One-level density of families of elliptic curves and the Ratios Conjecture

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Abstract

Using the Ratios Conjecture as introduced by Conrey, Farmer and Zirnbauer, we obtain closed formulas for the one-level density for two families of L-functions attached to elliptic curves, and we can then determine the underlying symmetry types of the families. The one-level scaling density for the first family corresponds to the orthogonal distribution as predicted by the conjectures of Katz and Sarnak, and the one-level scaling density for the second family is the sum of the Dirac distribution and the even orthogonal distribution. This is a new phenomenon for a family of curves with odd rank: the trivial zero at the central point accounts for the Dirac distribution, and also affects the remaining part of the scaling density which is then (maybe surprisingly) the even orthogonal distribution. The one-level density for this family was studied in the past for test functions with Fourier transforms of limited support, but since the Fourier transforms of the even orthogonal and odd orthogonal distributions are undistinguishable for small support, it was not possible to identify the distribution with those techniques. This can be done with the Ratios Conjecture, and it sheds more light on "independent" and "non-independent" zeroes, and the repulsion phenomenon.

1   Introduction

Since the work of Montgomery [22] on the pair correlation of the zeroes of the Riemann zeta function, it is known that there are many striking similarities between the statistics attached to zeroes of L-functions and eigenvalues of random matrices. The work of Montgomery was extended and generalised in many directions, in particular to the study of statistics of zeroes in families of L-functions, and their relation to the distribution laws for eigenvalues of random matrices. It is predicted by the Katz and Sarnak philosophy that in the limit (for large conductor), the statistics for the zeroes in families of L-functions follow distribution laws of random matrices.

We consider in this paper the one-level density for two families of L-functions attached to elliptic curves. Let $\mathcal{F}$ be such a family of elliptic curves, and let

$$\mathcal{F}(X) = \{E \in \mathcal{F} \ : \ N_E \leq X\} \tag{1.1}$$

be the set of curves of conductor N_E bounded by X.

For each $E \in \mathcal{F}$, its one-level density is the smooth counting function

$$D(E, \phi) = \sum_{\gamma_E} \phi(\gamma_E), \tag{1.2}$$

where the sum runs over the imaginary part of the zeroes γ_E of the L-function $L(s, E)$ of the curve E. We assume that the Generalized Riemann Hypothesis holds for the L-functions $L(s, E)$ which are normalised such that we can write the zeroes in the critical strip as $\rho_E = 1/2 + i\gamma_E$ with $\gamma_E \in \mathbb{R}$ (see Section 3 for details). Furthermore, ϕ is an even Schwartz test function.

The average of the one-level density over the family $\mathcal{F}(X)$ is then defined as

$$D(\mathcal{F}; \phi, X) := \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} D(E, \phi).$$

Katz and Sarnak predicted that the average one-level density should satisfy

$$\lim_{X \to \infty} D(\mathcal{F}; \phi, X) = \int_{-\infty}^{\infty} \phi(t) \mathcal{W}(t) \, dt, \tag{1.3}$$

where $\mathcal{W}(G)$ is the one-level scaling density of eigenvalues near 1 in the group of random matrices corresponding to the symmetry type of the family $\mathcal{F}$. Remarkably, it is believed that all natural families can be described by very few symmetry types, namely we have

$$\mathcal{W}(G)(t) = \begin{cases} 1 & \text{if } G = U; \\ 1 - \frac{\sin 2\pi t}{2\pi t} & \text{if } G = \text{Sp}; \\ 1 + \frac{1}{2}\delta_0(t) & \text{if } G = O; \\ 1 + \frac{\sin 2\pi t}{2\pi t} & \text{if } G = SO(\text{even}); \\ 1 + \delta_0(t) - \frac{\sin 2\pi t}{2\pi t} & \text{if } G = SO(\text{odd}); \end{cases} \tag{1.4}$$

where δ_0 is the Dirac distribution, and U, Sp, O, $SO(\text{even})$, $SO(\text{odd})$, are the groups of unitary, symplectic, orthogonal, even orthogonal and odd orthogonal matrices respectively. The function $\mathcal{W}(G)(t)$ is called the one-level scaling density of the group G. We refer the reader to [15] for details.

There has been extensive research dedicated to gathering evidence for the Katz and Sarnak conjecture for the one-level density for various families in the last few years. A standard approach is to compute the one-level density for test functions ϕ with limited support of the Fourier transform, i.e., supp $\hat{\phi} \subseteq (-a, a)$ for some $a \in \mathbb{R}$. In order to distinguish between the orthogonal symmetry types of (1.4), one needs to prove results for a test function ϕ with Fourier transform supported outside $[-1, 1]$. This approach was used in many papers, including [12] for various families, and [29] for the families of elliptic curves over $\mathbb{Q}$ with conductor up to X.

We are considering in this paper a different approach to study the one-level density of families of elliptic curves via the *Ratios Conjecture*, a powerful conjecture due to Conrey, Farmer and Zirnbauer [3] which predicts estimates for averages of quotients of (products of) L-functions evaluated at certain values. The Ratios Conjecture originated from the work of Farmer [5] about shifted moments of the Riemann zeta function, and the work of Nonnenmacher and Zirnbauer [23] about the Ratios of characteristic polynomials of random matrices. For the application to the one-level density of families of L-functions $L(s, E)$ attached to elliptic curves, it suffices to consider the ratio of the shifted L-functions

$$\frac{L(1/2 + \alpha, E)}{L(1/2 + \gamma, E)},$$

where α and γ are called the shifts. The first step of the "recipe" for obtaining the Ratios Conjecture for each family of L-functions is to use the approximate functional equation for each L-function to express the ratio as a principal sum and a dual sum, and then

replace both sums by their average over the family. This average of the L-functions Fourier coefficients over the family is the essential ingredient, and it will ultimately lead to the symmetry type of the family. The precise expression which is obtained by this procedure for the ratio of shifted L-functions of the family is called the Ratios Conjecture of the family (see Conjecture 3.7 and Conjecture 4.6 for the two families of elliptic curves considere in this paper). By differentiating with respect to the shift α, and then using $\alpha = \gamma = r$, we get an expression for the average of the ratio

$$\frac{L'(1/2 + r, E)}{L(1/2 + r, E)}$$

over the family, and by using Cauchy's theorem (3.2), this leads to an expression for the one-level density for each family of elliptic curves (Theorems 2.1 and 2.3). From this expression, we can identify without ambiguity the symmetry type (1.4) of the Katz-Sarnak predictions for those two families (Corollaries 2.2 and 2.4). We find that for the family of all elliptic curves, the one-level scaling density is given by

$$\mathcal{W}(t) = 1 + \frac{1}{2}\delta_0(t), \tag{1.5}$$

and for the one-parameter family of elliptic curves given by (2.1), the one-level scaling density is given by

$$\mathcal{W}(t) = 1 + \frac{\sin(2\pi t)}{2\pi t} + \delta_0(t). \tag{1.6}$$

The precise statements of those results can be found in Section 2, and the proofs in Sections 3 and Section 4 respectively. We also discuss in Section 5 a heuristic model based on the Birch and Swinnerton-Dyer conjectures which explains the symmetry type (1.6) of the second family.

2 Statement of the results

We now state the main results of this paper.

We first consider the family of all elliptic curves over $\mathbb{Q}$. Let $E_{a,b}$ be an elliptic curve over $\mathbb{Q}$ given by $E_{a,b} : y^2 = x^3 + ax + b$. We study the one-level density for the family

$$\mathcal{F}(X) = \{E = E_{a,b} : a \equiv r \bmod 6, b \equiv t \bmod 6, |a| \le X^{\frac{1}{3}}, |b| \le X^{\frac{1}{2}}, p^4 \mid a \Rightarrow p^6 \nmid b\},$$

for some fixed integers (r, t) such that $(r, 3) = 1$ and $(t, 2) = 1$. More details about this family are given in Section 3.

Theorem 2.1. *Fix $\varepsilon > 0$. Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N_E. Let ϕ be an even Schwartz function on $\mathbb{R}$ whose Fourier transform has compact support. Assuming GRH and the Ratios Conjecture 3.7, the one-level density for the zeros of the family $\mathcal{F}(X)$ of all elliptic curves is given by*

David *et al. Research in Number Theory* (2015) 1:6

Page 4 of 37

$$D(\mathcal{F};\phi,X) = \frac{1}{|\mathcal{F}(X)|}\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(t)\sum_{E\in\mathcal{F}(X)}\left[2\log\left(\frac{\sqrt{N_E}}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(1+it)\right.$$

$$+ \frac{\Gamma'}{\Gamma}(1-it) + 2\left(-\frac{\zeta'}{\zeta}(1+2it) + A_\alpha(it,it)\right.$$

$$\left.\left. - \omega_E\left(\frac{\sqrt{N_E}}{2\pi}\right)^{-2it}\frac{\Gamma(1-it)}{\Gamma(1+it)}\zeta(1+2it)A(-it,it)\right) - \frac{1-\omega_E}{it}\right]dt$$

$$+ \frac{\phi(0)}{|\mathcal{F}(X)|}\sum_{\substack{E\in\mathcal{F}(X)\\ \omega_E=-1}}1 + O(X^{-1/2+\varepsilon}),$$

where the function A_α is defined in (3.42).

According to the conjectures of Katz and Sarnak, one expects that the symmetry type of (1.4) is orthogonal for the family of all elliptic curves. In ([29], Theorem 3.1), Young showed that this is indeed the case for test functions ϕ with $\widehat{\phi} \subset \left(-\frac{7}{9},\frac{7}{9}\right)$. To see that Theorem 2.1 gives the same scaling density (without restrictions on the support of the Fourier transform, but under the the Ratios Conjecture for the given family), we have to make a change of variable to ensure that the sequence of low-lying zeroes γ_E has mean spacing 1 as E varies over the curves of the family $\mathcal{F}(X)$, and we then define ψ to be the normalized test function (see Section 3 for more details). The following corollary then follows from Theorem 2.1.

Corollary 2.2. *Assuming the Ratios Conjecture 3.7 and the equidistribution of the root number in the family $\mathcal{F}(X)$ [a], the one-level density of the family $\mathcal{F}(X)$ of all elliptic curves is given by*

$$\frac{1}{|\mathcal{F}(X)|}\sum_{E\in\mathcal{F}(X)}\psi\left(\frac{\gamma_E L}{\pi}\right) = \int_{-\infty}^{\infty}1 + \frac{1}{2}\delta_0(\tau) + O\left(\frac{1}{L}\right)d\tau,$$

where $L = \log(\sqrt{X}/2\pi e)$.

Then, according to (1.4), the underlying symmetry type is orthogonal and matches the conjectures of Katz and Sarnak for the family of all elliptic curves. The proofs of Theorem 2.1 and Corollary 2.2 are given in Section 3. Some lower order terms (for L^{-1} and L^{-2}) are also computed explicitly, and can be useful for experimental computations for small conductors.

We also use the Ratios Conjecture to study the one-level density of a one-parameter family of elliptic curves which was first considered by Washington [31], namely the family

$$E_t : y^2 = x^3 + tx^2 - (t+3)x + 1, \quad t \in \mathbb{Z}. \tag{2.1}$$

It was shown by Washington [31] (under some hypotheses and for a positive proportion of $t \in \mathbb{Z}$), and then by Rizzo [24] (unconditionally for all $t \in \mathbb{Z}$) that the sign of the functional equation is negative for the L-functions of the curves E_t, for all $t \in \mathbb{Z}$.

We study the one-level density for the family of curves

$$\mathcal{F}_1(X) = \left\{E_t : t \le X^{\frac{1}{4}}\right\}. \tag{2.2}$$

More details about this family can be found in Section 4.

David *et al. Research in Number Theory* (2015) 1:6

Page 5 of 37

Theorem 2.3. *Fix $\varepsilon > 0$. Let E_t be an elliptic curve defined over $\mathbb{Q}$ with conductor $C(t)$ defined in (2.1). Let ϕ be an even Schwartz function on $\mathbb{R}$ whose Fourier transform has compact support. Assuming GRH and the Ratios Conjecture 4.6, the one-level density for the zeros of the family $\mathcal{F}_1$ defined by (2.2) is given by*

$$D(\mathcal{F}_1; \phi, X) = \frac{1}{|\mathcal{F}_1(X)|} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u) \sum_{E_t \in \mathcal{F}_1(X)} \left[2 \log\left(\frac{\sqrt{C(t)}}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(1+iu) \right.$$

$$+ \frac{\Gamma'}{\Gamma}(1-iu) - 2\left(\frac{\zeta'}{\zeta}(1+2iu) + \frac{\zeta'}{\zeta}(1+iu)\right) + 2A_\alpha(iu, iu)$$

$$\left. + 2\left(\frac{\sqrt{C(t)}}{2\pi}\right)^{-2iu} \frac{\Gamma(1-iu)}{\Gamma(1+iu)} \frac{\zeta(1+2iu)\zeta(1+iu)}{\zeta(1-iu)} A(-iu, iu) - \frac{2}{iu} \right] du$$

$$+ \phi(0) + O(X^{-1/2+\varepsilon}),$$

where A and A_α are defined by (4.23) and (4.26) respectively.

Again, by using the appropriate change of variables to normalise the zeroes and the test function (see Section 4 for more details), we obtain the following result.

Corollary 2.4. *Assuming the Ratios Conjecture 4.6, the one-level density of the family $\mathcal{F}_1(X)$ is given by*

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E \in \mathcal{F}_1(X)} \psi\left(\frac{\gamma_E L}{\pi}\right) = \int_{-\infty}^{\infty} 1 + \frac{\sin(2\pi\tau)}{2\pi\tau} + \delta_0(\tau) + O\left(\frac{1}{L}\right) d\tau,$$

where $L = \log(\sqrt{X}/2\pi e)$.

The proofs of Theorem 2.3 and Corollary 2.4 are given in Section 4.

The scaling density of Corollary 2.4 is the sum of two densities $\mathcal{W}(G)(\tau)$ of (1.4), the Dirac distribution and $\mathcal{W}(SO(\text{even}))(\tau)$. This might seem surprising a priori, as $\mathcal{W}(SO(\text{even}))(\tau)$ usually corresponds to families of *even* rank, and we have a family of *odd* rank. This can be explained by the special behavior of the zero of the L-functions $L(s, E_t)$ at $s = 1/2$ forced by the sign of the functional equation. This phenomenon was also studied by Miller [19] for general one-parameter families of rank r. Then, by Silverman's specialization theorem [25], every curve in the family have rank at least r, and the r forced zeroes (from the Birch and Swinnerton-Dyer conjecture) are called the *family zeroes*. It was noticed by Miller, by computing the one-level density for test functions ϕ with Fourier transform $\hat{\phi}$ of limited support, that those zeroes act as if independent from the remaining zeroes, and should correspond to a sum of r Dirac functions in the density $\mathcal{W}(\tau)$ of

David *et al. Research in Number Theory* (2015) 1:6

Page 6 of 37

the family. But the density function could not be completely determined (even for the case $r = 1$ that we are considering here) because one can only take limited support for $\hat{\phi}$, and this does not allow us to differentiate $\mathcal{W}(G)(\tau)$ between $G = SO$(odd) and $G = SO$(even). See ([18], Section 6.1.3). By using the Ratios Conjecture, after removing the contribution $\delta_0(\tau)$ coming from the family zero, the one-level scaling density is then $\mathcal{W}(SO(\text{even}))(\tau)$. We give in Section 5 a heuristic based on the Birch and Swinnerton-Dyer conjectures explaining why once the zero is removed, the corresponding L-functions should indeed behave like a family of even rank.

