the result of Theorem 1.3 since convergence in mean square implies convergence in distribution.

Lemma 9.1. *Let $\mathcal{F}_d$ be any of the families under consideration. Assume that $K = \mathfrak{g}/\log\log(\mathfrak{g}|\mathcal{I}|)$, $\mathfrak{g} \to \infty$ and either $|\mathcal{I}|$ is fixed or $|\mathcal{I}| \to 0$ while $\mathfrak{g}|\mathcal{I}| \to \infty$. Then*

$$\left\langle \left| \frac{N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}| + S^{\pm}(K, C_f)}{\sqrt{(2(p-1)/\pi^2)\log(\mathfrak{g}|\mathcal{I}|)}} \right|^2 \right\rangle_{\mathcal{F}_d} \to 0.$$

Proof. From equation (7.4), using the Beurling–Selberg polynomials and the explicit formula (Lemma 7.1), we deduce that

$$\frac{-2\mathfrak{g}}{K+1} \le N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}| + S^-(K, C_f) \le S^-(K, C_f) - S^+(K, C_f) + \frac{2\mathfrak{g}}{K+1}$$

and

$$\frac{-2\mathfrak{g}}{K+1} \le -N_{\mathcal{I}}(C_f) + 2\mathfrak{g}|\mathcal{I}| - S^+(K, C_f) \le S^-(K, C_f) - S^+(K, C_f) + \frac{2\mathfrak{g}}{K+1}.$$

Using these two inequalities to bound the absolute value of the central term, we obtain

$$\begin{aligned}
&\left\langle \left(N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}| + S^{\pm}(K, C_f)\right)^2 \right\rangle_{\mathcal{F}_d} \\
&\quad \le \max\left\{ \left(\frac{2\mathfrak{g}}{K+1}\right)^2, \left\langle \left(S^-(K, C_f) - S^+(K, C_f) + \frac{2\mathfrak{g}}{K+1}\right)^2 \right\rangle_{\mathcal{F}_d} \right\} \\
&\quad \le \left(\frac{2\mathfrak{g}}{K+1}\right)^2 + \max\left\{0, \left\langle \left(S^-(K, C_f) - S^+(K, C_f)\right)^2 \right\rangle_{\mathcal{F}_d} \right. \\
&\qquad\qquad \left. + \frac{4\mathfrak{g}}{K+1} \left\langle S^-(K, C_f) - S^+(K, C_f) \right\rangle_{\mathcal{F}_d} \right\}.
\end{aligned}$$

Now Theorem 8.7 implies that

$$\left\langle S^-(K, C_f) - S^+(K, C_f) \right\rangle_{\mathcal{F}_d} = \left\langle S^-(K, C_f) \right\rangle_{\mathcal{F}_d} - \left\langle S^+(K, C_f) \right\rangle_{\mathcal{F}_d} = O(1).$$

For the remaining term we note that

$$\begin{aligned}
&\left\langle \left(S^-(K, C_f) - S^+(K, C_f)\right)^2 \right\rangle_{\mathcal{F}_d} \\
= &\left\langle \left(S^-(K, C_f)\right)^2 \right\rangle_{\mathcal{F}_d} + \left\langle \left(S^+(K, C_f)\right)^2 \right\rangle_{\mathcal{F}_d} \\
&-2 \left\langle \sum_{j_1, j_2 = 1}^{p-1} S^-(K, f, \psi^{j_1}) S^+(K, f, \psi^{j_2}) \right\rangle_{\mathcal{F}_d}.
\end{aligned}$$

By Corollary 8.14, this equals

$$\frac{4(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|) + O(1) - \frac{4(p-1)}{\pi^2}\log(\mathfrak{g}|\mathcal{I}|) + O(1) = O(1).$$

Therefore,

$$\left\langle \left(N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}| + S^{\pm}(K, C_f)\right)^2 \right\rangle = O\left(\left(\frac{2\mathfrak{g}}{K+1}\right)^2 \right)$$

and

$$\left\langle \left(\frac{N_{\mathcal{I}}(C_f) - 2\mathfrak{g}|\mathcal{I}| + S^{\pm}(K, C_f)}{\sqrt{(2(p-1)/\pi^2)\log(\mathfrak{g}|\mathcal{I}|)}} \right)^2 \right\rangle \to 0$$

when $\mathfrak{g}$ tends to infinity and $K = \mathfrak{g}/\log\log(\mathfrak{g}|\mathcal{I}|)$. □

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