THE MEAN VALUES OF CUBIC L-FUNCTIONS OVER FUNCTION FIELDS

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ABSTRACT. We obtain an asymptotic formula for the mean value of L–functions associated to cubic characters over $\mathbb{F}_q[T]$. We solve this problem in the non-Kummer setting when $q \equiv 2 \pmod 3$ and in the Kummer setting when $q \equiv 1 \pmod 3$. In the Kummer setting, the mean value over the complete family of cubic characters was never addressed in the literature (over number fields or function fields). The proofs rely on obtaining precise asymptotics for averages of cubic Gauss sums over function fields, which can be studied using the pioneer work of Kubota. In the non-Kummer setting, we display some explicit (and unexpected) cancellation between the main term and the dual term coming from the approximate functional equation of the L–functions. Exhibiting the cancellation involves evaluating sums of residues of a variant of the generating series of cubic Gauss sums.

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1. Introduction

The problem we consider in this paper is that of computing the mean value of Dirichlet L–functions $L_q(s,\chi)$ evaluated at the critical point $s = 1/2$ as χ varies over the primitive cubic Dirichlet characters of $\mathbb{F}_q[T]$. We will solve this problem in two different settings: when the base field $\mathbb{F}_q$ contains the cubic roots of unity (or equivalently when $q \equiv 1 \,(\mathrm{mod}\,3)$; we call this the Kummer setting) and when $\mathbb{F}_q$ does not contain the cubic roots of unity (when $q \equiv 2 \,(\mathrm{mod}\,3)$; we call this the non-Kummer setting.)

There are few papers in literature about moments of cubic Dirichlet twists over number fields, especially compared to the abundance of papers on quadratic twists. For the case of quadratic characters over $\mathbb{Q}$, the first moment was computed by Jutila [Jut81], and the second and third moments by Soundararajan [Sou00]. For the case of quadratic characters over $\mathbb{F}_q[T]$, the first 4 moments were computed by the second author of this paper [Flo17a, Flo17b, Flo17c]. In particular, the improvement of the error term for the first moment in [Flo17a] showed the existence of a secondary term (of size approximately the cube root of the main term) which was not predicted by any heuristic. A secondary term of size $X^{3/4}$ was explicitly computed by Diaconu and Whitehead in the number field setting [DW] for the cubic moment of quadratic L–functions and by Diaconu in the function field setting [Dia19].

For the case of cubic characters, Baier and Young [BY10] considered the cubic Dirichlet characters over $\mathbb{Q}$ and obtained for the smoothed first moment that

$$\sum_{(q,3)=1} \sum_{\substack{\chi \text{ primitive mod } q \\ \chi^3=\chi_0}} L(1/2,\chi) w\left(\frac{q}{Q}\right) = c\hat{w}(0)Q + O\left(Q^{37/38+\varepsilon}\right), \tag{1}$$

with an explicit constant c. Using an upper bound for higher moments of L–functions, Baier and Young also show that the number of primitive Dirichlet characters χ of order 3 with conductor less than or equal to Q for which $L(1/2,\chi) \neq 0$ is bounded below by $Q^{\frac{6}{7}-\varepsilon}$.

The first moment of the cubic Dirichlet twists over $\mathbb{Q}(\xi_3)$ was considered by Luo in [Luo04] and Friedberg, Hoffstein and Lieman in [FHL03] for a thin subset of the cubic characters, namely those given by the cubic residue symbols χ_c where $c \in \mathbb{Z}[\xi_3]$ is square-free. This does not count the conjugate characters $\chi_c^2 = \chi_{c^2}$, and in particular, the first moment of [Luo04] is not real.

The problem of computing the mean value of cubic L–functions over function fields was considered by Rosen in [Ros95], where he averages over all monic polynomials of a given degree. This problem is different than the one we consider, since the counting is not done by genus and obtaining an asymptotic formula relies on using a combinatorial identity.

Before stating our results, we first introduce some notation. Let q be an odd prime power, and let $\mathbb{F}_q[T]$ be the set of polynomials over the finite field $\mathbb{F}_q$. A Dirichlet character χ of modulus $m \in \mathbb{F}_q[T]$ is a multiplicative function from $(\mathbb{F}_q[T]/(m))^*$ to $\mathbb{C}^*$, extended to $\mathbb{F}_q[T]$ by periodicity if $(a,m) = 1$, and defined by $\chi(a) = 0$ if $(a,m) \neq 1$. A cubic Dirichlet character is a χ such that χ^3 equals the principal character χ_0 and $\chi \neq \chi_0$, and it takes values in μ_3, the cubic roots of 1 in $\mathbb{C}^*$. The smallest period of χ is called the conductor of the character. We say that χ is a primitive character of modulus m when m is the smallest period.

We denote by $L_q(s,\chi)$ the L–function attached to the character χ of $\mathbb{F}_q[T]$. We keep the index q in the notation to avoid confusion, as we will also work over the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$.

We can count primitive cubic characters ordering them by the degree of their conductor, or equivalently by the genus g of the cyclic cubic field extension of $\mathbb{F}_q[T]$ associated to such a character (see formula (8)).

The set of cubic characters differs when $\mathbb{F}_q$ contains the third roots of unity or not. If $q \equiv 1 \pmod 3$, $\mathbb{F}_q$ contains the third roots of unity. In this case we will be interested in odd characters, namely those χ that are nontrivial on $\mathbb{F}_q^*$. The number of odd primitive cubic Dirichlet characters with conductor of genus g is then asymptotic to $B_{\mathrm{K},1} g q^g + B_{\mathrm{K},2} q^g$ for some explicit constants $B_{\mathrm{K},1}, B_{\mathrm{K},2}$ (see Lemma 2.8). If $q \equiv 2 \pmod 3$, $\mathbb{F}_q$ does not contain the third roots of unity, and all characters are even, as they are trivial on $\mathbb{F}_q^*$. In this case the number of primitive cubic Dirichlet characters with conductor of genus g is asymptotic to $B_{\mathrm{nK}} q^g$ for some explicit constant B_{nK} (see Lemma 2.10).