Finally, we remark that one consequence of the scaling density of Corollary 2.4 associated to the family $\mathcal{F}_1(X)$ is that the forced zero of the L-functions $L(s, E_t)$ at $s = 1/2$ is independent in the limit from the other zeroes, and does not cause any repulsion. By contrast, in the family of odd rank quadratic twists of a fixed elliptic curve E over $\mathbb{Q}$, which is also a family where every L-function $L(s, E, \chi_d)$ has a zero at $s = 1/2$, the central zero is not independent, and causes some repulsion. Indeed, the (conjectural) one-level scaling densities are respectively

$$\mathcal{W}(t) = \begin{cases} \delta_0(\tau) + 1 - \frac{\sin(2\pi\tau)}{2\pi\tau} = \mathcal{W}(SO(\text{odd})) & \text{for } \mathcal{F}^{\text{odd}} = \{L(s, E, \chi_d)\} \\ \delta_0(\tau) + 1 + \frac{\sin(2\pi\tau)}{2\pi\tau} = \delta_0(\tau) + \mathcal{W}(SO(\text{even})) & \text{for } \mathcal{F}_1 = \{L(s, E_t)\}. \end{cases}$$

Since $\sin(2\pi\tau)/2\pi\tau \to 1$ as $\tau \to 0$, we have that $\mathcal{W}(\tau) - \delta_0(\tau)$ is close to 0 when τ is small in the first case, so the zero of $L(s, E, \chi_d)$ at $s = 1/2$ causes a repulsion of the zeroes with small imaginary part, while in the second case, $\mathcal{W}(\tau) - \delta_0(t)$ is close to 2 when τ is small, and there is no repulsion for the zeroes with small imaginary part.

In some work in progress (in collaboration with S. Bettin, C. Delaunay and S. J. Miller), we generalise Theorem 2.3 and Corollary 2.4 to arbitrary one-parameter families of elliptic curves over $\mathbb{Q}(t)$ with average rank not equal to 0. We also build many such families to illustrate the possible symmetry types occurring.

3 The family of all elliptic curves

Let $E_{a,b}$ be an elliptic curve over $\mathbb{Q}$ given by

$$E_{a,b} : y^2 = x^3 + ax + b. \tag{3.1}$$

We fix some integers (r, t) such that $(r, 3) = 1$ and $(t, 2) = 1$. We will use them to impose congruences modulo 6 on a, b to ensure that $E_{a,b}$ is minimal at $p = 2, 3$, so we remark that there are 12 choices of (r, s).

We study the family

$$\mathcal{F}(X) = \{E = E_{a,b} : a \equiv r \bmod 6, b \equiv t \bmod 6, |a| \le X^{\frac{1}{3}}, |b| \le X^{\frac{1}{2}}, p^4 \mid a \Rightarrow p^6 \nmid b\}$$

of all elliptic curves having discriminant of size $\ll X$. The conditions on $E_{a,b} \in \mathcal{F}(X)$ insure that $E_{a,b}$ is a minimal model at all primes p.

Let $L(s, E)$ denote the L-function attached to E, normalised in such a way that the center of the critical strip is the line $\text{Re}(s) = 1/2$. The average one-level density over the family is then

$$D(\mathcal{F}; \phi, X) = \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \sum_{\gamma_E} \phi(\gamma_E),$$

where γ_E runs over the ordinates of the non-trivial zeroes of $L(s, E)$.

David *et al. Research in Number Theory* (2015) 1:6

Page 7 of 37

By Cauchy's theorem, we can write the average one-level density as

$$D(\mathcal{F}; \phi, X) = \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'(s,E)}{L(s,E)} \phi(-i(s-1/2)) ds \qquad (3.2)$$

with $\frac{1}{2} < c < 1$.

Our strategy is to use the Ratios Conjecture to write a closed formula for the logarithmic derivative of $L(s,E)$ in (3.2). Following the approach of [3], we consider the ratio

$$\frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{L\left(\frac{1}{2} + \alpha, E\right)}{L\left(\frac{1}{2} + \gamma, E\right)} \qquad (3.3)$$

for $\alpha, \gamma \in \mathbb{C}$ with $\operatorname{Re}(\alpha), \operatorname{Re}(\gamma) > 0$.

For a minimal model $E = E_{a,b}$, we have that $\lambda_E(n) = \lambda_{a,b}(n)$ where for $p \neq 2$, $\lambda_{a,b}(p)$ is given by

$$\lambda_{a,b}(p) = \frac{1}{\sqrt{p}} (p + 1 - \#E_p(\mathbb{F}_p)) = -\frac{1}{\sqrt{p}} \sum_{x \bmod p} \left(\frac{x^3 + ax + b}{p} \right),$$

where $\left(\frac{\cdot}{p} \right)$ denotes the Legendre symbol. If $p = 2$ then (3.1) has a cusp and $\lambda_{a,b}\left(2^k\right) = 0$ for all $k \geq 1$.

We recall that the L-function attached to an elliptic curve E is given by

$$L(s,E) = \sum_{n=1}^{\infty} \frac{\lambda_E(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_E(p)}{p^s} + \frac{\psi_N(p)}{p^{2s}} \right)^{-1}, \qquad (3.4)$$

where ψ_N is the principal Dirichlet character modulo the conductor N_E of E, i.e.,

$$\psi_N(p) = \begin{cases} 1 & \text{if } p \nmid N_E, \\ 0 & \text{if } p \mid N_E. \end{cases}$$

It follows from (3.4) that $\lambda_E(n)$ is multiplicative, and prime powers can be computed by

$$\lambda_E(p^j) = \begin{cases} U_j\left(\frac{\lambda_E(p)}{2} \right) & \text{if } (p, N_E) = 1, \\ \lambda_E^j(p) & \text{if } (p, N_E) > 1, \end{cases} \qquad (3.5)$$

where $U_j(x)$ is the j-th Chebyshev polynomial of the second kind. The definition of the Chebyshev polynomials and their properties will be given shortly.

It was proven by Wiles *et al* [1,27,28] that

$$\Lambda(s,E) = \Gamma(s+1/2) \left(\frac{\sqrt{N_E}}{2\pi} \right)^s L(s,E)$$

satisfies the functional equation

$$\Lambda(s,E) = \omega_E \Lambda(1-s,E)$$

where $\omega_E = \pm 1$ is called the root number of E. It follows that we can write the values $L(s,E)$ as

$$L(s,E) = \sum_n \frac{\lambda_E(n)}{n^s} V_s \left(\frac{2\pi n}{Y \sqrt{N_E}} \right) + \omega_E X_E(s) \sum_n \frac{\lambda_E(n)}{n^{1-s}} V_{1-s} \left(\frac{2\pi n Y}{\sqrt{N_E}} \right), \qquad (3.6)$$

where

$$X_E(s) = \frac{\Gamma\left(\frac{3}{2} - s\right)}{\Gamma\left(\frac{1}{2} + s\right)} \left(\frac{\sqrt{N_E}}{2\pi} \right)^{1-2s} \qquad (3.7)$$

and $V_s(y)$ is a smooth function which decays rapidly for large values of y. The above identity is called the approximate functional equation for $L(s, E)$, and we refer the reader to ([13], Theorem 5.3) for the details.

One of the steps in the recipe leading to the Ratios Conjecture is to use the two sums of the approximate functional Eq. 3.6 at $s = \frac{1}{2} + \alpha$ ignoring questions of convergence, or error terms, i.e., the "principal sum"

$$\sum_n \frac{\lambda_{a,b}(n)}{n^{\frac{1}{2}+\alpha}} \tag{3.8}$$

and the "dual sum"

$$\omega_E X_E \left(\frac{1}{2} + \alpha\right) \sum_n \frac{\lambda_{a,b}(n)}{n^{\frac{1}{2}-\alpha}}. \tag{3.9}$$

Finally, we write

$$\frac{1}{L(s, E_{a,b})} = \prod_p \left(1 - \frac{\lambda_{a,b}(p)}{p^s} + \frac{\psi_N(p)}{p^{2s}}\right) = \sum_{n=1}^{\infty} \frac{\mu_{a,b}(n)}{n^s}, \tag{3.10}$$

where $\mu_{a,b}$ is a multiplicative function given by

$$\mu_{a,b}(p^k) = \begin{cases} -\lambda_{a,b}(p) & \text{if } k = 1, \\ \psi_N(p) & \text{if } k = 2, \\ 0 & \text{if } k > 2. \end{cases} \tag{3.11}$$

Following the standard recipe from [3] to derive the L-function Ratios Conjecture for our family (see also [4,10]), we replace the numerator of (3.3) with the principal sum (3.8) and the dual sum (3.9) of the approximate functional equation and the denominator of (3.3) with (3.10). We first focus on the principal sum which gives the sum

$$R_1(\alpha, \gamma) := \frac{1}{|\mathcal{F}(X)|} \sum_{E_{a,b} \in \mathcal{F}(X)} \sum_{m_1, m_2} \frac{\lambda_{a,b}(m_1)\mu_{a,b}(m_2)}{m_1^{\frac{1}{2}+\alpha} m_2^{\frac{1}{2}+\gamma}}. \tag{3.12}$$

We will consider in a second step the sum coming from the dual sum, namely the sum

$$R_2(\alpha, \gamma) := \frac{1}{|\mathcal{F}(X)|} \sum_{E_{a,b} \in \mathcal{F}(X)} \omega_{E_{a,b}} X_{E_{a,b}} \left(\frac{1}{2} + \alpha\right) \sum_{m_1, m_2} \frac{\lambda_{a,b}(m_1)\mu_{a,b}(m_2)}{m_1^{\frac{1}{2}-\alpha} m_2^{\frac{1}{2}+\gamma}}. \tag{3.13}$$

3.1 Average of Fourier coefficients over the family

To obtain the Ratios Conjecture for our family, we replace each (m_1, m_2)-summand in (3.12) and (3.13) by their averages

$$\lim_{X \to \infty} \frac{1}{|\mathcal{F}(X)|} \sum_{E_{a,b} \in \mathcal{F}(X)} \lambda_{a,b}(m_1)\mu_{a,b}(m_2) \tag{3.14}$$

over all curves in the family. This is similar to the work of Young in [30] where the author makes a conjecture on the moments of the central values $L(1/2, E)$ for the same family $\mathcal{F}$. He is then led to averages of the type

$$\lim_{X \to \infty} \frac{1}{|\mathcal{F}(X)|} \sum_{E_{a,b} \in \mathcal{F}(X)} \lambda_{a,b}(m_1) \cdots \lambda_{a,b}(m_k)$$

for the k-th moment. In the following, we will use some of the results of [30], and redo some of his computations in our setting for the sake of completeness.

David *et al. Research in Number Theory* (2015) 1:6

Page 9 of 37

Lemma 3.1. *Let*

$$\widetilde{Q}^*(m_1, m_2) := \frac{1}{(m^*)^2} \sum_{a,b \bmod m^*} \lambda_{a,b}(m_1)\mu_{a,b}(m_2) \tag{3.15}$$

and let m^* be the product of primes dividing $m = [m_1, m_2]$. Furthermore, set $m_i = \ell_i n_i$ where $(n_i, 6) = 1$ and $p \mid \ell_i \Rightarrow p = 2, 3$, and set

$$\widetilde{Q}^*_{r,t}(m_1, m_2) := \lambda_{r,t}(m_1)\mu_{r,t}(m_2)\widetilde{Q}^*(n_1, n_2) \prod_{\substack{p|m \\ p>3}} \left(1 - p^{-10}\right)^{-1}.$$

Then

$$\lim_{X \to \infty} \frac{1}{|\mathcal{F}(X)|} \sum_{E_{a,b} \in \mathcal{F}(X)} \lambda_{a,b}(m_1)\mu_{a,b}(m_2) \sim \widetilde{Q}^*_{r,t}(m_1, m_2). \tag{3.16}$$

Furthermore, $\widetilde{Q}^*_{r,t}$ is multiplicative in m_1 and m_2.

Proof. We will follow the proof of Lemma 3.2 in [30]. We have that

$$\lim_{X \to \infty} \frac{1}{|\mathcal{F}(X)|} \sum_{E_{a,b} \in \mathcal{F}(X)} \lambda_{a,b}(m_1)\mu_{a,b}(m_2)$$

$$= \lim_{X \to \infty} \frac{1}{|\mathcal{F}(X)|} \sum_{\substack{|a| \leq X^{\frac{1}{3}}, |b| \leq X^{\frac{1}{2}} \\ p^4|a \Rightarrow p^6 \nmid b \\ a \equiv r \bmod 6, b \equiv t \bmod 6}} \lambda_{a,b}(m_1)\mu_{a,b}(m_2). \tag{3.17}$$

We now need to extend the definition of $\lambda_{a,b}$ and $\mu_{a,b}$ for non-minimal curves $E_{a,b}$. We define

$$\lambda_{a,b}(p) := \begin{cases} \lambda_E(p) & \text{if } E_{a,b} \text{ is minimal at } p, \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi_{a,b}(p) := \begin{cases} 1, & \text{if } p \nmid -16(4a^3 + 27b^2)), \\ 0, & \text{otherwise.} \end{cases}$$

This defines $\mu_{a,b}$ at prime powers by (3.11), and $\lambda_{a,b}$ is defined at prime powers by the usual relation (3.5). We then extend to $\lambda_{a,b}(n), \mu_{a,b}(n)$ by multiplicativity.

We also have the usual power detector

$$\sum_{\substack{d^4|a \\ d^6|b}} \mu(d) = \begin{cases} 1, & \text{if there does not exist a } p \text{ such that } p^4 \mid a \text{ and } p^6 \mid b, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the left hand side of (3.17) can be rewritten as

$$\lim_{X \to \infty} \frac{1}{|\mathcal{F}(X)|} \sum_{\substack{d \leq X^{\frac{1}{12}} \\ (d,6)=1}} \mu(d) \sum_{\substack{|a| \leq d^{-4}X^{\frac{1}{3}}, |b| \leq d^{-6}X^{\frac{1}{2}} \\ a \equiv d^{-4}r \bmod 6, b \equiv d^{-6}t \bmod 6}} \lambda_{ad^4,bd^6}(m_1)\mu_{ad^4,bd^6}(m_2). \tag{3.18}$$

It follows from our definition of $\lambda_{a,b}$ and $\mu_{a,b}$ at non-minimal curves that

$$\lambda_{d^4a,d^6b}(n) = \begin{cases} \lambda_{a,b}(n) & \text{if } (n, d) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

David *et al. Research in Number Theory* (2015) 1:6

Page 10 of 37

and similarly

$$\mu_{d^4a,d^6b}(n) = \begin{cases} \mu_{a,b}(n) & \text{if } (n,d) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $(n,d) = 1$, we have

$$\lambda_{d^4a,d^6b}(m_1)\mu_{d^4a,d^6b}(m_2) = \lambda_{r,t}(\ell_1)\mu_{r,t}(\ell_2)\lambda_{a,b}(n_1)\mu_{a.b}(n_2),$$

and (3.18) becomes

$$\lim_{X\to\infty}\frac{1}{|\mathcal{F}(X)|}\lambda_{r,t}(\ell_1)\mu_{r,t}(\ell_2)\sum_{\substack{d\le X^{\frac{1}{12}}\\(d,6n)=1}}\mu(d)\sum_{\substack{|a|\le d^{-4}X^{\frac{1}{3}},|b|\le d^{-6}X^{\frac{1}{2}}\\a\equiv d^{-4}r\bmod 6\\b\equiv d^{-6}t\bmod 6}}\lambda_{a,b}(n_1)\mu_{a,b}(n_2).$$

Now $\lambda_{a,b}(n_1)\mu_{a,b}(n_2)$ is periodic in a and b with period equal to the product of primes dividing the least common multiple of n_1, n_2 say n^*. Breaking up the sum over a and b into arithmetic progressions modulo 6, we rewrite the last equation as

$$\lim_{X\to\infty}\frac{1}{|\mathcal{F}(X)|}\lambda_{r,t}(\ell_1)\mu_{r,t}(\ell_2)\frac{4X^{\frac{5}{6}}}{36}\sum_{\substack{d\le X^{\frac{1}{12}}\\(d,6n)=1}}\frac{\mu(d)}{d^{10}}\frac{1}{(m^*)^2}\sum_{\substack{\alpha\bmod n^*\\\beta\bmod n^*}}\lambda_{\alpha,\beta}(n_1)\mu_{\alpha,\beta}(n_2)$$

$$= \lim_{X\to\infty}\frac{1}{|\mathcal{F}(X)|}\lambda_{r,t}(\ell_1)\mu_{r,t}(\ell_2)\frac{X^{\frac{5}{6}}}{9\zeta_{6n}(10)}\widetilde{Q}^*(n_1,n_2),$$

where we define

$$\zeta_m(s) := \prod_{p|m}\left(1-\frac{1}{p^s}\right)^{-1}.$$

By ([30], Equation 3.22), we have

$$|\mathcal{F}(X)| \sim \frac{X^{\frac{5}{6}}}{9\zeta_6(10)},$$

and (3.18) becomes

$$\lambda_{r,t}(\ell_1)\mu_{r,t}(\ell_2)\widetilde{Q}^*(n_1,n_2)\frac{\zeta_6(10)}{\zeta_{6n}(10)}.$$

Since

$$\frac{\zeta_6(10)}{\zeta_{6n}(10)} = \prod_{\substack{p|n\\p>3}}\left(1-p^{-10}\right)^{-1} = \prod_{\substack{p|m\\p>3}}\left(1-p^{-10}\right)^{-1},$$

this completes the proof of (3.16). □

Now we replace each term of (3.12) by its average value $\widetilde{Q}^*_{r,t}(m_1,m_2)$, and using Lemma 3.1, we are led to consider

$$H(\alpha,\gamma) := \sum_{m_1,m_2}\frac{\widetilde{Q}^*_{r,t}(m_1,m_2)}{m_1^{\frac{1}{2}+\alpha}m_2^{\frac{1}{2}+\gamma}} = \prod_p\sum_{m_1,m_2}\frac{\widetilde{Q}^*_{r,t}(p^{m_1},p^{m_2})}{p^{m_1(\frac{1}{2}+\alpha)+m_2(\frac{1}{2}+\gamma)}}$$

$$= \left(\prod_{2\le p\le 3}\sum_{m_1,m_2}\frac{\lambda_{r,t}(p^{m_1})\mu_{r,t}(p^{m_2})}{p^{m_1(\frac{1}{2}+\alpha)+m_2(\frac{1}{2}+\gamma)}}\right)\left(\prod_{p>3}\sum_{m_1,m_2}\frac{\delta(p)\widetilde{Q}^*(p^{m_1},p^{m_2})}{p^{m_1(\frac{1}{2}+\alpha)+m_2(\frac{1}{2}+\gamma)}}\right),$$

$$(3.19)$$

where $\delta(p) = \left(1-p^{-10}\right)^{-1}$ if $m_1+m_2 > 0$ and $\delta(p) = 1$ otherwise.

Thus, it suffices to consider $\widetilde{Q}^*_{r,t}(p^{m_1}, p^{m_2})$ at a prime p and integers m_1, m_2. Notice that we switched notation, and we are now using m_1, m_2 for the exponents of the prime powers. By the definition of the Möebius function in (3.11) only the terms with $m_2 = 0, 1$ and 2 in (3.19) contribute. For $p = 2, 3$, we denote by $E_p(\alpha, \gamma)$ the Euler factor

$$E_p(\alpha, \gamma) := \sum_{m_1, m_2} \frac{\lambda_{r,t}(p^{m_1}) \mu_{r,t}(p^{m_2})}{p^{m_1(\frac{1}{2}+\alpha)+m_2(\frac{1}{2}+\gamma)}} \tag{3.20}$$

at p in $H(\alpha, \gamma)$.

So we have that

$$H(\alpha, \gamma) = E_2(\alpha, \gamma) E_3(\alpha, \gamma) \prod_{p>3} \sum_{m_1, m_2} \frac{\delta(p) \widetilde{Q}^*(p^{m_1}, p^{m_2})}{p^{m_1(\frac{1}{2}+\alpha)+m_2(\frac{1}{2}+\gamma)}}$$

$$= E_2(\alpha, \gamma) E_3(\alpha, \gamma) \prod_{p>3} \left(1 + \left(1 - p^{-10}\right)^{-1} \right.$$

$$\left. \times \left(\sum_{m_1 \geq 1} \frac{\widetilde{Q}^*(p^{m_1}, p^0)}{p^{m_1(\frac{1}{2}+\alpha)}} + \sum_{m_1 \geq 0} \frac{\widetilde{Q}^*(p^{m_1}, p^1)}{p^{m_1(\frac{1}{2}+\alpha)+\frac{1}{2}+\gamma}} + \sum_{m_1 \geq 0} \frac{\widetilde{Q}^*(p^{m_1}, p^2)}{p^{m_1(\frac{1}{2}+\alpha)+1+2\gamma}} \right) \right),$$

where

$$\widetilde{Q}^*(p^{m_1}, p^0) = \frac{1}{p^2} \sum_{a,b \bmod p} \lambda_{a,b}(p^{m_1}), \tag{3.21}$$

$$\widetilde{Q}^*(p^{m_1}, p^1) = -\frac{1}{p^2} \sum_{a,b \bmod p} \lambda_{a,b}(p^{m_1}) \lambda_{a,b}(p), \tag{3.22}$$

$$\widetilde{Q}^*(p^{m_1}, p^2) = \frac{1}{p^2} \sum_{\substack{a,b \bmod p \\ p \nmid N_E}} \lambda_{a,b}(p^{m_1}). \tag{3.23}$$

In the following theorem, we write a closed formula for $H(\alpha, \gamma)$ in terms of the trace of the Hecke operators T_p, using the Eichler-Selberg Trace Formula, following [29] (see Lemma 3.3 below). We first need some notation. Let $Tr_j(p)$ denote the trace of the Hecke operator T_p acting on the space of weight j holomorphic cusp forms on the full modular group. The normalized trace $Tr_j^*(p)$ is given by

$$Tr_j^*(p) = p^{(1-j)/2} Tr_j(p). \tag{3.24}$$

We recall that we have that $Tr_j^*(p) = 0$ for $j < 12$. Now $H(\alpha, \gamma)$ can be rewritten in terms of $Tr_j^*(p)$.

Theorem 3.2. *Let $\alpha, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 0$, and let H be given by (3.19). Then*

$$H(\alpha, \gamma) = E_2(\alpha, \gamma) E_3(\alpha, \gamma) \prod_{p>3} \left[1 + \left(1 - \frac{p^9 - 1}{p^{10} - 1} \right) \left(\frac{1}{p^{1+2\gamma}} - \frac{1}{p^{1+\alpha+\gamma}} \right. \right.$$

$$\left. \left. + \frac{p^{-(2+\alpha+\gamma)} - p^{-(2+2\gamma)}}{p^{2+2\alpha} - 1} + \left(\frac{p^{1+2\alpha+\gamma} - p^{1+\alpha+2\gamma} + p^\gamma - p^\alpha}{p^{\frac{3}{2}+\alpha+2\gamma}} \right) \sum_{\substack{m_1 \geq 10 \\ m_1 \text{ even}}} \frac{Tr_{m_1+2}^*(p)}{p^{m_1(\frac{1}{2}+\alpha)}} \right) \right].$$

David *et al. Research in Number Theory* (2015) 1:6

Page 12 of 37

Furthermore, H has the form

$$H(\alpha, \gamma) = \frac{\zeta(1 + 2\gamma)}{\zeta(1 + \alpha + \gamma)} A(\alpha, \gamma)$$

where $A(\alpha, \gamma)$ is holomorphic and non-zero for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/4$. We also have that $A(r, r) = 1$ in this region.

Before proving Theorem 3.2, we make some observations and state some useful lemmata. First we consider the Chebyshev polynomials $U_n(x)$ appearing in (3.5) and their properties. The polynomials $U_n(x)$ satisfy the recursion formula

$$U_{n+2}(x) - 2x U_{n+1}(x) + U_n(x) = 0, \quad \text{for } n \geq 0, \tag{3.25}$$

which is equivalent to the formal identity

$$\sum_{n \geq 0} U_n(x) t^n = \frac{1}{1 - 2xt + t^2}. \tag{3.26}$$

The first few Chebyshev polynomials are $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, $U_3(x) = 8x^3 - 4x$, etc. Also, the Chebyshev polynomials satisfy

$$U_n(-x) = (-1)^n U_n(x), \tag{3.27}$$

i.e., $U_n(x)$ is odd when n is odd, and even when n is even. We also define the coefficients $c_\ell(m_1, m_2)$ by

$$U_{m_1}(x) U_{m_2}(x) = \sum_{\ell \geq 0} c_\ell(m_1, m_2) U_\ell(x).$$

From the properties of the Chebyshev polynomials above we have that if $m_1 + m_2$ is odd (and $p > 2$), then $Q^*(p^{m_1}, p^{m_2})$ is 0. We can see this by first making the change of variables $a = d^2 a', b = d^3 b'$ where d is a quadratic nonresidue modulo p. Then we have that $\lambda_{a,b}(p) = -\lambda_{a',b'}(p)$. Now if $m_2 = 0, 1$ then

$$
\begin{aligned}
\widetilde{Q}^*(p^{m_1}, p^{m_2}) &= \frac{(-1)^{m_2}}{p^2} \sum_{a,b \bmod p} \lambda_{a,b}(p^{m_1}) \lambda_{a,b}^{m_2}(p) \\
&= \frac{(-1)^{m_2}}{p^2} \sum_{\substack{a,b \bmod p \\ p|N_E}} \lambda_{a,b}^{m_1+m_2}(p) + \frac{(-1)^{m_2}}{p^2} \sum_{\substack{a,b \bmod p \\ p \nmid N_E}} U_{m_1}\left(\frac{\lambda_{a,b}(p)}{2}\right) \lambda_{a,b}^{m_2}(p) \\
&= \frac{(-1)^{m_1}}{p^2}\left(\sum_{\substack{a',b' \bmod p \\ p|N_E}} \lambda_{a',b'}^{m_1+m_2}(p) + \sum_{\substack{a',b' \bmod p \\ p \nmid N_E}} U_{m_1}\left(\frac{\lambda_{a',b'}(p)}{2}\right) \lambda_{a',b'}^{m_2}(p)\right) \\
&= (-1)^{m_1+m_2} \widetilde{Q}^*(p^{m_1}, p^{m_2}),
\end{aligned}
$$

where we have used the property (3.27) of the Chebyshev polynomials. If $m_2 = 2$ then

$$
\begin{aligned}
\widetilde{Q}^*(p^{m_1}, p^2) &= \frac{1}{p^2} \sum_{\substack{a,b \bmod p \\ p \nmid N_E}} U_{m_1}\left(\frac{\lambda_{a,b}(p)}{2}\right) = \frac{(-1)^{m_1}}{p^2} \sum_{\substack{a',b' \bmod p \\ p \nmid N_E}} U_{m_1}\left(\frac{\lambda_{a',b'}(p)}{2}\right) \\
&= (-1)^{m_1} \widetilde{Q}^*(p^{m_1}, p^2).
\end{aligned}
$$

Hence, we have for $p > 3$, each Euler factor in $H(\alpha, \gamma)$ can be written as

$$1 + \frac{1}{1 - p^{-10}} \left(\sum_{\substack{m_1 \geq 2 \\ m_1 \text{ even}}} \frac{\widetilde{Q}^*(p^{m_1}, p^0)}{p^{m_1(\frac{1}{2} + \alpha)}} + \sum_{\substack{m_1 \geq 1 \\ m_1 \text{ odd}}} \frac{\widetilde{Q}^*(p^{m_1}, p^1)}{p^{m_1(\frac{1}{2} + \alpha) + \frac{1}{2} + \gamma}} + \sum_{\substack{m_1 \geq 0 \\ m_1 \text{ even}}} \frac{\widetilde{Q}^*(p^{m_1}, p^2)}{p^{m_1(\frac{1}{2} + \alpha) + 1 + 2\gamma}} \right).$$

$$(3.28)$$

We will use the following result from [30].

Lemma 3.3 (Proposition 4.2, [30]). *Let*

$$Q^*(p^{m_1}, p^{m_2}) = \frac{1}{p^2} \sum_{a,b \pmod{p}} \lambda_{a,b}(p^{m_1}) \lambda_{a,b}(p^{m_2}). \tag{3.29}$$

Then for $p > 3$ and $m_1 + m_2$ even and positive, we have

$$Q^*(p^{m_1}, p^{m_2}) = c_0(m_1, m_2) \frac{p-1}{p} + \frac{p-1}{p^2} p^{-(m_1 + m_2)/2}$$
$$- \sum_{\ell \geq 1} c_\ell(m_1, m_2) \left(\frac{p-1}{p^{3/2}} Tr^*_{\ell+2}(p) + \frac{p-1}{p^2} p^{-\ell/2} \right). \tag{3.30}$$

If $m_1 + m_2$ is odd, or $p = 2$, then $Q^*(p^{m_1}, p^{m_2}) = 0$.

Lemma 3.4. *Let $p > 3$ and $m_1 \geq 2$ even. Then*

$$\widetilde{Q}^*(p^{m_1}, p^0) = -\frac{p-1}{p^{3/2}} Tr^*_{m_1+2}(p). \tag{3.31}$$

Proof. This follows immediately from (3.21) by specializing Lemma 3.3 since $c_\ell(m_1, 0) = 1$ for $\ell = m_1$ and 0 otherwise. □

Lemma 3.5. *Let $p > 3$ and $m_1 \geq 1$ odd. Then, for $m_1 \geq 3$,*

$$\widetilde{Q}^* \left(p^{m_1}, p^1 \right) = \frac{p-1}{p^2} p^{-(m_1-1)/2} + \frac{p-1}{p^{3/2}} \left(Tr^*_{m_1+1}(p) + Tr^*_{m_1+3}(p) \right)$$

and

$$\widetilde{Q}^*(p, p) = \frac{1-p}{p}.$$

Proof. From (3.22), we have that $\widetilde{Q}^*(p^{m_1}, p^1) = -Q^*(p^{m_1}, p)$, and then it follows immediately by specializing Lemma 3.3 that

$$\widetilde{Q}^*(p^{m_1}, p^1) = -c_0(m_1, 1) \frac{p-1}{p} + \sum_{\ell \geq 1} c_\ell(m_1, 1) \left[\frac{p-1}{p^{3/2}} Tr^*_{\ell+2}(p) + \frac{p-1}{p^2} p^{-\ell/2} \right]$$
$$- \frac{p-1}{p^2} p^{-(m_1+1)/2}. \tag{3.32}$$

For $m_1 \geq 1$, the recursion relation (3.25) gives

$$U_{m_1}(x) U_1(x) = 2x U_{m_1}(x) = U_{m_1+1}(x) + U_{m_1-1}(x)$$

and

$$c_\ell(m_1, 1) = \begin{cases} 1 & \text{for } \ell = m_1 - 1, m_1 + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

David *et al. Research in Number Theory* (2015) 1:6

Page 14 of 37

Replacing in (3.32), this gives the result for $m_1 \geq 3$. For $m_1 = 1$, we also use the fact that $Tr_4^*(p) = 0$. $\qquad\square$

Lemma 3.6. *If $p > 3$, and $m_1 \geq 2$ is even, then*

$$\widetilde{Q}^*(p^{m_1}, p^2) = -\frac{p-1}{p^{3/2}} Tr_{m_1+2}^*(p) - \frac{p-1}{p^2} p^{-m_1/2}. \qquad (3.34)$$

Furthermore,

$$\tilde{Q}^*(p^0, p^2) = \frac{(p-1)}{p}.$$

Proof. Let $m_1 \geq 2$. From (3.23), we have that

$$\widetilde{Q}^*(p^{m_1}, p^2) = \widetilde{Q}^*(p^{m_1}, p^0) - \frac{1}{p^2} \sum_{\substack{a,b \mod p \\ p \mid N_E}} \lambda_{a,b}(p_1^m).$$

For $p > 3$, one shows that

$$\sum_{\substack{a,b \mod p \\ p \mid N_E}} \lambda_{a,b}(p_1^m) = p^{-m_1/2}(p-1)$$

by parameterizing all pairs $(a, b) \in \mathbb{F}_p^2$ such that $\Delta \equiv 0 \mod p$ (see the proof of Proposition 4.2 in [29]). The result then follows from Lemma 3.4. If $m_1 = 0$, then

$$\tilde{Q}^*(p^0, p^2) = \frac{1}{p^2} \sum_{\substack{a,b \mod p \\ p \nmid N_E}} 1 = \frac{p(p-1)}{p^2}.$$