We compute the first moment of cubic L–functions for the two settings. In the non-Kummer case, we have the following.

Theorem 1.1. *Let q be an odd prime power such that $q \equiv 2 \pmod 3$. Then, for $\varepsilon > 0$,*

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L_q(1/2,\chi) = \frac{\zeta_q(3/2)}{\zeta_q(3)} \mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2}, \frac{1}{q^{3/2}}\right) q^{g+2} + O(q^{\frac{7g}{8}+\varepsilon g}),$$

with $\mathcal{A}_{\mathrm{nK}}(q^{-2}, q^{-3/2})$ given in Lemma 4.1, and the implicit constant in the error term depends on both q and ε.

In the Kummer case, we have the following.

Theorem 1.2. *Let q be an odd prime power such that $q \equiv 1 \pmod 3$, and let χ_3 be the cubic character on $\mathbb{F}_q^*$ given by* (3)*. Then, for $\varepsilon > 0$,*

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L_q(1/2,\chi) = C_{\mathrm{K},1} g q^{g+1} + C_{\mathrm{K},2} q^{g+1} + O\left(q^{g\frac{1+\sqrt{7}}{4}+\varepsilon g}\right),$$

where $C_{\mathrm{K},1}$ and $C_{\mathrm{K},2}$ are given by equations (86) *and* (87) *respectively, and the implicit constant in the error term depends on both q and ε.*

The hypothesis that χ restricts to the character χ_3 on $\mathbb{F}_q$ is not important, but simplifies the computations by ensuring that the L–functions have the same functional equation. It is analogous to the restriction in the case of quadratic characters to those with conductor of degree either $2g$ or $2g+1$.

Since L-functions satisfy the Lindelöf hypothesis over function fields (see Lemma 2.5), one can easily bound the second moment, and we get the following corollary.

Corollary 1.3. *Let q be an odd prime power. Then, for $\varepsilon > 0$,*

$$\#\{\chi \text{ cubic, primitive of genus } g \ : \ L_q(1/2,\chi) \neq 0\} \gg_\varepsilon q^{(1-\varepsilon)g}.$$

Translating from the function field to the number field setting, we associate q^g with Q. Note that Theorem 1.1 is the function field analog of (1), and the proof of our Theorem 1.1

has many similarities with the work of [BY10]. The better quality of our error term can be explained in part by the fact that we can use the Riemann Hypothesis to bound the error term. In the number field case, the same quality of error term can be obtained without the Riemann Hypothesis for some families using the appropriate version of the large sieve (for example in the case of the family of quadratic characters, with the quadratic large sieve due to Heath-Brown [HB95]). However the cubic large sieve, also due to Heath-Brown [HB00], provides a weaker upper bound. There is also an asymmetry between the sum over the cubic characters, which is naturally a sum over $\mathbb{Q}(\xi_3)$, and the truncated Dirichlet series of the L-function, which is a sum over $\mathbb{Z}$. The asymmetry of the sums also exists in the function field setting.

Another difference from the work of Baier and Young is that we explicitly exhibit cancellation between the main term and the dual term coming from using the approximate functional equation for the L–functions. In their work Baier and Young [BY10] prove an upper bound for the dual term without obtaining an asymptotic formula for it, which is what we do in the function field case.

The first steps of our proofs are the usual ones, using the approximate functional equation to write the special value

$$L_q(1/2,\chi) \quad = \quad \sum_{f\in\mathcal{M}_q} \frac{\chi(f)}{|f|_q^{1/2}}, \tag{2}$$

as a sum of two terms (the principal sum and the dual sum), where for a polynomial $f \in \mathbb{F}_q[T]$ the norm is defined by $|f|_q = q^{\deg(f)}$. Inspired by the work of Florea [Flo17c] to improve the quality of the error term, we evaluate exactly the dual sum and the secondary term of the main sum (corresponding to taking f cube in the approximate functional equation) in order to obtain cancellation of those terms. This is similar to the work of Florea for the first moment of quadratic Dirichlet characters over functions fields, replacing quadratic Gauss sums by cubic Gauss sums. Of course, this is not a trivial difference, as the behavior of quadratic Gauss sums is very regular since they are multiplicative functions. However cubic Gauss sums are different as they are no longer multiplicative. Handling the cubic Gauss sums is significantly more difficult than working with quadratic Gauss sums. This is one of the main focuses of our paper.

The distribution of Gauss sums over number fields was adressed by Heath-Brown and Patterson [HBP79], using the deep work of Kubota for automorphic forms associated to the metaplectic group. This was generalised by Hoffstein [Hof92] and Patterson [Pat07] for the function field case. In Section 3 we review their work and further develop the results concerning the generating series for cubic Gauss sums using only its functional equation and the periodicity condition on the residues. The main goal of Section 3 is to obtain an exact formula for the residues of the generating series

$$\tilde{\Psi}_q(f,u) = \sum_{\substack{F\in\mathcal{M}_q\\(F,f)=1}} G_q(f,F)u^{\deg(F)},$$

where $G_q(f,F)$ is the generalized shifted Gauss sum over $\mathbb{F}_q$ as defined by (21). With those residues in hand, we can evaluate precisely the main term of the dual sum, and indeed we can show that it (magically!) cancels with the secondary term of the principal sum. Unfortunately obtaining the cancellation is not enough to improve the error term, as we do not have good bounds for $\tilde{\Psi}_q(f,u)$ beyond the pole at $u^3 = 1/q^4$. We prove that the convexity bound in Lemma 3.11 holds, and any improvement of the convexity bound would allow an improvement of the error term of Theorem 1.1 coming from the cancellation that we exhibit.

Proving Theorem 1.2 is more difficult than obtaining the asymptotic formula in the non-Kummer case, and our error term is not as good as that in Theorem 1.1. To our knowledge, Theorem 1.2 is the first result when one considers all the primitive cubic characters (with the technical restriction that $\chi|_{\mathbb{F}_q^*} = \chi_3$, which does not change the size of the family). This explains the (maybe surprising) asymptotic for the first moment in Theorem 1.2, which is of the shape $gq^g P(1/g)$ where P is a polynomial of degree 1.