$\qquad\square$

Proof of Theorem 3.2. Starting from (3.28), we have

$$H(\alpha, \gamma) = E_2(\alpha, \gamma) E_3(\alpha, \gamma) \prod_{p>3} \left[1 + (1 - p^{-10})^{-1} \left(\sum_{\substack{m_1>0 \\ m_1 \text{ even}}} \frac{\widetilde{Q}^*(p^{m_1}, p^0)}{p^{m_1(\frac{1}{2}+\alpha)}} \right. \right.$$

$$\left. \left. + \frac{1}{p^{\frac{1}{2}+\gamma}} \sum_{\substack{m_1>0 \\ m_1 \text{ odd}}} \frac{\widetilde{Q}^*(p^{m_1}, p^1)}{p^{m_1(\frac{1}{2}+\alpha)}} + \frac{\widetilde{Q}^*(1, p^2)}{p^{1+2\gamma}} + \frac{1}{p^{1+2\gamma}} \sum_{\substack{m_1>0 \\ m_1 \text{ even}}} \frac{\widetilde{Q}^*(p^{m_1}, p^2)}{p^{m_1(\frac{1}{2}+\alpha)}} \right) \right].$$

Let $p > 3$. We consider each of the four terms in the Euler factors $E_p(\alpha, \gamma)$ separately. From Lemma 3.4, we have that

$$\sum_{\substack{m_1>0 \\ m_1 \text{ even}}} \frac{\widetilde{Q}^*(p^{m_1}, p^0)}{p^{m_1(\frac{1}{2}+\alpha)}} = \frac{-(p-1)}{p^{\frac{3}{2}}} \sum_{\substack{m_1>0 \\ m_1 \text{ even}}} \frac{Tr_{m_1+2}^*(p)}{p^{m_1(\frac{1}{2}+\alpha)}}. \qquad (3.35)$$

From Lemma 3.5, we have that

$$
\frac{1}{p^{1/2+\gamma}} \sum_{\substack{m_1 \geq 1 \\ m_1 \text{odd}}} \frac{\widetilde{Q}^*(p^{m_1}, p^1)}{p^{m_1(\frac{1}{2}+\alpha)}} = \frac{-(p-1)}{p^{2+\gamma+\alpha}} + \sum_{\substack{m_1 \geq 3 \\ m_1 \text{odd}}} \frac{p-1}{p^{2+\gamma+m_1(1+\alpha)}}
$$

$$
+ \frac{p-1}{p^{2+\gamma}} \sum_{\substack{m_1 \geq 9 \\ m_1 \text{odd}}} \frac{Tr^*_{m_1+1}(p) + Tr^*_{m_1+3}(p)}{p^{m_1(1/2+\alpha)}}.
$$

We compute that

$$
\sum_{\substack{m_1 \geq 3 \\ m_1 \text{odd}}} \frac{p-1}{p^{2+\gamma+m_1(1+\alpha)}} = \frac{p-1}{p^{3+\alpha+\gamma}} \sum_{m_1 \geq 1} \left(\frac{1}{p^{1+\alpha}} \right)^{2m_1} = \frac{p-1}{p^{3+\alpha+\gamma}(p^{2+2\alpha}-1)}.
$$

Finally, from Lemma 3.6 we have that

$$
\frac{\widetilde{Q}^*(1, p^2)}{p^{1+2\gamma}} = \frac{p-1}{p^{2+2\gamma}} \tag{3.36}
$$

and

$$
\sum_{\substack{m_1 > 0 \\ m_1 \text{ even}}} \frac{\widetilde{Q}^*(p^{m_1}, p^2)}{p^{m_1(\frac{1}{2}+\alpha)+1+2\gamma}} = \frac{-(p-1)}{p^{\frac{5}{2}+2\gamma}} \sum_{\substack{m_1 > 0 \\ m_1 \text{ even}}} \frac{Tr^*_{m_1+2}(p)}{p^{m_1(\frac{1}{2}+\alpha)}} - \frac{p-1}{p^{3+2\gamma}} \sum_{\substack{m_1 > 0 \\ m_1 \text{ even}}} \frac{1}{p^{m_1(1+\alpha)}}
$$

$$
= \frac{-(p-1)}{p^{\frac{5}{2}+2\gamma}} \sum_{\substack{m_1 > 0 \\ m_1 \text{ even}}} \frac{Tr^*_{m_1+2}(p)}{p^{m_1(\frac{1}{2}+\alpha)}} - \frac{p-1}{p^{3+2\gamma}(p^{2+2\alpha}-1)}.
$$

Since $Tr^*_j(p) = 0$ for $j < 12$, summing the four terms above and collecting terms gives

$$
H(\alpha, \gamma) = E_2(\alpha, \gamma) E_3(\alpha, \gamma) \prod_{p>3} \left[1 + (1-p^{-10})^{-1} \left(\frac{p-1}{p^{2+2\gamma}} - \frac{p-1}{p^{2+\alpha+\gamma}} \right. \right.
$$

$$
+ \frac{p-1}{p^{3+\gamma}(p^{2+2\alpha}-1)} \left(\frac{1}{p^\alpha} - \frac{1}{p^\gamma} \right) - \frac{p-1}{p^{\frac{3}{2}}} \left(\frac{1}{p^{1+2\gamma}} + 1 \right) \sum_{\substack{m_1 \geq 10 \\ m_1 \text{ even}}} \frac{Tr^*_{m_1+2}(p)}{p^{m_1(\frac{1}{2}+\alpha)}}
$$

$$
\left. \left. + \frac{p-1}{p^{2+\gamma}} \sum_{\substack{m_1 \geq 9 \\ m_1 \text{ odd}}} \frac{Tr^*_{m_1+1}(p) + Tr^*_{m_1+3}(p)}{p^{m_1(\frac{1}{2}+\alpha)}} \right) \right],
$$

and factoring out $\frac{p-1}{p}$ gives

$$
H(\alpha, \gamma) = E_2(\alpha, \gamma) E_3(\alpha, \gamma) \prod_{p>3} \left[1 + \left(1 - \frac{p^9-1}{p^{10}-1} \right) \left(\frac{1}{p^{1+2\gamma}} - \frac{1}{p^{1+\alpha+\gamma}} \right. \right.
$$

$$
+ \frac{1}{p^{2+\gamma}(p^{2+2\alpha}-1)} \left(\frac{1}{p^\alpha} - \frac{1}{p^\gamma} \right) - \left(\frac{1}{p^{\frac{3}{2}+2\gamma}} + \frac{1}{p^{\frac{1}{2}}} \right) \sum_{\substack{m_1 \geq 10 \\ m_1 \text{ even}}} \frac{Tr^*_{m_1+2}(p)}{p^{m_1(\frac{1}{2}+\alpha)}}
$$

$$
\left. \left. + sum_{\substack{m_1 \geq 9 \\ m_1 \text{ odd}}} \frac{Tr^*_{m_1+1}(p) + Tr^*_{m_1+3}(p)}{p^{m_1(\frac{1}{2}+\alpha)+1+\gamma}} \right) \right]. \tag{3.37}
$$

Finally, we compute that

$$\left(\frac{-1}{p^{\frac{3}{2}+2\gamma}} - \frac{1}{p^{\frac{1}{2}}}\right) \sum_{\substack{m_1 \geq 10 \\ m_1 \text{ even}}} \frac{Tr^*_{m_1+2}(p)}{p^{m_1(\frac{1}{2}+\alpha)}} + \sum_{\substack{m_1 \geq 9 \\ m_1 \text{ odd}}} \frac{Tr^*_{m_1+1}(p) + Tr^*_{m_1+3}(p)}{p^{m_1(\frac{1}{2}+\alpha)+1+\gamma}}$$

$$= \left(\frac{p^{1+2\alpha+\gamma} + p^\gamma - p^\alpha - p^{1+\alpha+2\gamma}}{p^{\frac{3}{2}+\alpha+2\gamma}}\right) \sum_{\substack{m_1 \geq 10 \\ m_1 \text{ even}}} \frac{Tr^*_{m_1+2}(p)}{p^{m_1(\frac{1}{2}+\alpha)}}.$$

Now, looking at the contributions of the Euler factors in (3.37), the term $p^{-(1+2\gamma)}$ contributes a pole, and the term $-p^{-(1+\alpha+\gamma)}$ contributes a zero, and we can write

$$H(\alpha, \gamma) = \frac{\zeta(1+2\gamma)}{\zeta(1+\alpha+\gamma)} A(\alpha, \gamma) \tag{3.38}$$

as required where $A(\alpha, \gamma)$ converges uniformly and absolutely for $\text{Re}(\alpha), \text{Re}(\gamma) > -1/4$ since $|Tr^*_j(p)| \leq 1$.

Finally, by setting $\alpha = \gamma = r$, we also have that

$$A(r, r) = H(r, r) = 1, \tag{3.39}$$

and in fact, each of the Euler factors $E_p(r, r) = 1$. If $p > 3$, it can be seen directly by setting $\alpha = \gamma = r$ in (3.37), as

$$\left(\frac{1}{p^{1+2r}} - \frac{1}{p^{1+2r}} + \frac{p^{-(2+2r)} - p^{-(2+2r)}}{p^{2+2r} - 1}\right.$$

$$\left. + \left(\frac{p^{1+3r} - p^{1+3r} + p^r - p^r}{p^{\frac{3}{2}+3r}}\right) \sum_{\substack{m_1 \geq 10 \\ m_1 \text{ even}}} \frac{Tr^*_{m_1+2}(p)}{p^{m_1(\frac{1}{2}+r)}}\right) = 0.$$

For $p = 2, 3$ we use a more general result, and show directly that for all Euler factors $E_p(\alpha, \beta)$ as given by (3.20), that $E_p(r, r) = 1$ because of the Hecke relations. This is done in Lemma 4.7 of the next section for the Euler factors in the second family, where we have no closed formula, and it shows the result for $E_2(r, r)$ and $E_3(r, r)$ by adapting the proof for this family. □

3.2 The ratios conjecture for the family $\mathcal{F}(X)$

By replacing each (m_1, m_2)-summand in (3.12) by its average over the family, we effectively replace $R_1(\alpha, \gamma)$ by $H(\alpha, \gamma)$ as given by (3.38). We also set

$$Y(\alpha, \gamma) := \frac{\zeta(1+2\gamma)}{\zeta(1+\alpha+\gamma)}. \tag{3.40}$$

We now consider the sum $R_2(\alpha, \gamma)$ of (3.13) coming from the dual sum of the approximate functional equation. Working similarly, replacing each (m_1, m_2)-summand in (3.13) by its average over the family, we rewrite $R_2(\alpha, \gamma)$ as

$$\omega_E \left(\frac{\sqrt{N}}{2\pi}\right)^{-2\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} Y(-\alpha, \gamma) A(-\alpha, \gamma). \tag{3.41}$$

Then, replacing each summand in (3.3), we get the Ratios Conjecture for our family.

David *et al. Research in Number Theory* (2015) 1:6

Page 17 of 37

Conjecture 3.7 (Ratios Conjecture). *Let $\varepsilon > 0$. Let $\alpha, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > -1/4$, $\mathrm{Re}(\gamma) \gg \frac{1}{\log X}$ and $\mathrm{Im}(\alpha), \mathrm{Im}(\gamma) \ll_{\varepsilon} X^{1-\varepsilon}$. Then*

$$
\frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{L\left(\frac{1}{2} + \alpha, E\right)}{L\left(\frac{1}{2} + \gamma, E\right)} = \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \left[Y(\alpha, \gamma) A(\alpha, \gamma) \right.
$$
$$
\left. + \omega_E \left(\frac{\sqrt{N_E}}{2\pi}\right)^{-2\alpha} \frac{\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)} Y(-\alpha, \gamma) A(-\alpha, \gamma) \right] + O\left(X^{-1/2 + \varepsilon}\right),
$$

where $Y(\alpha, \gamma)$ is defined in (3.40) and $A(\alpha, \gamma)$ in (3.38).

We remark that the error term $O\left(X^{-1/2+\varepsilon}\right)$ is part of the statement of the Ratios Conjecture, and the power on X is not suggested by any of the steps leading to the main expression in Conjecture 3.7, and the original motivation for the exponent in the error term is the general philosophy of square-root cancelation. The quality of this error term was tested in recent work of Fiorilli and Miller [6], who uncover new lower order terms in the one-level density for the family of Dirichlet L-functions of modulus q, and also obtain some result for the natural accuracy of the error term in the Ratios Conjecture. Other papers investigating the quality of the error term of the Ratios Conjecture include [7,11,20,21].

The lower bound for $\mathrm{Re}(\gamma)$ and the upper bound for $\mathrm{Im}(\alpha), \mathrm{Im}(\gamma)$ are also part of the statement of the Ratios Conjecture, and should be thought as reasonable conditions under which the Conjecture 3.7 should hold. For more details, we refer the reader to [4] (see for example the conditions (2.11b) and (2.11c) on page 6). Ignoring issues about the error term and uniformity, there should of course be a condition of the type $\mathrm{Re}(\gamma) \geq \delta$ for some $\delta > 0$.

To get the one-level density for our family, we have to differentiate the result of Conjecture 3.7 with respect to α and use (3.2). We then obtain

Theorem 3.8. *Let $\varepsilon > 0$, and $r \in \mathbb{C}$. Assuming the Ratios Conjecture 3.7, $\mathrm{Re}(r) \gg \frac{1}{\log X}$ and $\mathrm{Im}(r) \ll X^{1-\varepsilon}$, we have*

$$
\frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{L'\left(\frac{1}{2} + r, E\right)}{L\left(\frac{1}{2} + r, E\right)} = \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \left[-\frac{\zeta'}{\zeta}(1 + 2r) + A_{\alpha}(r, r) \right.
$$
$$
\left. - \omega_E \left(\frac{\sqrt{N_E}}{2\pi}\right)^{-2r} \frac{\Gamma(1 - r)}{\Gamma(1 + r)} \zeta(1 + 2r) A(-r, r) \right] + O(X^{-1/2 + \varepsilon}),
$$

where $A_{\alpha}(r, r)$ is defined in (3.42).

Proof. We set

$$
A_{\alpha}(r, r) := \frac{\partial}{\partial \alpha} A(\alpha, \gamma) \bigg|_{\alpha = \gamma = r}. \tag{3.42}
$$

Then

$$
\frac{\partial}{\partial \alpha} Y(\alpha, \gamma) A(\alpha, \gamma) \bigg|_{\alpha = \gamma = r} = -\frac{\zeta'}{\zeta}(1 + 2r) + A_{\alpha}(r, r), \tag{3.43}
$$

David *et al. Research in Number Theory* (2015) 1:6

Page 18 of 37

and

$$\frac{\partial}{\partial \alpha}\left[\omega_E\left(\frac{\sqrt{N}}{2\pi}\right)^{-2\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}Y(-\alpha,\gamma)A(-\alpha,\gamma)\right]_{\alpha=\gamma=r}$$

$$=\omega_E\left(\frac{\sqrt{N}}{2\pi}\right)^{-2\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}A(-\alpha,\gamma)\frac{d}{d\alpha}\left[Y(-\alpha,\gamma)\right]_{\alpha=\gamma=r}$$

$$=-\omega_E\left(\frac{\sqrt{N}}{2\pi}\right)^{-2r}\frac{\Gamma(1-r)}{\Gamma(1+r)}\zeta(1+2r)A(-r,r). \tag{3.44}$$

Replacing (3.43) and (3.44) in Conjecture 3.7, and using $\alpha = \gamma = r$, we get the desired formula for

$$\frac{1}{|\mathcal{F}(X)|}\sum_{E\in\mathcal{F}(X)}\frac{L'\left(\frac{1}{2}+r,E\right)}{L\left(\frac{1}{2}+r,E\right)}.$$

We still have to justify that the error term remains the same under differentiation. We claim that

$$\frac{1}{|\mathcal{F}(X)|}\left\{\sum_{E\in\mathcal{F}(X)}\left[\frac{L'\left(\frac{1}{2}+r,E\right)}{L\left(\frac{1}{2}+r,E\right)}-\left[-\frac{\zeta'}{\zeta}(1+2r)+A_\alpha(r,r)\right.\right.\right.$$

$$\left.\left.\left.-\omega_E\left(\frac{\sqrt{N_E}}{2\pi}\right)^{-2r}\frac{\Gamma(1-r)}{\Gamma(1+r)}\zeta(1+2r)A(-r,r)\right]\right]\right\}=O\left(X^{-1/2+\varepsilon}\right). \tag{3.45}$$