Because of the size of the family of cubic twists in the Kummer case, we are not able to obtain cancellation between the dual term and the error term from the main term. Certain cross-terms seem to contribute to the cancellation, but we cannot obtain an asymptotic formula for these cross terms. Instead we bound them using the convexity bound for $\tilde{\Psi}_q(f,u)$, which explains the bigger error term from Theorem 1.2.

We remark that the results of Theorems 1.1 and 1.2 both correspond to a family with unitary symmetry, as expected. Note that for our results, we fix the size q of the finite field and let the genus g go to infinity. If instead one fixes the genus and lets q go to infinity, it should be possible to obtain asymptotic formulas for moments using equidistribution results as in the work of Katz and Sarnak [KS99] and then a random matrix theory computation as in the work of Keating and Snaith [KS00].

As mentioned before, a lower order term of size the cube root of the main term was computed in [Flo17c] in the case of the mean value of quadratic L–functions. We remark that in the case of the mean value of cubic L–functions, we can explicitly compute a term of size $q^{5g/6}$ in the non-Kummer case and a term of size $gq^{5g/6}$ in the Kummer setting (see remarks 4.5 and 5.6 respectively). Due to the size of the error terms, these terms do not appear in the asymptotic formulas in Theorems 1.1 and 1.2. However, we suspect these terms do persist in the asymptotic formulas. Improving the convexity bound on $\tilde{\Psi}_q(f,u)$ would allow us to improve the error terms, and maybe to detect the lower order terms. However, since $\tilde{\Psi}_q(f,u)$ is a function with no Euler product, we do not know if there is a solid basis to hope to improve the convexity bound which follows from the functional equation and the Phragmén–Lindelöf principle. We remark that a similar sized lower order term term was conjectured by Heath-Brown and Patterson [HBP79] for the average of the arguments of cubic Gauss sums in the number field setting. We believe the matching size of these terms is not a coincidence, as the source of our $q^{5g/6}$ comes from averaging cubic Gauss sums over function fields. A lower order term of size $X^{5/6}$ has also been identified when counting cubic number fields with discriminant less than X, as in [BST13, TT13], and the work of [FHL03, Dia04] gives some evidence for such a secondary term in the first and second moments of cubic L-functions.

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2. Notation and Setting

Let q be an odd prime power such that $q \equiv 1 \pmod 3$. We denote by $\mathcal{M}_q$ the set of monic polynomials of $\mathbb{F}_q[T]$, by $\mathcal{M}_{q,d}$ the set of monic polynomials of degree exactly d, by $\mathcal{M}_{q,\leq d}$ the set of monic polynomials of degree smaller than or equal to d, by $\mathcal{H}_q$ the set of monic square-free polynomials of $\mathbb{F}_q[T]$ and analogously for $\mathcal{H}_{q,d}$ and $\mathcal{H}_{q,\leq d}$. Note that $|\mathcal{M}_{q,d}| = q^d$ and for $d \geq 2$, we have that $|\mathcal{H}_{q,d}| = q^d(1 - \frac{1}{q})$.

In general, unless stated otherwise, all polynomials are monic. As for the L–functions in the introduction, we keep the index q in the notation to avoid confusion, as we will have to consider polynomials over the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$ when $q \equiv 2 \pmod 3$.

We define the norm of a polynomial $f(T) \in \mathbb{F}_q[T]$ over $\mathbb{F}_q[T]$ by

$$|f|_q = q^{\deg(f)}.$$

Then, if $f(T) \in \mathbb{F}_q[T]$, we have $|f|_{q^n} = q^{n \deg(f)}$, for any positive integer n.

For $q \equiv 1 \pmod 3$ we fix once and for all an isomorphism Ω between μ_3, the cubic roots of 1 in $\mathbb{C}^*$, and the cubic roots of 1 in $\mathbb{F}_q^*$. We also fix a cubic character χ_3 on $\mathbb{F}_q^*$ by

$$\chi_3(\alpha) = \Omega^{-1}\left(\alpha^{\frac{q-1}{3}}\right). \tag{3}$$

For any character χ on $\mathbb{F}_q[T]$, we say that χ is even if it is trivial on $\mathbb{F}_q^*$, and odd otherwise. Then, when q is an odd prime power such that $q \equiv 1 \pmod 3$, any cubic character on $\mathbb{F}_q[T]$ falls in three natural classes depending on its restriction to $\mathbb{F}_q^*$ which is either χ_3, χ_3^2 or the trivial character (in the first 2 cases, the character is odd, and in the last case, the character is even).[1]

For any odd character χ on $\mathbb{F}_q[T]$, we denote by $\tau(\chi)$ the Gauss sum of the restriction of χ to $\mathbb{F}_q^*$ (which is either χ_3 or χ_3^2), i.e.

$$\tau(\chi) = \sum_{a \in \mathbb{F}_q^*} \chi(a) e^{2\pi i \operatorname{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)/p}. \tag{4}$$

Then, $|\tau(\chi)| = q^{1/2}$, and we denote the sign of the Gauss sum by

$$\epsilon(\chi) = q^{-1/2}\tau(\chi). \tag{5}$$

When χ is even, we set $\epsilon(\chi) = 1$.

Finally, we recall Perron's formula over $\mathbb{F}_q[T]$ which we will use many times throughout the paper.