Let the left hand side of (3.45) be denoted by $R(\alpha)$. Let $\alpha_0 \in \mathbb{C}$ such that $\mathrm{Re}(\alpha_0) > 0$. Assume R is analytic in a neighborhood of α_0 and let C be a circle of radius $r_0 \approx 1$ around α_0. Then by Cauchy's integral formula we have

$$|R'(\alpha_0)|=\left|\frac{1}{2\pi i}\oint_C\frac{R(\alpha)}{(\alpha-\alpha_0)^2}d\alpha\right|\leq\max_{\alpha\in C}\left|\frac{R(\alpha)}{2\pi i(\alpha-\alpha_0)^2}\right|(2\pi r_0)$$

$$\leq\max_{\alpha\in C}\left|\frac{1}{(\alpha-\alpha_0)^2}\right|O\left(X^{-1/2+\varepsilon}\right)=O\left(X^{-1/2+\varepsilon}\right) \tag{3.46}$$

from our assumption in Conjecture 3.7. This completes the proof. $\qquad\square$

3.3 Proof of Theorem 2.1 and Corollary 2.2

We now use the Ratios Conjecture as stated in Theorem 3.8 to rewrite the one-level density $D(\mathcal{F};\phi,X)$ for the family of all elliptic curves. As in [4], we assume that $\phi(s)$ is holomorphic in the strip $|\mathrm{Im}(s)| < 2$, is real on the real line and even, and that $\phi(x) \ll 1/(1+x^2)$ as $x \to \infty$. With the change of variable $s \to 1-s$ in (3.2) (noting that ϕ is even), we get that

$$\frac{1}{|\mathcal{F}(X)|}\sum_{E\in\mathcal{F}(X)}\frac{1}{2\pi i}\int_{(1-c)}\frac{L'(s,E)}{L(s,E)}\phi(-i(s-\tfrac{1}{2}))ds$$

$$=\frac{1}{|\mathcal{F}(X)|}\sum_{E\in\mathcal{F}(X)}\frac{1}{2\pi i}\int_{(c)}\frac{L'(1-s,E)}{L(1-s,E)}\phi\left(-i\left(s-\tfrac{1}{2}\right)\right)ds. \tag{3.47}$$

The functional equation

$$L(s,E)=\omega_E X(s,E)L(1-s,E) \tag{3.48}$$

David *et al. Research in Number Theory* (2015) 1:6

Page 19 of 37

where $X(s, E)$ is defined in (3.7) gives

$$\frac{L'(s, E)}{L(s, E)} = \frac{X'(s, E)}{X(s, E)} - \frac{L'(1 - s, E)}{L(1 - s, E)}. \tag{3.49}$$

Using (3.49) in (3.47) gives that the second integral of (3.2) can be rewritten as

$$\frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{1}{2\pi i} \int_{(c)} \left[\frac{L'(s, E)}{L(s, E)} - \frac{X'(s, E)}{X(s, E)} \right] \phi\left(-i\left(s - \tfrac{1}{2}\right)\right) ds,$$

and then

$$D(\mathcal{F}; \phi, X)$$
$$= \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'(s, E)}{L(s, E)} \phi\left(-i\left(s - \tfrac{1}{2}\right)\right) ds$$
$$= \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{1}{2\pi i} \int_{(c)} \left[2\frac{L'(s, E)}{L(s, E)} - \frac{X'(s, E)}{X(s, E)} \right] \phi\left(-i\left(s - \tfrac{1}{2}\right)\right) ds$$
$$= \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{1}{2\pi i} \int_{(c-1/2)} \left[2\frac{L'(1/2 + r, E)}{L(1/2 + r, E)} - \frac{X'(1/2 + r, E)}{X(1/2 + r, E)} \right] \phi(-ir) dr$$

with the change of variable $s = 1/2 + r$. We bring the summation inside the integral and substitute

$$\frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{L'\left(\tfrac{1}{2} + r, E\right)}{L\left(\tfrac{1}{2} + r, E\right)}$$

with the expression of Theorem 3.8, and we pull the summation back outside of the integral. This gives

$$D(\mathcal{F}; \phi, X)$$
$$= \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{1}{2\pi i} \int_{(c-1/2)} \left[-2\frac{\zeta'}{\zeta}(1 + 2r) + 2A_\alpha(r, r) \right.$$
$$\left. - 2\omega_E \left(\frac{\sqrt{N_E}}{2\pi} \right)^{-2r} \frac{\Gamma(1 - r)}{\Gamma(1 + r)} \zeta(1 + 2r) A(-r, r) - \frac{X'(1/2 + r, E)}{X(1/2 + r, E)} \right] \phi(-ir) dr$$
$$+ O\left(X^{-1/2 + \varepsilon} \right).$$

We now move the integral from $\mathrm{Re}(r) = c - 1/2 = c'$ to $\mathrm{Re}(r) = 0$ by integrating over the rectangle R from $c' - iT$ to $c' + iT$ to iT to $-iT$ and back to $c' - iT$, and letting $T \to \infty$. The two horizontal integrals tend to 0, and we only have to consider the vertical integrals. We have to distinguish 2 cases, as the integrand

$$F(r) = -\frac{X'(1/2 + r, E)}{X(1/2 + r, E)} - 2\frac{\zeta'}{\zeta}(1 + 2r) + 2A_\alpha(r, r)$$
$$- 2\omega_E \left(\frac{\sqrt{N_E}}{2\pi} \right)^{-2r} \frac{\Gamma(1 - r)}{\Gamma(1 + r)} \zeta(1 + 2r) A(-r, r)$$

has a pole at $r = 0$ with residue 2 on the boundary of the rectangle R when $\omega_E = -1$. We have that the function

$$F(r) - \frac{1 - \omega_E}{r}$$

David *et al. Research in Number Theory* (2015) 1:6

Page 20 of 37

is analytic inside and on the contour R. Hence from Cauchy's Theorem we have that

$$
\begin{aligned}
D(\mathcal{F}; \phi, X) = \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Bigg[& -2\frac{\zeta'}{\zeta}(1 + 2it) + 2A_\alpha(it, it) \\
& - 2\omega_E \left(\frac{\sqrt{N_E}}{2\pi} \right)^{-2it} \frac{\Gamma(1 - it)}{\Gamma(1 + it)} \zeta(1 + 2it) A(-it, it) \\
& + 2\log\left(\frac{\sqrt{N_E}}{2\pi} \right) + \frac{\Gamma'}{\Gamma}(1 - it) + \frac{\Gamma'}{\Gamma}(1 + it) - \frac{1 - \omega_E}{it} \Bigg] \phi(t)\, dt \\
& + \frac{1}{|\mathcal{F}(X)|} \sum_{\substack{E \in \mathcal{F}(X) \\ \omega_E = -1}} \frac{1}{2\pi i} \int_{(c')} \frac{2\phi(-ir)}{r}\, dr + O(X^{-1/2 + \varepsilon}),
\end{aligned}
$$

where we used the change of variable $r = it$ in the first integral, and

$$
\frac{X'(1/2 + it, E)}{X(1/2 + it, E)} = -2\log\left(\frac{\sqrt{N_E}}{2\pi} \right) - \frac{\Gamma'}{\Gamma}(1 - it) - \frac{\Gamma'}{\Gamma}(1 + it).
$$

If $\omega_E = 1$, then $1 - \omega_E = 0$, and the second sum is zero. If $\omega_E = -1$, then by Cauchy's Theorem

$$
2\phi(0) = \frac{1}{2\pi i} \int_{(c')} \frac{2\phi(-ir)}{r}\, dr - \frac{1}{2\pi i} \int_{(-c')} \frac{2\phi(-ir)}{r}\, dr = \frac{2}{2\pi i} \int_{(c')} \frac{2\phi(-ir)}{r}\, dr,
$$

which completes the proof of Theorem 2.1.

We now proceed to the proof of Corollary 2.2. We first make the change of variable

$$
\tau = \frac{tL}{\pi} \quad \text{with } L = \log\left(\frac{\sqrt{X}}{2\pi e} \right), \tag{3.50}
$$

where L is chosen so that for $N_E \approx X$, the sequence γ_E of low-lying zeros arising from (say) $\gamma_E \leq 1$ has essentially constant mean spacing one. Recall that by a Riemann-von Mangold type theorem, as in for example ([13], Thm. 5.8), $L(s, E)$ has approximately $\log(N_E/(2\pi e)^2)/2\pi$ zeros in the region $0 < \Im(\rho_E) \leq 1$. We then define the normalized test function ψ by

$$
\phi(t) = \psi\left(\frac{tL}{\pi} \right). \tag{3.51}
$$

We know from the work of ([29], Lemma 5.1) that the *conductor condition* holds for the family $\mathcal{F}(X)$, and we can write

$$
\frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \log\left(\frac{\sqrt{N_E}}{2\pi} \right) = \log\left(\frac{\sqrt{X}}{2\pi} \right) \left\{ 1 + O\left(\frac{1}{\log X} \right) \right\}. \tag{3.52}
$$

We now make the standard hypothesis that that the root number ω_E is equidistributed in the family of all elliptic curves, i.e. half of the elliptic curves given by (3.1) have $\omega_E = 1$, and half have $\omega_E = -1$, as predicted by the Katz-Sarnak philosophy [14,15]. The natural expectation is that the root number ω_E is equidistributed in general families of elliptic curves, when the family has at least one place of multiplicative reduction. [b] This was investigated by Helfgott in his Ph. D. thesis [8,9], and he showed that under two standard arithmetical conjectures, this is indeed the case. Helfgott also showed that the equidistribution of the root number holds unconditionally in some families of curves, as for example the family of elliptic curves over $\mathbb{Q}$ with rational 2-torsion $y^2 = x(x + a)(a + b)$, for $a, b \in \mathbb{Z}$.

David *et al. Research in Number Theory* (2015) 1:6

Page 21 of 37

Then, using the change of variables (3.50), we rewrite the statement of Theorem 2.1 as

$$\frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \sum_{\gamma_E} \psi\left(\frac{\gamma_E L}{\pi}\right) \sim \int_{-\infty}^{\infty} \psi(\tau) g(\tau)\, d\tau + \frac{\phi(0)}{2}$$

$$= \int_{-\infty}^{\infty} \psi(\tau) h(\tau)\, d\tau,$$

where

$$g(\tau) = h(\tau) - \frac{1}{2}\delta_0(\tau)$$

$$= \frac{1}{2L}\left[2\log\left(\frac{\sqrt{X}}{2\pi}\right) + \frac{\Gamma'}{\Gamma}\left(1 + \frac{\pi i\tau}{L}\right) + \frac{\Gamma'}{\Gamma}\left(1 - \frac{\pi i\tau}{L}\right) \right.$$

$$\left. -2\frac{\zeta'}{\zeta}\left(1 + \frac{2\pi i\tau}{L}\right) + 2A_\alpha\left(\frac{\pi i\tau}{L}, \frac{\pi i\tau}{L}\right) - \frac{L}{\pi i\tau} \right].$$

We define the partial derivatives $A_{\alpha\alpha}(r,r) := \left.\frac{d}{d\alpha}A_\alpha(\alpha,\gamma)\right|_{\alpha=\gamma=r}$ and $A_{\alpha\gamma}(r,r) := \left.\frac{d}{d\gamma}A_\alpha(\alpha,\gamma)\right|_{\alpha=\gamma=r}$ and denote the Stieltjes constants by γ_n. Then the Taylor expansion of $h(\tau)$ in L^{-1} is

$$h(\tau) = \frac{1}{2L}\left[2L - 2\gamma_0 - 2\left(\frac{-L}{2\pi i\tau} + \gamma_0 - (\gamma_0^2 + \gamma_1)\frac{2\pi i\tau}{L}\right) \right.$$

$$\left. + 2\left(A_\alpha(0,0) + (A_{\alpha\alpha}(0,0) + A_{\alpha\gamma}(0,0))\frac{\pi i\tau}{L}\right) - \frac{L}{\pi i\tau} + O\left(L^{-2}\right) \right] + \frac{1}{2}\delta_0(\tau)$$

$$= 1 + \frac{1}{2}\delta_0(\tau) + \frac{A_\alpha(0,0) - 2\gamma_0}{L} + \frac{\left(A_{\alpha\alpha}(0,0) + A_{\alpha\gamma}(0,0) + 2\left(\gamma_0^2 + \gamma_1\right)\right)\pi i\tau}{L^2}$$

$$+ O\left(\frac{1}{L^3}\right).$$

Then, the leading terms for the one-level scaling density associated to the families of all elliptic curves give $\mathcal{W}(\tau) = 1 + \frac{1}{2}\delta_0(\tau)$ which corresponds to the density $\mathcal{W}(O)(\tau)$ associated with the orthogonal group O as predicted by the conjectures of Katz and Sarnak. We also get lower order terms for the one-level scaling density which are particular to this family, and could be used to refine experimental statistics for small conductor. This completes the proof of Corollary 2.2.

4 A one-parameter family of elliptic curves

We now consider another family of elliptic curves, the one-parameter family of elliptic curves

$$E_t : y^2 = x^3 + tx^2 - (t+3)x + 1. \tag{4.1}$$

This family was first studied by Washington [31], who proved that the rank of E_t is odd for $t^2 + 3t + 9$ square-free, assuming the finiteness of the Tate-Shafarevic group. Rizzo [24] then proved that the root number $W(E_t)$ is equal to -1 for all $t \in \mathbb{Z}$ using the tables of local root numbers due to Rohrlich and Halberstadt. The one-level density for this family was also studied by Miller [17,18]. The discriminant of the curves E_t is $\Delta(t) = 2^4\left(t^2 + 3t + 9\right)^2$. Replacing t with $12t + 1$ gives $\Delta(12t+1) = 2^4(144t^2 + 60t + 13)^2$.

David *et al. Research in Number Theory* (2015) 1:6

Page 22 of 37

As proven in [18], if $144t^2 + 60t + 13$ is square-free then the conductor is $C(t) = 2^3(t^2 + 3t + 9)^2$.

In this section, we study the one-level density of the family

$$\mathcal{F}_1(X) = \{E_t : t \leq X^{\frac{1}{4}}\}. \tag{4.2}$$

Let

$$L(s, E_t) = \sum_{n=1}^{\infty} \frac{\lambda_t(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_t(p)}{p^s} + \frac{\psi_t(p)}{p^{2s}}\right)^{-1} \tag{4.3}$$

denote the L-function attached to E_t where ψ_t is the principal Dirichlet character modulo the conductor $C(t)$ of E, i.e.,

$$\psi_t(p) = \begin{cases} 1 & \text{if } p \nmid C(t), \\ 0 & \text{if } p \mid C(t). \end{cases}$$

For $p \neq 2$, $\lambda_t(p)$ is given by

$$\lambda_t(p) = -\frac{1}{\sqrt{p}} \sum_{x \bmod p} \left(\frac{x^3 + tx^2 - (t+3)x + 1}{p}\right).$$

If $p = 2$ then E_t has a cusp and $\lambda_t(2^k) = 0$ for all positive k. We recall that $\lambda_t(n)$ are multiplicative, and prime powers are computed by the Hecke relations

$$\lambda_t(p^j) = \begin{cases} U_j\left(\frac{\lambda_t(p)}{2}\right) & \text{if } (p, C(t)) = 1, \\ \lambda_t^j(p) & \text{if } (p, C(t)) > 1, \end{cases}$$

where $U_j(x)$ are the Chebyshev polynomials.