Lemma 2.1 (Perron's Formula)**.** *If the generating series $\mathcal{A}(u) = \sum_{f \in \mathcal{M}_q} a(f) u^{\deg(f)}$ is absolutely convergent in $|u| \leq r < 1$, then*

$$\sum_{f \in \mathcal{M}_{q,n}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u^n} \frac{du}{u}$$

[1] We will see in Section 2.2 that when $q \equiv 2 \pmod 3$, any cubic character on $\mathbb{F}_q[T]$ is even.

and

$$\sum_{f\in\mathcal{M}_{q,\leq n}} a(f) = \frac{1}{2\pi i}\oint_{|u|=r} \frac{\mathcal{A}(u)}{u^n(1-u)}\,\frac{du}{u},$$

where, in the usual notation, we take $\oint$ *to signify the integral over the circle oriented counterclockwise.*

2.1. **Zeta functions and the approximate functional equation.** The *affine zeta function* over $\mathbb{F}_q[T]$ is defined by

$$\mathcal{Z}_q(u) = \sum_{f\in\mathcal{M}_q} u^{\deg(f)}$$

for $|u|<1/q$. By grouping the polynomials according to the degree, it follows that

$$\mathcal{Z}_q(u) = \sum_{n=0}^{\infty} u^n q^n = \frac{1}{1-qu},$$

and this provides a meromorphic continuation of $\mathcal{Z}_q(u)$ to the entire complex plane. We remark that $\mathcal{Z}_q(u)$ has a simple pole at $u=1/q$ with residue $-\frac{1}{q}$. We also define

$$\zeta_q(s) = \mathcal{Z}_q(q^{-s}).$$

Note that $\mathcal{Z}_q(u)$ can be expressed in terms of an Euler product as follows

$$\mathcal{Z}_q(u) = \prod_P (1-u^{\deg(P)})^{-1},$$

where the product is over monic irreducible polynomials in $\mathbb{F}_q[T]$.

Let C be a curve over $\mathbb{F}_q(T)$ whose function field is a cyclic cubic extension of $\mathbb{F}_q(T)$. From the Weil conjectures, the zeta function of the curve C can be written as

$$\mathcal{Z}_C(u) = \frac{\mathcal{P}_C(u)}{(1-u)(1-qu)},$$

where

$$\mathcal{P}_C(u) = \prod_{j=1}^{g}\left(1-\sqrt{q}ue^{2\pi i\theta_j}\right)\prod_{j=1}^{g}\left(1-\sqrt{q}ue^{-2\pi i\theta_j}\right)$$

for some eigenangles θ_j, $j=1,\dots,g$.

We can write $\mathcal{P}_C(u)$ in terms of the L-functions of the two cubic Dirichlet characters χ and $\overline{\chi}$ of the function field of C. Let h be the conductor of the non-principal character χ. Define

$$\mathcal{L}_q(u,\chi) := \sum_{f\in\mathcal{M}_q}\chi(u)u^{\deg(f)} = \sum_{d<\deg(h)} u^d \sum_{f\in\mathcal{M}_{q,d}}\chi(f), \tag{6}$$

where the second equality follows from the orthogonality relations.

We remark that setting $u=q^{-s}$, we have $L_q(s,\chi)=\mathcal{L}_q(u,\chi)$. From now on we will mainly use the notation $\mathcal{L}_q(u,\chi)$. The L–function has the following Euler product

$$\mathcal{L}_q(u,\chi) = \prod_{P\nmid h}(1-\chi(P)u^{\deg(P)})^{-1},$$

where the product is again over monic irreducible polynomials P in $\mathbb{F}_q[T]$. From now on, the Euler products we consider are over monic, irreducible polynomials and if there is an ambiguity as to whether the polynomials belong to $\mathbb{F}_q[T]$ or $\mathbb{F}_{q^2}[T]$ we will indicate so.

Considering the prime at infinity, we write

$$\mathcal{L}_C(u,\chi) = \begin{cases} \mathcal{L}_q(u,\chi) & \text{if } \chi \text{ is odd,} \\ \dfrac{\mathcal{L}_q(u,\chi)}{1-u} & \text{if } \chi \text{ is even.} \end{cases} \tag{7}$$

Then we have

$$\mathcal{P}_C(u) = \mathcal{L}_C(u,\chi)\mathcal{L}_C(u,\overline{\chi}).$$

Furthermore, using the Riemann–Hurwitz formula, we have that

$$\deg(h) = g + 2 - \begin{cases} 0 & \text{if } \chi \text{ is even,} \\ 1 & \text{if } \chi \text{ is odd.} \end{cases} \tag{8}$$

Lemma 2.2. *Let χ be a primitive cubic character to the modulus h.*

If χ is odd, then $\mathcal{L}_q(u,\chi)$ satisfies the functional equation

$$\mathcal{L}_q(u,\chi) = \omega(\chi)(\sqrt{q}u)^{\deg(h)-1}\mathcal{L}_q\left(\frac{1}{qu},\overline{\chi}\right), \tag{9}$$

where the sign of the functional equation is

$$\omega(\chi) \;=\; q^{-(\deg(h)-1)/2} \sum_{f\in\mathcal{M}_{q,\deg(h)-1}} \chi(f). \tag{10}$$

If χ is even, then $\mathcal{L}_q(u,\chi)$ satisfies the functional equation

$$\mathcal{L}_q(u,\chi) = \omega(\chi)(\sqrt{q}u)^{\deg(h)-2}\frac{1-u}{1-\frac{1}{qu}}\mathcal{L}_q\left(\frac{1}{qu},\overline{\chi}\right),$$

where the sign of the functional equation is

$$\omega(\chi) = -q^{-(\deg(h)-2)/2} \sum_{f\in\mathcal{M}_{q,\deg(h)-1}} \chi(f). \tag{11}$$

Proof. From (7) and (8), if χ is odd, then $g = \deg(h)-1$, $\mathcal{L}_q(u,\chi) = \mathcal{L}_C(u,\chi)$, and the functional equation follows from the Weil conjectures, since we have

$$\begin{aligned} \mathcal{L}_C(u,\chi) &= (u\sqrt{q})^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} \left((u\sqrt{q})^{-1} - e^{2\pi i\theta_j}\right) \\ &= (u\sqrt{q})^{\deg(h)-1}(-1)^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} e^{2\pi i\theta_j} \prod_{j=1}^{\deg(h)-1} \left(1 - \frac{e^{-2\pi i\theta_j}}{u\sqrt{q}}\right) \\ &= (u\sqrt{q})^{\deg(h)-1}(-1)^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} e^{2\pi i\theta_j}\mathcal{L}_C\left(\frac{1}{qu},\overline{\chi}\right). \end{aligned} \tag{12}$$