As in the previous section, we use the principal and the dual sums of the approximate functional Eq. 3.6 at $s = 1/2 + \alpha$, ignoring questions of convergence or error terms. The principal sum is

$$\sum_n \frac{\lambda_t(n)}{n^{\frac{1}{2}+\alpha}} \tag{4.4}$$

and since the sign of the functional equation is always negative, the dual sum is

$$-X_t\left(\frac{1}{2} + \alpha\right) \sum_n \frac{\lambda_t(n)}{n^{\frac{1}{2}-\alpha}}, \tag{4.5}$$

where

$$X_t(s) = \frac{\Gamma\left(\frac{3}{2} - s\right)}{\Gamma\left(\frac{1}{2} + s\right)} \left(\frac{\sqrt{C(t)}}{2\pi}\right)^{1-2s}. \tag{4.6}$$

Finally, we write

$$\frac{1}{L(s, E_t)} = \prod_p \left(1 - \frac{\lambda_t(p)}{p^s} + \frac{\psi_t(p)}{p^{2s}}\right) = \sum_{n=1}^{\infty} \frac{\mu_t(n)}{n^s}, \tag{4.7}$$

where μ_t is multiplicative and given by

$$\mu_t(p^k) = \begin{cases} -\lambda_t(p) & \text{if } k = 1, \\ \psi_t(p) & \text{if } k = 2, \\ 0 & \text{if } k > 2. \end{cases} \tag{4.8}$$

We are now ready to derive the L-function Ratios Conjecture for this family following the same recipe that was used in the first family. We keep in this section all the notation

David *et al. Research in Number Theory* (2015) 1:6

Page 23 of 37

of the previous section, but the objects are now attached to the new family. Using (4.4), (4.5) and (4.7), we set

$$R_1(\alpha, \gamma) := \frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \sum_{m_1, m_2} \frac{\lambda_t(m_1)\mu_t(m_2)}{m_1^{\frac{1}{2}+\alpha} m_2^{\frac{1}{2}+\gamma}}, \tag{4.9}$$

$$R_2(\alpha, \gamma) := \frac{-1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} X_t \left(\frac{1}{2} + \alpha\right) \sum_{m_1, m_2} \frac{\lambda_t(m_1)\mu_t(m_2)}{m_1^{\frac{1}{2}+\alpha} m_2^{\frac{1}{2}+\gamma}}. \tag{4.10}$$

and we approximate

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \frac{L(1/2 + \alpha, E_t)}{L(1/2 + \gamma, E_t)} \tag{4.11}$$

by $R_1(\alpha, \gamma) + R_2(\alpha, \gamma)$.

4.1 Average of the Fourier coefficients over the family

As in Section 3, the main step to obtain the Ratios Conjecture for this family is to replace each (m_1, m_2)-summand in (4.9) by its average over the family.

Lemma 4.1. *Let $m_1, m_2 \geq 0$ be fixed integers, and let*

$$\widetilde{Q}^*(m_1, m_2) := \frac{1}{m^*} \sum_{t \bmod m^*} \lambda_t(m_1)\mu_t(m_2)$$

where $m^* = \prod_{p|m_1 m_2} p$. Then

$$\lim_{X \to \infty} \frac{1}{|\mathcal{F}(X)|} \sum_{E \in \mathcal{F}(X)} \lambda_t(m_1)\mu_t(m_2) = \widetilde{Q}^*(m_1, m_2).$$

Furthermore, $\widetilde{Q}^*(m_1, m_2)$ is multiplicative.

Proof. This is completely similar to the proof of Lemma 3.1. □

Replacing each term in (4.9) by its average value $\widetilde{Q}^*(m_1, m_2)$, and using Lemma 4.1, we are led as before to consider

$$H(\alpha, \gamma) := \sum_{m_1, m_2} \frac{\widetilde{Q}^*(m_1, m_2)}{m_1^{\frac{1}{2}+\alpha} m_2^{\frac{1}{2}+\gamma}} = \prod_p \sum_{m_1, m_2} \frac{\widetilde{Q}^*(p^{m_1}, p^{m_2})}{p^{m_1(\frac{1}{2}+\alpha)+m_2(\frac{1}{2}+\gamma)}}. \tag{4.12}$$

As in the previous case, we switched notation, and we are now using m_1, m_2 for the exponents of the prime powers. By the definition of the Möebius function in (4.8) only the terms with $m_2 = 0, 1, 2$ in (4.12) contribute. So we have

$$\sum_{m_1, m_2} \frac{\widetilde{Q}^*(m_1, m_2)}{p^{m_1(\frac{1}{2}+\alpha)+m_2(\frac{1}{2}+\gamma)}}$$

$$= \sum_{m_1 \geq 0} \frac{\widetilde{Q}^*(p^{m_1}, p^0)}{p^{m_1(\frac{1}{2}+\alpha)}} + \sum_{m_1 \geq 0} \frac{\widetilde{Q}^*(p^{m_1}, p^1)}{p^{m_1(\frac{1}{2}+\alpha)+\frac{1}{2}+\gamma}} + \sum_{m_1 \geq 0} \frac{\widetilde{Q}^*(p^{m_1}, p^2)}{p^{m_1(\frac{1}{2}+\alpha)+1+2\gamma}} \tag{4.13}$$

David *et al. Research in Number Theory* (2015) 1:6

Page 24 of 37

where

$$\widetilde{Q}^*(p^{m_1}, p^0) = \frac{1}{p} \sum_{t \bmod p} \lambda_t(p^{m_1}),$$

$$\widetilde{Q}^*(p^{m_1}, p^1) = -\frac{1}{p} \sum_{t \bmod p} \lambda_t(p^{m_1})\lambda_t(p),$$

$$\widetilde{Q}^*(p^{m_1}, p^2) = \frac{1}{p} \sum_{\substack{t \bmod p \\ p \nmid C(t)}} \lambda_t(p^{m_1}).$$

Let $\chi_4(n)$ denote the non-principal character modulo 4.

Lemma 4.2. *For $p > 2$ we have that*

$$\widetilde{Q}^*(p, 1) = -\widetilde{Q}^*(1, p) = -\left(\frac{1 + \chi_4(p)}{\sqrt{p}}\right). \tag{4.14}$$

Proof. We have

$$\widetilde{Q}^*(p, 1) = \frac{1}{p} \sum_{t \bmod p} \lambda_t(p) = -\widetilde{Q}^*(1, p)$$

$$= \frac{1}{p} \sum_{t \bmod p} \left(\frac{-1}{\sqrt{p}} \sum_{x \bmod p} \left(\frac{(x^2 - x)t + (x^3 - 3x + 1)}{p}\right)\right)$$

$$= \frac{-1}{p^{\frac{3}{2}}} \sum_{x \bmod p} \left(\sum_{t \bmod p} \left(\frac{(x^2 - x)t + (x^3 - 3x + 1)}{p}\right)\right).$$

We have that

$$\sum_{t \bmod p} \left(\frac{(x^2 - x)t + (x^3 - 3x + 1)}{p}\right) = \begin{cases} p\left(\frac{x^3 - 3x + 1}{p}\right) & \text{if } x^2 \equiv x \bmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\widetilde{Q}^*(p, 1) = \frac{-1}{\sqrt{p}}\left(\left(\frac{1}{p}\right) + \left(\frac{-1}{p}\right)\right) = \frac{-(1 + \chi_4(p))}{\sqrt{p}}.$$

$\square$

Lemma 4.3. *We have for $p > 2$ that*

$$\widetilde{Q}^*(p, p) = -1 + O\left(\frac{1}{p}\right). \tag{4.15}$$

Proof. We have that

$$\widetilde{Q}^*(p, p) = \frac{-1}{p} \sum_{t \bmod p} \left(\frac{-1}{\sqrt{p}} \sum_{x \bmod p} \left(\frac{(x^2 - x)t + (x^3 - 3x + 1)}{p}\right)\right)^2$$

$$= \frac{-1}{p^2} \sum_{t,x,y \bmod p} \left(\frac{(x^2 - x)t + (x^3 - 3x + 1)}{p}\right)\left(\frac{(y^2 - y)t + (y^3 - 3y + 1)}{p}\right).$$

David *et al. Research in Number Theory* (2015) 1:6

Page 25 of 37

Let

$$a := (x^2 - x)(y^2 - y),$$
$$b := (x^2 - x)(y^3 - 3y + 1) + (x^3 - 3x + 1)(y^2 - y),$$
$$c := (x^3 - 3x + 1)(y^3 - 3y + 1),$$
$$d := (x - y)(xy - x + 1)(xy - y + 1),$$

then

$$b^2 - 4ac = \left[(x^2 - x)(y^3 - 3y + 1) - (x^3 - 3x + 1)(y^2 - y) \right]^2 = d^2.$$

From ([26], Exercise 1.1.9) we have for $a \not\equiv 0 \bmod p$ that

$$\sum_{t \bmod p} \left(\frac{at^2 + bt + c}{p} \right) = \begin{cases} \left(\frac{a}{p} \right)(p - 1) & \text{if } b^2 - 4ac \equiv 0 \bmod p, \\ -\left(\frac{a}{p} \right) & \text{if } b^2 - 4ac \not\equiv 0 \bmod p. \end{cases} \tag{4.16}$$

Now let

$$S := \sum_{\substack{x,y \bmod p \\ a \not\equiv 0 \bmod p \\ d \equiv 0 \bmod p}} \left(\frac{(x^2 - x)(y^2 - y)}{p} \right) = \sum_{\substack{x,y \bmod p \\ d \equiv 0 \bmod p}} \left(\frac{(x^2 - x)(y^2 - y)}{p} \right).$$

If $a \equiv 0 \bmod p$ then there are four cases to consider, $(x, y) = (0, 0), (0, 1), (1, 0)$ or $(1, 1)$. In all 4 cases $b \equiv 0 \bmod p$. Then we have that

$$\begin{aligned} \widetilde{Q}^*(p, p) = &\frac{-S}{p} + \frac{1}{p^2} \sum_{\substack{x,y \bmod p \\ a \not\equiv 0 \bmod p \\ d \not\equiv 0 \bmod p}} \left(\frac{(x^2 - x)(y^2 - y)}{p} \right) \\ &- \frac{1}{p} \sum_{\substack{x,y \bmod p \\ a \equiv b \equiv 0 \bmod p}} \left(\frac{(x^3 - 3x + 1)(y^3 - 3y + 1)}{p} \right) \\ = &\frac{-1}{p} S + \frac{1}{p^2} \sum_{\substack{x,y \bmod p \\ a \not\equiv 0 \bmod p}} \left(\frac{(x^2 - x)(y^2 - y)}{p} \right) \\ &- \frac{1}{p} \sum_{\substack{x,y \bmod p \\ a \equiv b \equiv 0 \bmod p}} \left(\frac{(x^3 - 3x + 1)(y^3 - 3y + 1)}{p} \right). \tag{4.17} \end{aligned}$$

First we consider the third sum in (4.17) and compute

$$\sum_{\substack{x,y \bmod p \\ a \equiv b \equiv 0 \bmod p}} \left(\frac{(x^3 - 3x + 1)(y^3 - 3y + 1)}{p} \right)$$
$$= \sum_{x,y \equiv 0,1 \bmod p} \left(\frac{(x^3 - 3x + 1)(y^3 - 3y + 1)}{p} \right) = 2 \left[1 + \left(\frac{-1}{p} \right) \right]. \tag{4.18}$$

David *et al. Research in Number Theory* (2015) 1:6

Page 26 of 37

Next we consider the second sum in (4.17) and compute

$$\sum_{\substack{x,y \bmod p \\ a \not\equiv 0 \bmod p}} \left(\frac{(x^2-x)(y^2-y)}{p} \right) = \sum_{x,y \bmod p} \left(\frac{(x^2-x)(y^2-y)}{p} \right)$$

$$= \left[\sum_{x \bmod p} \left(\frac{x^2-x}{p} \right) \right]^2 = \left[-\left(\frac{1}{p} \right) \right]^2 = 1. \qquad (4.19)$$

If $d = (x-y)(xy-x+1)(xy-y+1) \equiv 0 \bmod p$ then either $x \equiv y \bmod p$, $xy-x+1 \equiv 0 \bmod p$ or $xy-y+1 \equiv 0 \bmod p$. All three equations are satisfied when $x^2-x+1 \equiv 0 \bmod p$, which has at most two solutions $\bmod\, p$. This gives

$$S = \sum_{x \bmod p} \left(\frac{(x^2-x)^2}{p} \right)$$

$$+ \left[\sum_{\substack{xy-x+1\equiv 0 \bmod p \\ x\equiv\not\equiv 0 \ \bmod p}} + \sum_{\substack{xy-y+1\equiv 0 \bmod p \\ x\equiv\not\equiv 0 \ \ \bmod p \\ xy-x+1\not\equiv 0 \bmod p}} \right] \left(\frac{(x^2-x)(y^2-y)}{p} \right). \qquad (4.20)$$

For the the second two sums in (4.20) for each $x \bmod p$ there is at most one y satisfying the equation $x \equiv y \bmod p$, $xy-x+1 \equiv 0 \bmod p$ or $xy-y+1 \equiv 0 \bmod p$ which gives $S = p + O(1)$.

Then substituting (4.18), (4.19) in (4.17) gives Lemma 4.3. □

We remark that for any one-parameter family of elliptic curves over $\mathbb{Q}(t)$ with non-constant $j(E_t)$, we have the estimate $\widetilde{Q}^*(p,p) = -1 + O\left(p^{-1/2}\right)$ due to Michel [16], which is used for example in ([18], Section 6.1.3) for the same family.

Lemma 4.4. *Let $p > 2$. Then,*

$$\widetilde{Q}^*(1,p^2) = 1 + O\left(\frac{1}{p} \right) \quad and \quad \widetilde{Q}^*(p^2,1) = O\left(\frac{1}{p} \right).$$

Proof. For $p > 2$, we compute

$$\widetilde{Q}^*(1,p^2) = \frac{1}{p} \sum_{\substack{t \bmod p \\ p \nmid C(t)}} 1 = 1 - \frac{1}{p} \sum_{\substack{t \bmod p \\ p \mid C(t)}} 1.$$

Since there are at most 2 solutions to the congruence $C(t) \equiv 0 \bmod p$, this gives

$$\widetilde{Q}^*(1,p^2) = 1 + O\left(\frac{1}{p} \right).$$

We also have that

$$\widetilde{Q}^*(p^2,1) = \frac{1}{p} \sum_{t \bmod p} \lambda_t\left(p^2\right),$$

and from (3.5)

$$\lambda_t(p^2) = \begin{cases} U_2\left(\frac{\lambda_t(p)}{2} \right) = \lambda_t^2(p) - 1 & \text{if } (p, C(t)) = 1, \\ \lambda_t^2(p) & \text{if } (p, C(t)) > 1. \end{cases}$$

Hence we compute

$$\widetilde{Q}^*(p^2, 1) = \frac{1}{p} \sum_{\substack{t \bmod p \\ p \nmid C(t)}} (\lambda_t^2(p) - 1) + \frac{1}{p} \sum_{\substack{t \bmod p \\ p \mid C(t)}} \lambda_t^2(p) = -\widetilde{Q}^*(p, p) - \widetilde{Q}^*(1, p^2)$$

$$= -\left(-1 + O\left(\frac{1}{p}\right)\right) - \left(1 + O\left(\frac{1}{p}\right)\right) = O\left(\frac{1}{p}\right).$$

□

Finally, if $p = 2$, we have $\widetilde{Q}^*(2^{m_1}, 2^{m_2}) = 0$ if $(m_1, m_2) \neq (0, 0)$. Now we are ready to prove the following result.

Theorem 4.5. *Let H be given by (4.12). Then H has the form*

$$H(\alpha, \gamma) = \frac{\zeta(1 + 2\gamma)\zeta(1 + \gamma)}{\zeta(1 + \alpha + \gamma)\zeta(1 + \alpha)} A(\alpha, \gamma) \tag{4.21}$$

where $A(\alpha, \gamma)$ is holomorphic and non-zero for $Re(\alpha), Re(\gamma) > -1/4$.