Since

$$\sum_{n=0}^{\deg(h)-1} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f) = \prod_{j=1}^{\deg(h)-1} (1-u\sqrt{q}e^{2\pi i\theta_j}),$$

comparing the coefficients of $u^{\deg(h)-1}$, it follows that

$$\sum_{f\in\mathcal{M}_{q,\deg(h)-1}} \chi(f) = (-1)^{\deg(h)-1} q^{(\deg(h)-1)/2} \prod_{j=1}^{\deg(h)-1} e^{2\pi i\theta_j},$$

which gives that

$$\omega(\chi) = q^{-(\deg(h)-1)/2} \sum_{f\in\mathcal{M}_{q,\deg(h)-1}} \chi(f).$$

From (7) and (8), if χ is even, then $g = \deg(h)-2$, $\mathcal{L}_q(u,\chi) = (1-u)\mathcal{L}_C(u,\chi)$, and we have

$$\begin{aligned}
\mathcal{L}_q(u,\chi) &= (1-u)(u\sqrt{q})^{\deg(h)-2} \prod_{j=1}^{\deg(h)-2} \left((u\sqrt{q})^{-1} - e^{2\pi i\theta_j}\right) \\
&= (1-u)(u\sqrt{q})^{\deg(h)-2}(-1)^{\deg(h)-2} \prod_{j=1}^{\deg(h)-2} e^{2\pi i\theta_j} \prod_{i=1}^{\deg(h)-2} \left(1-\frac{e^{-2\pi i\theta_j}}{u\sqrt{q}}\right) \\
&= \left(\frac{1-u}{1-\frac{1}{qu}}\right)(u\sqrt{q})^{\deg(h)-2}(-1)^{\deg(h)-2} \prod_{j=1}^{\deg(h)-2} e^{2\pi i\theta_j} \mathcal{L}_q\left(\frac{1}{qu},\overline{\chi}\right). \qquad (13)
\end{aligned}$$

Since

$$\sum_{n=0}^{\deg(h)-1} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f) = (1-u) \prod_{j=1}^{\deg(h)-2} (1-u\sqrt{q}e^{2\pi i\theta_j}),$$

comparing the coefficients of $u^{\deg(h)-1}$, it follows that

$$\sum_{f\in\mathcal{M}_{q,\deg(h)-1}} \chi(f) = (-1)^{\deg(h)-1} q^{(\deg(h)-2)/2} \prod_{j=1}^{\deg(h)-2} e^{2\pi i\theta_j},$$

which gives that

$$\omega(\chi) = -q^{-(\deg(h)-2)/2} \sum_{f\in\mathcal{M}_{q,\deg(h)-1}} \chi(f).$$

□

It is more natural to rewrite the sign of the functional equation in terms of Gauss sums over $\mathbb{F}_q[T]$. In particular, it is not obvious from (10) and (11) that $|\omega(\chi)| = 1$.

As in [Flo17c], we will use the exponential function which was introduced by D. Hayes [Hay66]. For any $a \in \mathbb{F}_q((1/T))$, we define

$$e_q(a) = e^{\frac{2\pi i \, \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)}{p}}, \qquad (14)$$

with a_1 the coefficient of $1/T$ in the Laurent expansion of a. We then have that $e_q(a+b) = e_q(a)e_q(b)$, and $e_q(a) = 1$ for $a \in \mathbb{F}_q[T]$. Also, if $a, b, h \in \mathbb{F}_q[T]$ are such that $a \equiv b \pmod{h}$, then $e_q(a/h) = e_q(b/h)$.

For χ a primitive character of modulus h on $\mathbb{F}_q[T]$, let

$$G(\chi) = \sum_{a \pmod{h}} \chi(a) e_q\left(\frac{a}{h}\right)$$

be the Gauss sum of the primitive Dirichlet character χ over $\mathbb{F}_q[T]$. The following corollary expresses the root number in terms of Gauss sums.

Corollary 2.3. *Let χ be a primitive character of modulus h on $\mathbb{F}_q[T]$. Then*

$$\omega(\chi) = \begin{cases} \frac{1}{\tau(\chi)} q^{-(\deg(h)-1)/2} G(\chi) & \text{if } \chi \text{ odd,} \\ \frac{1}{\sqrt{q}} q^{-(\deg(h)-1)/2} G(\chi) & \text{if } \chi \text{ even.} \end{cases}$$

Proof. We prove the following relation

$$G(\chi) = \begin{cases} \tau(\chi) \sum_{f \in \mathcal{M}_{q,\deg(h)-1}} \chi(h) & \text{if } \chi \text{ odd,} \\ -q \sum_{f \in \mathcal{M}_{q,\deg(h)-1}} \chi(h) & \text{if } \chi \text{ even,} \end{cases}$$

which clearly implies the corollary. Writing

$$\mathcal{L}_q(u, \chi) = \sum_{j=0}^{\deg(h)-1} a_j u^j, \qquad a_j = \sum_{\ell \in \mathcal{M}_{q,j}} \chi(\ell),$$

we have

$$\sum_{\ell \pmod{h}} \chi(\ell) e_q\left(\frac{\ell}{h}\right) = \sum_{j=0}^{\deg(h)-2} a_j \sum_{a \in \mathbb{F}_q^*} \chi(a) + a_{\deg(h)-1} \sum_{a \in \mathbb{F}_q^*} \chi(a) e^{2\pi i \operatorname{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)/p}.$$

$$= \begin{cases} \tau(\chi) a_{\deg(h)-1} & \text{if } \chi \text{ odd,} \\ (q-1) \sum_{j=0}^{\deg(h)-2} a_j - a_{\deg(h)-1} & \text{if } \chi \text{ even.} \end{cases}$$

When χ is even, 1 is a root of $\mathcal{L}_q(u, \chi)$ and therefore $\sum_{j=0}^{\deg(h)-1} a_j = 0$. The result follows. $\square$

The following result allows us to replace the sum (6) by two shorter sums of lengths A and $g - A - 1$, where A is a parameter that can be chosen later, where the relationship between g and $\deg(h)$ is given by (8).