Proof. We have that $\widetilde{Q}^*(1, 1) = 1$, and from (4.12) and (4.13), we have

$$H(\alpha, \gamma) = \prod_p \left(1 + \sum_{m_1 \geq 1} \frac{\widetilde{Q}^*(p^{m_1}, p^0)}{p^{m_1(\frac{1}{2} + \alpha)}} + \sum_{m_1 \geq 0} \frac{\widetilde{Q}^*(p^{m_1}, p^1)}{p^{m_1(\frac{1}{2} + \alpha) + \frac{1}{2} + \gamma}} + \sum_{m_1 \geq 0} \frac{\widetilde{Q}^*(p^{m_1}, p^2)}{p^{m_1(\frac{1}{2} + \alpha) + 1 + 2\gamma}}\right) \cdot$$

$$= \prod_p \left(1 + \frac{\widetilde{Q}^*(p, 1)}{p^{\frac{1}{2} + \alpha}} + \frac{\widetilde{Q}^*(p^2, 1)}{p^{1 + 2\alpha}} + \frac{\widetilde{Q}^*(1, p)}{p^{\frac{1}{2} + \gamma}} + \frac{\widetilde{Q}^*(p, p)}{p^{1 + \alpha + \gamma}} + \frac{\widetilde{Q}^*(1, p^2)}{p^{1 + 2\gamma}} + T_p(\alpha, \gamma)\right),$$

where

$$T_p(\alpha, \gamma) := \sum_{\substack{m_1 \geq 3, m_2 = 0 \\ m_1 \geq 2, m_2 = 1 \\ m_1 \geq 1, m_2 = 2}} \frac{\widetilde{Q}^*(p^{m_1}, p^{m_2})}{p^{m_1(\frac{1}{2} + \alpha) + m_2(\frac{1}{2} + \gamma)}}.$$

Using the formulas from Lemma 4.2, 4.3 and 4.4 in $H(\alpha, \gamma)$, we obtain

$$H(\alpha, \gamma) = \prod_{p > 2} \left(1 + (1 + \chi_4(p)) \left(\frac{1}{p^{1+\gamma}} - \frac{1}{p^{1+\alpha}}\right) - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}}\right.$$

$$\left. + T_p(\alpha, \gamma) + O\left(\frac{1}{p^{2+2\alpha}} + \frac{1}{p^{2+2\gamma}} + \frac{1}{p^{2+\alpha+\gamma}}\right)\right).$$

Then

$$H(\alpha, \gamma) = A(\alpha, \gamma) \prod_{p > 2} \left(1 + (1 + \chi_4(p)) \left(\frac{1}{p^{1+\gamma}} - \frac{1}{p^{1+\alpha}}\right) - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}}\right),$$

$$\tag{4.22}$$

where $A(\alpha, \gamma)$ is analytic for $Re(\alpha), Re(\gamma) > -1/4$.

Let $\zeta_K(s)$ be the Dedekind zeta function of $K = \mathbb{Q}(i)$, i.e.,

$$\zeta_K(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \prod_{p \equiv 1 \bmod 4} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^{2s}}\right)^{-1}.$$

Since $\zeta_K(s)$ has a simple pole at $s = 1$, the factors

$$\frac{1 + \chi_4(p)}{p^{1+\gamma}} \quad \text{and} \quad -\frac{1 + \chi_4(p)}{p^{1+\alpha}}$$

David *et al. Research in Number Theory* (2015) 1:6

Page 28 of 37

contribute respectively a pole and a zero to $H(\alpha, \gamma)$. So now we factor out the zeta factors. We can then write (renaming $A(\alpha, \gamma)$)

$$H(\alpha, \gamma) = \frac{\zeta(1 + 2\gamma)\zeta(1 + \gamma)}{\zeta(1 + \alpha + \gamma)\zeta(1 + \alpha)} A(\alpha, \gamma), \tag{4.23}$$

where $A(\alpha, \gamma)$ is analytic for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/4$. $\qquad \square$

Finally, we define

$$Y(\alpha, \gamma) := \frac{\zeta(1 + 2\gamma)\zeta(1 + \gamma)}{\zeta(1 + \alpha + \gamma)\zeta(1 + \alpha)}. \tag{4.24}$$

4.2 The ratios conjecture for the family $\mathcal{F}_1(X)$

Working as in Section 3, we replace $R_1(\alpha, \gamma)$ in (4.9) with $Y(\alpha, \gamma)A(\gamma, \alpha)$, and $R_2(\alpha, \gamma)$ in (4.10) with

$$-\left(\frac{\sqrt{C(t)}}{2\pi}\right)^{-2\alpha} \frac{\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)} Y(-\alpha, \gamma)A(-\alpha, \gamma).$$

Then, replacing each summand in (4.11), we obtain the following conjecture.

Conjecture 4.6 (Ratios Conjecture). *Let $\varepsilon > 0$. Let $\alpha, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > -1/4$, $\mathrm{Re}(\gamma) \gg \frac{1}{\log X}$ and $\mathrm{Im}(\alpha), \mathrm{Im}(\gamma) \ll_\varepsilon X^{1-\varepsilon}$. Then*

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \frac{L(\frac{1}{2} + \alpha, E_t)}{L(\frac{1}{2} + \gamma, E_t)}$$

$$= \frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \left[Y(\alpha, \gamma)A(\alpha, \gamma) - \left(\frac{\sqrt{C(t)}}{2\pi}\right)^{-2\alpha} \frac{\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)} Y(-\alpha, \gamma)A(-\alpha, \gamma) \right]$$

$$+ O(X^{-1/2+\varepsilon})$$

where $Y(\alpha, \gamma)$ is defined in (4.24) and $A(\alpha, \gamma)$ in (4.23).

We now use $\alpha = \gamma = r$. We first show that $H(r, r) = A(r, r) = 1$. For the family of Section 3, we had a closed form for $H(\alpha, \gamma)$ that we used to show that $H(r, r) = 1$, but this is in fact true for any family by the Hecke relations as we show in the next lemma.

Lemma 4.7. *We have*

$$H(r, r) = A(r, r) = 1. \tag{4.25}$$

Proof. Using (4.12) and the definition of $\widetilde{Q}^*(m_1, m_2)$ from Lemma 4.1, we have that

$$H(r, r) = \prod_p \frac{1}{p} \sum_{t \bmod p} \sum_{m_1} \frac{\lambda_t(p^{m_1})}{p^{m_1(\frac{1}{2}+r)}} \sum_{m_2} \frac{\mu_t(p^{m_2})}{p^{m_2(\frac{1}{2}+r)}}.$$

From the Hecke relations (4.3) and (4.7), the m_1-sum is

$$\left(1 - \frac{\lambda_t(p)}{p^s} + \frac{\psi_t(p)}{p^{2s}}\right)^{-1}$$

and the m_2-sum is

$$\left(1 - \frac{\lambda_t(p)}{p^s} + \frac{\psi_t(p)}{p^{2s}}\right).$$

David *et al. Research in Number Theory* (2015) 1:6

Page 29 of 37

This proves that $H(r,r) = 1$, and $A(r,r) = 1$ by (4.23). $\qquad\square$

To get the one-level density for the family $\mathcal{F}_1$, we have to differentiate the result of Conjecture 4.6 with respect to α and use (3.2). We define

$$A_\alpha(r,r) := \frac{\partial}{\partial\alpha} A(\alpha,\gamma)\Big|_{\alpha=\gamma=r}, \tag{4.26}$$

and we obtain the following theorem.

Theorem 4.8. *Let $\varepsilon > 0$, and $r \in \mathbb{C}$. Assuming the Ratios Conjecture 4.6, $\mathrm{Re}(r) \gg \frac{1}{\log X}$ and $\mathrm{Im}(r) \ll X^{1-\varepsilon}$, we have*

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \frac{L'\left(\frac{1}{2}+r, E_t\right)}{L\left(\frac{1}{2}+r, E_t\right)}$$

$$= \frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \left[-\frac{\zeta'}{\zeta}(1+2r) - \frac{\zeta'}{\zeta}(1+r) + A_\alpha(r,r) \right.$$

$$\left. + \left(\frac{\sqrt{C(t)}}{2\pi}\right)^{-2r} \frac{\Gamma(1-r)}{\Gamma(1+r)} \frac{\zeta(1+2r)\zeta(1+r)}{\zeta(1-r)} A(-r,r) \right] + O(X^{-1/2+\varepsilon})$$

where $A_\alpha(r,r)$ is defined in (4.26).

Proof. We have that

$$Y_\alpha(r,r) = \frac{\partial}{\partial\alpha} Y(\alpha,\gamma)\Big|_{\alpha=\gamma=r} = -\frac{\zeta'}{\zeta}(1+2r) - \frac{\zeta'}{\zeta}(1+r).$$

Differentiating the first term in the sum in Conjecture 4.6 gives

$$-\frac{\zeta'}{\zeta}(1+2r) - \frac{\zeta'}{\zeta}(1+r) + A_\alpha(r,r).$$

For the second term, we compute that

$$Y_\alpha(-r,r) = \frac{\partial}{\partial\alpha} Y(-\alpha,\gamma)\Big|_{\alpha=\gamma=r} = -\frac{\zeta(1+2r)\zeta(1+r)}{\zeta(1-r)}.$$

Since $Y(-r,r) = 0$, differentiating the second term in Conjecture 4.6 gives

$$\frac{\partial}{\partial\alpha} R_2(\alpha,\gamma)\Big|_{\alpha=\gamma=r} = \left(\frac{\sqrt{C(t)}}{2\pi}\right)^{-2r} \frac{\Gamma(1-r)\zeta(1+2r)\zeta(1+r)A(-r,r)}{\Gamma(1+r)\zeta(1-r)}.$$

$\qquad\square$

4.3 Proof of Theorem 2.3 and Corollary 2.4

We now use the Ratios Conjectures for the family $\mathcal{F}_1$ to prove Theorem 2.3. Working as in Section 3, we write the one-level density for the family $\mathcal{F}_1$ as

$$D(\mathcal{F}_1; \phi, X)$$

$$= \frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \frac{1}{2\pi i} \int_{(c-1/2)} \left[2\frac{L'(1/2+r, E)}{L(1/2+r, E)} - \frac{X'(1/2+r, E_t)}{X(1/2+r, E_t)} \right] \phi(-ir)dr$$

using the relation

$$\frac{L'(s, E_t)}{L(s, E_t)} = \frac{X'_t(s)}{X_t(s)} - \frac{L'(1-s, E_t)}{L(1-s, E_t)},$$

where $X_t(s)$ is defined by (4.6).

David *et al. Research in Number Theory* (2015) 1:6

Page 30 of 37

We bring the summation inside the integral, and substitute the sum

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \frac{L'(1/2 + r, E_t)}{L(1/2 + r, E_t)}$$

with the expression of Theorem 4.8 to obtain

$$
\begin{aligned}
D(\mathcal{F}_1; \phi, X) = \frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \frac{1}{2\pi i} \int_{(c-\frac{1}{2})} & \left[-2 \left[\frac{\zeta'}{\zeta}(1 + 2r) + \frac{\zeta'}{\zeta}(1 + r) \right] \right. \\
& + 2A_\alpha(r, r) + 2 \left(\frac{\sqrt{C(t)}}{2\pi} \right)^{-2r} \frac{\Gamma(1 - r)}{\Gamma(1 + r)} \frac{\zeta(1 + 2r)\zeta(1 + r)}{\zeta(1 - r)} A(-r, r) \\
& \left. - \frac{X'\left(\frac{1}{2} + r, E_t\right)}{X\left(\frac{1}{2} + r, E_t\right)} \right] \phi(-ir)\,dr + O(X^{-1/2+\varepsilon}).
\end{aligned}
$$

We also compute

$$\frac{X_t'(s)}{X_t(s)} \bigg|_{s=\frac{1}{2}+r} = -2 \log \left(\frac{\sqrt{C(t)}}{2\pi} \right) - \frac{\Gamma'}{\Gamma}(1 - r) - \frac{\Gamma'}{\Gamma}(1 + r). \tag{4.27}$$

As in Section 3, we move the integral from $\operatorname{Re}(s) = c - \frac{1}{2} = c'$ to $\operatorname{Re}(s) = 0$ by integrating over the rectangle R from $c' - iT$ to $c' + iT$ to iT to $-iT$ and back to $c' - iT$, and letting $T \to \infty$. The two horizontal integrals tend to 0, and we only have to consider the vertical integrals. Using (4.27), the integrand is

$$
\begin{aligned}
F(r) = & \left[-2 \left[\frac{\zeta'}{\zeta}(1 + 2r) + \frac{\zeta'}{\zeta}(1 + r) \right] \right. \\
& + 2A_\alpha(r, r) + 2 \left(\frac{\sqrt{C(t)}}{2\pi} \right)^{-2r} \frac{\Gamma(1 - r)}{\Gamma(1 + r)} \frac{\zeta(1 + 2r)\zeta(1 + r)}{\zeta(1 - r)} A(-r, r) \\
& \left. - \left(-2 \log \left(\frac{\sqrt{C(t)}}{2\pi} \right) - \frac{\Gamma'}{\Gamma}(1 - r) - \frac{\Gamma'}{\Gamma}(1 + r) \right) \right].
\end{aligned}
$$

Using the Laurent series for $\zeta(1 - s)^{-1}$ gives

$$
\begin{aligned}
F(r) = & -2 \left(\frac{-1}{2r} + O(1) \right) - 2 \left(\frac{-1}{r} + O(1) \right) \\
& + 2 \left(1 + O(r) \right) \left(\frac{1}{2r} + O(1) \right) \left(\frac{1}{r} + O(1) \right) \left(-r + O(r^2) \right) + O(1) \\
= & \frac{2}{r} + O(1).
\end{aligned}
$$

There is a pole at $r = 0$ with residue 2 on the boundary of the rectangle R, and $F(r) - \frac{2}{r}$ is an analytic function inside and on the contour R. Setting $r = iu$, and using

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \frac{1}{2\pi i} \int_{(c')} \frac{2\phi(-ir)}{r}\,dr = 2\phi(0),$$

David *et al. Research in Number Theory* (2015) 1:6

Page 31 of 37

we obtain that

$$
D(\mathcal{F}_1; \phi, X) = \frac{1}{|\mathcal{F}_1(X)|} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u) \sum_{E_t \in \mathcal{F}_1(X)} \left[2 \log\left(\frac{\sqrt{C(t)}}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(1+iu) \right.
$$

$$
+ \frac{\Gamma'}{\Gamma}(1-iu) - 2\left(\frac{\zeta'}{\zeta}(1+2iu) + \frac{\zeta'}{\zeta}(1+iu)\right) + 2A_\alpha(iu, iu)
$$

$$
\left. + 2\left(\frac{\sqrt{C(t)}}{2\pi}\right)^{-2iu} \frac{\Gamma(1-iu)}{\Gamma(1+iu)} \frac{\zeta(1+2iu)\zeta(1+iu)}{\zeta(1-iu)} A(-iu, iu) - \frac{2}{iu} \right] du
$$

$$
+ \phi(0) + O(X^{-1/2+\varepsilon}),
$$

which is the statement of Theorem 2.3.

We now prove Corollary 2.4. We make the usual change of variables

$$
\tau = \frac{uL}{\pi} \quad \text{with} \quad L = \log\left(\frac{\sqrt{X}}{2\pi}\right), \tag{4.28}
$$

and we define the test function ψ by (3.51).