Proposition 2.4 (Approximate Functional Equation)**.** *Let χ be a primitive cubic character of modulus h. If χ is odd, then*

$$\mathcal{L}_q\left(\frac{1}{\sqrt{q}}, \chi\right) = \sum_{f \in \mathcal{M}_{q,\leq A}} \frac{\chi(f)}{q^{\deg(f)/2}} + \omega(\chi) \sum_{f \in \mathcal{M}_{q,\leq g-A-1}} \frac{\overline{\chi}(f)}{q^{\deg(f)/2}},$$

where $g = \deg(h) - 1$ by (8).

If χ is even, then

$$\mathcal{L}_q\left(\frac{1}{\sqrt{q}},\chi\right) = \sum_{f\in\mathcal{M}_{q,\leq A}} \frac{\chi(f)}{q^{\deg(f)/2}} + \omega(\chi) \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{\overline{\chi}(f)}{q^{\deg(f)/2}} + \frac{1}{1-\sqrt{q}} \sum_{f\in\mathcal{M}_{q,A+1}} \frac{\chi(f)}{q^{\deg(f)/2}} + \frac{\omega(\chi)}{1-\sqrt{q}} \sum_{f\in\mathcal{M}_{q,g-A}} \frac{\overline{\chi}(f)}{q^{\deg(f)/2}},$$

where $g = \deg(h) - 2$ by (8).

Proof. For χ odd, we use Lemma 2.2 for χ and then we have that

$$\mathcal{L}_q(u,\chi) = \omega(\chi)(\sqrt{q}u)^g \mathcal{L}_q\left(\frac{1}{qu},\overline{\chi}\right).$$

Using equation (6) and the functional equation above, it follows that

$$\sum_{f\in\mathcal{M}_{q,n}} \chi(f) = \omega(\chi) q^{n-\frac{g}{2}} \sum_{f\in\mathcal{M}_{q,g-n}} \overline{\chi}(f). \tag{15}$$

Writing

$$\mathcal{L}_q(u,\chi) = \sum_{n=0}^{A} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f) + \sum_{n=A+1}^{g} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f),$$

and using (15) for the second sum, it follows that

$$\mathcal{L}_q(u,\chi) = \sum_{f\in\mathcal{M}_{q,\leq A}} \chi(f) u^{\deg(f)} + \omega(\chi)(\sqrt{q}u)^g \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{\overline{\chi}(f)}{(qu)^{\deg(f)}}.$$

Plugging in $u = 1/\sqrt{q}$ finishes the proof.

For χ even we have

$$\mathcal{L}_q(u,\chi) = \sum_{n=0}^{g+1} a_n u^n, \qquad a_n = \sum_{f\in\mathcal{M}_{q,n}} \chi(f).$$

We write

$$\mathcal{L}_C(u,\chi) = \prod_{j=1}^{g} \left(1 - u\sqrt{q}e^{2\pi i\theta_j}\right) = \sum_{n=0}^{g} b_n u^n.$$

By the functional equation (13),

$$\begin{aligned}\sum_{n=0}^{g} b_n u^n =& \omega(\chi)(\sqrt{q}u)^g \sum_{n=0}^{g} \overline{b_n} q^{-n} u^{-n} \\ =& \omega(\chi) \sum_{n=0}^{g} \overline{b_n} q^{g/2-n} u^{g-n} = \omega(\chi) \sum_{m=0}^{g} \overline{b_{g-m}} q^{m-g/2} u^m,\end{aligned}$$

from where

$$b_n = \omega(\chi)\overline{b_{g-n}} q^{n-g/2}.$$

Thus, we can write

$$\mathcal{L}_C(u,\chi) = \sum_{n=0}^{A} b_n u^n + \omega(\chi)(\sqrt{q}u)^g \sum_{n=0}^{g-A-1} \frac{\overline{b_n}}{q^n u^n}.$$

Now since $\mathcal{L}_q(u,\chi) = (1-u)\mathcal{L}_C(u,\chi)$, we get that

$$a_n = b_n - b_{n-1}$$

for $n = 0, \ldots, g$ and $a_{g+1} = -b_g$. Hence

$$b_n = a_0 + \ldots + a_n \tag{16}$$

for $n = 0, \ldots, g$. Now plugging in $u = 1/\sqrt{q}$, we get that

$$\mathcal{L}_q\left(\frac{1}{\sqrt{q}},\chi\right) = \sum_{n=0}^{A} \frac{b_n}{q^{n/2}}\left(1-\frac{1}{\sqrt{q}}\right) + \omega(\chi)\sum_{n=0}^{g-A-1} \frac{\overline{b_n}}{q^{n/2}}\left(1-\frac{1}{\sqrt{q}}\right).$$

Now using equation (16) for b_n and b_{n+1}, substracting the two equations and using the functional equation for b_n, we get that

$$\overline{a_0} + \ldots + \overline{a_{g-n-1}} = \frac{1}{q-1} a_{n+1}\overline{\omega(\chi)} q^{\frac{g}{2}-n} + \frac{\overline{a_{g-n}}}{q-1},$$

and hence

$$a_0 + \ldots + a_{g-n-1} = \frac{1}{q-1}\overline{a_{n+1}}\omega(\chi) q^{\frac{g}{2}-n} + \frac{a_{g-n}}{q-1}.$$

Now we use the equations above for $n = g-1-A$ and $n = A$ and after some manipulations, we get that

$$\mathcal{L}_q\left(\frac{1}{\sqrt{q}},\chi\right) = \sum_{f\in\mathcal{M}_{q,\le A}} \frac{\chi(f)}{q^{\deg(f)/2}} + \omega(\chi)\sum_{f\in\mathcal{M}_{q,\le g-A-1}} \frac{\overline{\chi}(f)}{q^{\deg(f)/2}}$$
$$+ \frac{a_{A+1}}{(1-\sqrt{q})q^{\frac{A+1}{2}}} + \omega(\chi)\frac{\overline{a_{g-A}}}{(1-\sqrt{q})q^{\frac{g-A}{2}}},$$

and the result follows. ∎

The following lemmas provide upper and lower bounds for L–functions.

Lemma 2.5. *Let χ be a primitive cubic character of conductor h defined over $\mathbb{F}_q[T]$. Then, for $\mathrm{Re}(s) \ge 1/2$ and for all $\varepsilon > 0$,*

$$|L_q(s,\chi)| \ll_\varepsilon q^{\varepsilon\deg(h)}.$$

Proof. This is the Lindelöf hypothesis in function fields. It is Theorem 5.1 in [BCD+18]. For the quadratic case see also the proof of Corollary 8.2 in [Flo17a] and Theorem 3.3 in [AT14]. ∎

Lemma 2.6. *Let χ be a primitive cubic character of conductor h defined over $\mathbb{F}_q[T]$. Then, for $\mathrm{Re}(s) \ge 1$ and for all $\varepsilon > 0$,*

$$|L_q(s,\chi)| \gg_\varepsilon q^{-\varepsilon\deg(h)}.$$

Proof. First assume that χ is an odd character. Recall that $g = \deg(h) - 1$. Then

$$L_q(s,\chi) = \prod_{j=1}^{g} \left(1 - q^{\frac{1}{2}-s} e^{2\pi i \theta_j}\right),$$

and

$$\frac{1}{\log q} \frac{L_q'}{L_q}(s,\chi) = -g + \sum_{j=1}^{g} \frac{1}{1 - q^{\frac{1}{2}-s} e^{2\pi i \theta_j}}.$$

From the above it follows that if $\mathrm{Re}(s) \geq 1$ then

$$\left|\frac{L_q'}{L_q}(s,\chi)\right| \ll \deg(h). \tag{17}$$

Now for $\mathrm{Re}(s) = \sigma > 1$ we have

$$\log L_q(s,\chi) = \sum_{f \in \mathcal{M}_q} \frac{\Lambda(f)\chi(f)}{|f|^s \deg(f)},$$

where $\Lambda(f)$ is the von Mangoldt function, equal to $\deg(P)$ when $f = P^n$ for P prime, and zero otherwise.

Hence

$$|\log L_q(s,\chi)| \leq \sum_{f \in \mathcal{M}_q} \frac{\Lambda(f)}{|f|^\sigma \deg(f)} = \log \zeta_q(\sigma) = -\log\left(1 - q^{1-\sigma}\right).$$

If $\sigma \geq 1 + \frac{1}{\deg h}$ then it follows that

$$|\log L_q(s,\chi)| \ll \log(\deg(h)). \tag{18}$$

Now if $s = 1 + it$ and $s_1 = 1 + \frac{1}{\deg(h)} + it$, we have that

$$\log L_q(s,\chi) - \log L_q(s_1,\chi) = \int_{s_1}^{s} \frac{L_q'}{L_q}(z)\, dz \ll |s_1 - s| \deg(h) \ll 1,$$

where the first inequality follows from (17). Combining the above and (18) it follows that when $\mathrm{Re}(s) = 1$ we have

$$|\log L_q(s,\chi)| \ll \log(\deg(h)).$$

Now

$$\left|\log \frac{1}{|L_q(s,\chi)|}\right| = |\mathrm{Re} \log L_q(s,\chi)| \leq |\log L_q(s,\chi)| \ll \log(\deg(h)),$$

and then

$$|L_q(s,\chi)| \gg \deg(h)^{-1} \gg q^{-\varepsilon \deg(h)}.$$

When χ is an even character, the L-function has an extra factor of $1 - q^{-s}$ which does not affect the bound.

□

Note that using ideas as in the work of Carneiro and Chandee [CC11] one could prove that

$$|L_q(s,\chi)| \gg \frac{1}{\log(\deg(h))},$$

when $\mathrm{Re}(s) = 1$. For our purposes the lower bound $\gg \deg(h)^{-1}$ is enough and we do not have to follow the method in [CC11].

2.2. **Primitive cubic characters over $\mathbb{F}_q[T]$.** Let q be an odd power of a prime. In this section we describe the cubic characters over $\mathbb{F}_q[T]$ when $q \equiv 1 \pmod 3$ (the Kummer case) and $q \equiv 2 \pmod 3$ (the non-Kummer case).

We first suppose that q is odd and $q \equiv 1 \pmod 3$.

We define the cubic residue symbol χ_P, for P an irreducible monic polynomial in $\mathbb{F}_q[T]$. Let $a \in \mathbb{F}_q[T]$. If $P \mid a$, then $\chi_P(a) = 0$, and otherwise $\chi_P(a) = \alpha$, where α is the unique root of unity in $\mathbb{C}$ such that

$$a^{\frac{q^{\deg(P)}-1}{3}} \equiv \Omega(\alpha) \pmod P.$$

We extend the definition by multiplicativity to any monic polynomial $F \in \mathbb{F}_q[T]$ by defining for $F = P_1^{e_1} \dots P_s^{e_s}$, with distinct primes P_i,

$$\chi_F = \chi_{P_1}^{e_1} \cdots \chi_{P_s}^{e_s}.$$

Then, χ_F is a cubic character modulo $P_1 \dots P_s$. It is primitive if and only if all the e_i are 1 or 2. Then it follows that the conductors of the primitive cubic characters are the square-free monic polynomials $F \in \mathbb{F}_q[T]$, and for each such conductor, there are $2^{\omega(F)}$ characters, where $\omega(F)$ is the number of primes dividing F. More precisely, for any conductor $F = F_1F_2$ with $(F_1, F_2) = 1$ we have the primitive character of modulus F given by

$$\chi_{F_1F_2^2} = \chi_{F_1}\chi_{F_2}^2 = \chi_{F_1}\overline{\chi_{F_2}}.$$

We will often use the fact that when $q \equiv 1 \pmod 6$, the cubic reciprocity law is very simple.