Lemma 4.9. *Let $\mathcal{F}_1$ be the family defined by defined by (4.2). Then*

$$
\frac{1}{|\mathcal{F}_1(X)|} \sum_{E_t \in \mathcal{F}_1(X)} \log\left(\frac{\sqrt{C(t)}}{2\pi}\right) \sim \log\left(\frac{\sqrt{X}}{2\pi}\right). \tag{4.29}
$$

Proof. Let $T = X^{\frac{1}{4}}$. Showing (4.29) is equivalent to showing that

$$
\frac{1}{T} \sum_{t \leq T} \log(C(t)) \sim \log X.
$$

We begin by noting that since $\Delta(t) \sim X$ we have that

$$
\frac{1}{T} \sum_{t \leq T} \log(\Delta(t)) \sim \log X.
$$

We write

$$
\frac{1}{T} \sum_{E_t \in \mathcal{C}} \log C(t) = \frac{1}{T} \sum_{t \leq T} \log \Delta(t) - \frac{1}{T} \sum_{t \leq T} \log\left(\frac{\Delta(t)}{C(t)}\right),
$$

and we will show that the second term on the right hand side is in the error term. Let $v_p(f(t))$ denote the function such that $p^{v_p(f(t))} || f(t)$ then we have that

$$
\frac{1}{T} \sum_{t \leq T} \log\left(\frac{\Delta(t)}{C(t)}\right) = \frac{1}{T} \sum_{t \leq T} \sum_{p^{v_p(\Delta(t))} || \Delta(t)} \log(p^{v_p(\Delta(t)) - v_p(C(t))}).
$$

Since $\Delta(t) = 2^4(t^2 + 3t + 9)^2$ we have that $v_p(\Delta(t)) \geq 2$ for primes $p > 2$ and for primes $p > 2, 3$ we have that $p \mid C(t)$ implies that $p \mid \Delta(t)$. Now suppose $v_p(\Delta(t)) = 2$ then $p || t^2 + 3t + 9$ and thus $p^2 || C(t)$. Hence

$$
\sum_{t \leq T} \frac{1}{\log X} \sum_{\substack{p | \Delta(t) \\ v_p(\Delta(t)) = 2}} \log(p^{v_p(\Delta(t)) - v_p(C(t))}) = 0.
$$

David *et al. Research in Number Theory* (2015) 1:6

Page 32 of 37

Thus

$$\frac{1}{T} \sum_{t \leq T} \log\left(\frac{\Delta(t)}{C(t)}\right) = \frac{1}{T} \sum_{t \leq T} \sum_{\substack{p^{v_p(\Delta(t))} || \Delta(t) \\ v_p(\Delta(t)) > 2}} \log(p^{v_p(\Delta(t)) - v_p(C(t))})$$

$$\leq \frac{1}{T} \sum_{t \leq T} \sum_{\substack{p^{v_p(\Delta(t))} || \Delta(t) \\ v_p(\Delta(t)) > 2}} \log(p^{v_p(\Delta(t)) - 2})$$

$$\ll \frac{1}{T} \sum_{\substack{p^{v_p(\Delta(t))} \ll T^2 \\ v_p(\Delta(t)) > 2}} \log(p^{v_p(\Delta(t)) - 2}) \sum_{t \leq \frac{T}{p^{v_p(\Delta(t))}}} 1$$

$$\ll \sum_{\substack{p^{v_p(\Delta(t))} \ll T^2 \\ v_p(\Delta(t)) > 2}} \frac{\log(p^{v_p(\Delta(t)) - 2})}{p^{v_p(\Delta(t))}} + \frac{1}{T} \sum_{\substack{p^{v_p(\Delta(t))} \ll T^2 \\ v_p(\Delta(t)) > 2}} \log(p^{v_p(\Delta(t)) - 2})$$

$$\ll T^{\frac{-1}{3}} = o(\log X)$$

by partial summation. $\square$

Then, using the change of variables (4.28), we rewrite the statement of Theorem 2.3 as

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E \in \mathcal{F}_1(X)} \sum_{\gamma_E} \psi\left(\frac{\gamma_E L}{\pi}\right) \sim \int_{-\infty}^{\infty} \psi(\tau) g(\tau) \, d\tau + \phi(0)$$

$$= \int_{-\infty}^{\infty} \psi(\tau) h(\tau) \, d\tau$$

where

$$g(\tau) = h(\tau) - \delta_0(\tau)$$

$$= \frac{1}{2L}\left[2\log\left(\frac{\sqrt{X}}{2\pi}\right) + \frac{\Gamma'}{\Gamma}\left(1 + \frac{\pi i \tau}{L}\right) + \frac{\Gamma'}{\Gamma}\left(1 - \frac{\pi i \tau}{L}\right) + 2\left\{-\frac{\zeta'}{\zeta}\left(1 + \frac{2\pi i \tau}{L}\right)\right.\right.$$

$$- \frac{\zeta'}{\zeta}\left(1 + \frac{\pi i \tau}{L}\right) + A_\alpha\left(\frac{\pi i \tau}{L}, \frac{\pi i \tau}{L}\right) + \exp\left(-\frac{2\pi i \tau}{L}\log\left(\frac{\sqrt{X}}{2\pi}\right)\right)$$

$$\left.\left. \times \frac{\Gamma(1 - \frac{\pi i \tau}{L})}{\Gamma(1 + \frac{\pi i \tau}{L})} \frac{\zeta(1 + \frac{2\pi i \tau}{L})\zeta(1 + \frac{\pi i \tau}{L})}{\zeta(1 - \frac{\pi i \tau}{L})} A\left(-\frac{\pi i \tau}{L}, \frac{\pi i \tau}{L}\right) \right\} - \frac{2L}{\pi i \tau}\right].$$

We then compute the Taylor expansion of $h(\tau)$ in L^{-1} which gives

$$h(\tau) = \frac{1}{2L}\left[2L - 2\gamma_0 + 2\left[-\left(\frac{-L}{2\pi i \tau} + \frac{-L}{\pi i \tau} + 2\gamma_0\right) + A_\alpha(0, 0) + O(L^{-1})\right.\right.$$

$$+ e^{-2\pi i \tau}\left(1 + \frac{2\pi \gamma_0 i \tau}{L} + O(L^{-2})\right)\left(\frac{L}{2\pi i \tau} + \gamma_0 - \frac{2\pi \gamma_0 i \tau}{L} + O(L^{-2})\right)$$

$$\times \left(\frac{L}{\pi i \tau} + \gamma_0 - \frac{\pi \gamma_0 i \tau}{L} + O(L^{-2})\right)\left(\frac{-\pi i \tau}{L} + \frac{\pi^2 \gamma_0 \tau^2}{L^2} + O(L^{-3})\right)$$

$$\left.\left. \times \left(1 + (A_\gamma(0,0) - A_\alpha(0,0))\frac{\pi i \tau}{L} + O(L^{-2})\right)\right] - \frac{2L}{\pi i \tau}\right] + \delta_0(\tau)$$

$$= 1 + \frac{\sin(2\pi \tau)}{2\pi \tau} + \frac{1 - \cos(2\pi \tau)}{2\pi i \tau} + \delta_0(\tau) + \frac{A_1(\tau)}{L} + O\left(\frac{1}{L^2}\right)$$

David *et al. Research in Number Theory* (2015) 1:6

Page 33 of 37

where

$$A_1(\tau) = A_\alpha(0,0) - 3\gamma_0 + e^{-2\pi i\tau}\left(\frac{1}{2}\left(A_\alpha(0,0) - A_\gamma(0,0)\right) - 3\gamma_0\right).$$

Since ψ is an even function we have that

$$\int_{-\infty}^{\infty} \psi(\tau)\left(\frac{1 - \cos(2\pi\tau)}{2\pi i\tau}\right) d\tau = 0 ,$$

and hence

$$\frac{1}{|\mathcal{F}_1(X)|} \sum_{E \in \mathcal{F}_1(X)} \sum_{\gamma_E} \psi\left(\frac{\gamma_E L}{\pi}\right)$$

$$\sim \int_{-\infty}^{\infty} \psi(\tau)\left[1 + \delta_0(\tau) + \frac{\sin(2\pi\tau)}{2\pi\tau} + A_1(\tau)L^{-1} + O(L^{-2})\right] d\tau.$$

Then, the leading terms for the one-level scaling density associated to the family $\mathcal{F}_1$ given by (4.2) is

$$\mathcal{W}(\tau) = 1 + \delta_0(\tau) + \frac{\sin(2\pi\tau)}{2\pi\tau} = \delta_0(\tau) + \mathcal{W}(SO(\text{even}))(\tau),$$

which proves Corollary 2.4.

5 Heuristic for the one-level density for the family $\mathcal{F}_1$

We give in this section a heuristic for the scaling density

$$\mathcal{W}(\tau) = 1 + \delta_0(\tau) + \frac{\sin(2\pi\tau)}{2\pi\tau}$$

of the one-parameter family $\mathcal{F}_1$. There are two pieces for this density, the first one corresponding to the contribution of the family zero at the central point, and we write

$$\mathcal{W}(\tau) = \mathcal{W}_1(\tau) + \mathcal{W}_2(\tau),$$

where $\mathcal{W}_1(\tau) = \delta_0(\tau)$.

We first review the steps that led to Theorem 2.3. Using the Ratios Conjecture, we computed in Section 4 the average value of

$$\frac{L(1/2 + \alpha, E_t)}{L(1/2 + \gamma, E_t)}$$

for the family $\mathcal{F}_1$. Writing

$$L(s, E_t) = \prod_p \left(1 - \frac{\lambda_t(p)}{p^s} + \frac{\psi(p)}{p^{2s}}\right)^{-1},$$

we have by Lemma 4.2 that the average of $\lambda_t(p)$ over the family is

$$-\frac{1 + \chi_4(p)}{\sqrt{p}},$$

and we then define $\lambda_t^*(p)$ by

$$\lambda_t(p) = \lambda_t^*(p) - \frac{1 + \chi_4(p)}{\sqrt{p}}.$$

David *et al. Research in Number Theory* (2015) 1:6

Page 34 of 37

For Re$(s) > 1$,

$$
\begin{aligned}
L(s, E_t) &= \prod_p \left(1 - \frac{\lambda_t^*(p)}{p^s} + \frac{\psi(p)}{p^{2s}} + \frac{1 + \chi_4(p)}{p^{1/2+s}}\right)^{-1} \\
&= \prod_p \left(1 + \frac{\lambda_t^*(p)}{p^s} - \frac{\psi(p)}{p^s} - \frac{1 + \chi_4(p)}{p^{1/2+s}} + \frac{\lambda_t^*(p)^2}{p^{2s}} + \text{h.o.t.}\right) \\
&= \prod_p \left(1 + \frac{\lambda_t^*(p)}{p^s} - \frac{1 + \chi_4(p)}{p^{1/2+s}} + \frac{\lambda_t^*(p)^2 - \psi(p)}{p^{2s}} + \text{h.o.t.}\right)
\end{aligned}
$$

and

$$
L(s, E_t)^{-1} = \prod_p \left(1 - \frac{\lambda_t^*(p)}{p^s} + \frac{\psi(p)}{p^{2s}} + \frac{1 + \chi_4(p)}{p^{1/2+s}} + \text{h.o.t.}\right),
$$

where the higher order terms are bounded by $p^{-2\sigma - 1/2 + \varepsilon}$. We will use $\sigma = 1/2$ below, so the higher order terms do not affect the convergence.

By Lemma 4.2, the average over the family of $\lambda_t^*(p)$ is 0, and by Lemma 4.3, the average over the family of $\lambda_t^*(p)^2$ is 1. Then, replacing each expression in the Euler product by its average over the family, we obtained from (4.22) the "average Euler product"

$$
\prod_p \left(1 - \frac{1 + \chi_4(p)}{p^{1+\alpha}} + \frac{1}{p^{1+2\gamma}} + \frac{1 + \chi_4(p)}{p^{1+\gamma}} - \frac{1}{p^{1+\alpha+\gamma}} + \text{h.o.t}\right),
$$

where the higher order terms give an absolutely convergent product in the neighborhood of $(0, 0)$, so the above behaves like

$$
\frac{\zeta(1 + \gamma)\zeta(1 + 2\gamma)}{\zeta(1 + \alpha)\zeta(1 + \alpha + \gamma)},
$$

which is the result of Theorem 4.5. In order to isolate the family zero from the previous argument, we first write

$$
L^*(s, E_t) = L(s, E_t)L(s), \tag{5.1}
$$

where

$$
L^*(s, E_t) = \prod_p \left(1 - \frac{\lambda_t^*(p)}{p^s} + \frac{\psi(p)}{p^{2s}}\right)^{-1},
$$

and

$$
L(s) = \frac{L^*(s, E_t)}{L(s, E_t)} = \prod_p \left(1 + \frac{1 + \chi_4(p)}{p^{1/2+s}}\right) F(s), \tag{5.2}
$$

where $F(s)$ converges absolutely for Re$(s) \geq 1/2$, and has no zeroes in this region. Furthermore,

$$
\prod_p \left(1 + \frac{1 + \chi_4(p)}{p^{s+1/2}}\right)
$$

is related to the Dedekind zeta function $\zeta_K(s + 1/2)$, where $K = \mathbb{Q}(i)$, and we can rewrite (5.2) as

$$
L(s) = \frac{L^*(s, E_t)}{L(s, E_t)} = \zeta_K(s + 1/2)F(s), \tag{5.3}
$$

where $F(s)$ converges absolutely for Re$(s) \geq 1/2$, and has no zeroes in this region (renaming F). From (5.1) and (5.3), the set of zeroes of $L(s, E_t)$ for Re$(s) = 1/2$ is the union of the zeroes of $L^*(s, E_t)$ and the poles of $\zeta_K(s + 1/2)$ for Re$(s) = 1/2$. In other words,

$$\mathcal{W}(\tau) = \mathcal{W}_1(\tau) + \mathcal{W}_2(\tau), \tag{5.4}$$

where $\mathcal{W}_2(\tau)$ is the density corresponding to the L-functions $L^*(s, E_t)$ on average for $E_t \in \mathcal{F}$ for the family $\mathcal{F}$ of (4.2), and $\mathcal{W}_1(\tau)$ is the density corresponding to the zeroes coming from the poles of $L(s) = \zeta_K(s + 1/2)F(s)$ for $\mathrm{Re}(s) = 1/2$. Then, $\mathcal{W}_1(\tau)$ does not depend of the family, and since there is only one pole at $s = 1/2$, this gives

$$\mathcal{W}_1(\tau) = \delta_0(\tau).$$

We now study the zeroes of $L^*(s, E_t)$. Of course, these are not the L-functions associated to any elliptic curve, but we can predict the "rank" of those L-functions assuming the Birch and Swinnerton-Dyer conjecture for the original L-functions $L(s, E_t)$. In a nutshell, if the original L-functions have odd rank, then the L-functions $L^*(s, E_t)$ have even rank, since $\lambda_t(p) = \lambda_t^*(p) - \dfrac{1 + \chi_4(p)}{\sqrt{p}}$.

More precisely, let E be an elliptic curve of rank r. With the usual notation, we have

$$\lambda_E(p) = \frac{a_E(p)}{\sqrt{p}},$$

and the Birch and Swinnerton-Dyer conjecture [2] predicts that

$$\prod_{p \le x} \frac{p + 1 - a_E(p)}{p} = \prod_{p \le x} 1 + \frac{1 - \lambda_E(p)\sqrt{p}}{p} \sim C(\log x)^r$$

for some constant C depending on E. Then, for the L-functions $L^*(s, E_t)$, the Birch and Swinnerton-Dyer conjecture predicts that

$$\prod_{p \le x} \frac{p + 1 - a_E^*(p)}{p} = \prod_{p \le x} \frac{p + 1 - a_E(p) - (1 + \chi_4(p))}{p} \sim C'(\log x)^{r-1}$$

where r is the rank of the original curve E and C' depends on E, since

$$\prod_{p \le x} \frac{p + 1 - a_E(p) - (1 + \chi_4(p))}{p}$$

$$= \prod_{p \le x} \frac{p + 1 - a_E(p)}{p} \prod_{p \le x} \frac{p + 1 - a_E(p) - (1 + \chi_4(p))}{p + 1 - a_E(p)}$$

$$= \prod_{p \le x} \frac{p + 1 - a_E(p)}{p} \prod_{p \le x} \frac{p - 1}{p} \prod_{p \le x} \frac{p + 1 - a_E(p) - (1 + \chi_4(p))}{p + 1 - a_E(p)} \frac{p}{p - 1}$$

$$= \prod_{p \le x} \frac{p + 1 - a_E(p)}{p} \prod_{p \le x} \frac{p - 1}{p} \prod_{p \le x} \frac{p^2 - p a_E(p) - p \chi_4(p)}{p^2 - p a_E(p) + a_E(p) - 1}$$

$$= \prod_{p \le x} \frac{p + 1 - a_E(p)}{p} \prod_{p \le x} \frac{p - 1}{p} \prod_{p \le x} 1 + \frac{-p \chi_4(p) - a_E(p) + 1}{p^2 - p a_E(p) + a_E(p) - 1}$$

$$\sim C'(\log x)^{r-1}.$$

Then, since r was odd for the original family, the L-functions $L^*(s, E_t)$ behave like a family of *even* rank, and we should have

$$\mathcal{W}_2(\tau) = 1 + \frac{\sin(2\pi\tau)}{2\pi\tau},$$

in (5.4), and

$$\mathcal{W}(\tau) = \mathcal{W}_1(\tau) + \mathcal{W}_2(\tau) = \delta_0(\tau) + 1 + \frac{\sin(2\pi\tau)}{2\pi\tau}$$

$$= \delta_0(\tau) + \mathcal{W}(SO(\text{even}))(\tau).$$

Endnotes

[a]The equidistribution of the root number is the standard conjecture that half of the elliptic curves have root number $\omega_E = -1$ and half have root number $\omega_E = 1$. We refer the reader to Section 3.3 for a discussion on the equidistribution of the root number.

[b]When the family has no place of multiplicative reduction, the root number is not necessarily equidistributed. The one-parameter family of elliptic curves of Section 4 is such an example; in that case, the root number of each curve is $\omega_E = -1$.

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David *et al. Research in Number Theory*   (2015) 1:6

Page 37 of 37

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