Lemma 2.7 (Cubic Reciprocity)**.** *Let $a, b \in \mathbb{F}_q[T]$ be relatively prime monic polynomials, and let χ_a and χ_b be the cubic residue symbols defined above. If $q \equiv 1 \pmod 6$, then*

$$\chi_a(b) = \chi_b(a).$$

Proof. This is Theorem 3.5 in [Ros02] in the case where a and b are monic and $q \equiv 1 \pmod 6$. ☐

Lemma 2.8. *Suppose q is odd and $q \equiv 1 \pmod 3$, and let $N_{\mathrm{K}}(g)$ be the number of odd primitive cubic characters with conductor of genus g. Then, for all $\varepsilon > 0$,*

$$N_{\mathrm{K}}(g) = B_{\mathrm{K},1}gq^g + B_{\mathrm{K},2}q^g + O\left(q^{(1/2+\varepsilon)g}\right),$$

where $B_{\mathrm{K},1} = q\mathcal{F}_{\mathrm{K}}(1/q)$, $B_{\mathrm{K},2} = (2q\mathcal{F}_{\mathrm{K}}(1/q) - \mathcal{F}_{\mathrm{K}}'(1/q))$, and $\mathcal{F}_{\mathrm{K}}$ is given by (19), and the implicit constant in the error term depends on both q and ε.

Proof. Let $a(F)$ be the number of cubic primitive characters of conductor F. By the above discussion, the generating series for $a(F)$ is given by

$$\mathcal{G}_{\mathrm{K}}(u) = \sum_{F \in \mathcal{M}_q} a(F)u^{\deg(F)} = \prod_P \left(1 + 2u^{\deg(P)}\right),$$

which is analytic for $|u| < 1/q$ with a double pole at $u = 1/q$. We write

$$\mathcal{F}_{\mathrm{K}}(u) = \mathcal{G}_{\mathrm{K}}(u)(1 - qu)^2 = \prod_P \left(1 - 3u^{2\deg(P)} + 2u^{3\deg(P)}\right). \tag{19}$$

We seek a formula for the coefficient of u^{g+1} in $\mathcal{G}_{\mathrm{K}}(u)$, since odd characters have conductor of degree $g+1$. Then, using Perron's formula (Lemma 2.1), and moving the integral from $|u| = q^{-2}$ to $|u| = q^{-(1/2+\varepsilon)}$ while picking the residue of the (double) pole at $u = q^{-1}$, we have

$$\begin{aligned} N_{\mathrm{K}}(g) &= \frac{1}{2\pi i}\oint_{|u|=q^{-2}} \frac{\mathcal{F}_{\mathrm{K}}(u)}{u^{g+1}(1-qu)^2}\frac{du}{u} \\ &= \mathcal{F}_{\mathrm{K}}(1/q) g q^{g+1} + \left(2q\mathcal{F}_{\mathrm{K}}(1/q) - \mathcal{F}_{\mathrm{K}}'(1/q)\right) q^g + O\left(q^{(1/2+\varepsilon)g}\right). \end{aligned}$$

□

For each primitive cubic character $\chi_{F_1F_2^2}$, we have that for $\alpha \in \mathbb{F}_q^*$,

$$\chi_{F_1F_2^2}(\alpha) = \Omega^{-1}\left(\alpha^{\frac{q-1}{3}(\deg(F_1)+2\deg(F_2))}\right),$$

and $\chi_{F_1F_2^2}$ is even if and only if $\deg(F_1)+2\deg(F_2) \equiv 0 \pmod 3$. If $\chi_{F_1F_2^2}$ is odd, the restriction to $\mathbb{F}_q^*$ is χ_3 when $\deg(F_1) + 2\deg(F_2) \equiv 1 \pmod 3$, and χ_3^2 when $\deg(F_1) + 2\deg(F_2) \equiv 2 \pmod 3$, where χ_3 is defined by (3).

Then, since the conductor of $\chi_{F_1F_2^2}$ is $F = F_1F_2$, we have from (8) that

$$\deg(F_1) + \deg(F_2) = \begin{cases} g+2 & \deg(F_1) + 2\deg(F_2) \equiv 0 \pmod 3, \\ g+1 & \deg(F_1) + 2\deg(F_2) \not\equiv 0 \pmod 3. \end{cases}$$

For convenience, recall that we restrict to the odd cubic primitive characters such that the restriction to $\mathbb{F}_q^*$ is χ_3.

We have then showed the following.

Lemma 2.9. *Suppose q is odd and $q \equiv 1 \pmod 3$. Then,*

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \mathrm{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L_q(1/2,\chi) = \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2\equiv 1 \pmod 3}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ F_2\in\mathcal{H}_{q,d_2} \\ (F_1,F_2)=1}} L_q\left(\frac{1}{2}, \chi_{F_1}\overline{\chi_{F_2}}\right),$$

and the sign of the functional equation of $L_q(s,\chi_{F_1}\overline{\chi_{F_2}})$ is equal to

$$\omega(\chi_{F_1}\overline{\chi_{F_2}}) = \overline{\epsilon(\chi_3)}\, q^{-(d_1+d_2)/2} G(\chi_{F_1}\overline{\chi_{F_2}}),$$

where χ_3 is the cubic residue symbol on $\mathbb{F}_q^*$ defined by (3) *and $\epsilon(\chi_3)$ is defined by* (5).

We now suppose that $q \equiv 2 \pmod 3$. Then there are no cubic characters modulo P for primes of odd degree since $3 \nmid q^{\deg(P)} - 1$. For each prime P of even degree and $a \in \mathbb{F}_q[T]$, we have the cubic residue symbol $\chi_P(a) = \alpha$, where α is the unique cubic root of unity in $\mathbb{C}$ such that

$$a^{\frac{q^{\deg(P)}-1}{3}} \equiv \Omega(\alpha) \pmod P,$$

where Ω takes values in the cubic roots of unity in $\mathbb{F}_{q^2}$.

We extend the definition by multiplicativity to any monic polynomial $F \in \mathbb{F}_q[T]$ supported on primes of even degree by defining for $F = P_1^{e_1} \dots P_s^{e_s}$, with distinct primes P_i of even degree,

$$\chi_F = \chi_{P_1}^{e_1} \cdots \chi_{P_s}^{e_s}.$$

Then, χ_F is a cubic character modulo $P_1 \dots P_s$, and it is primitive if and only if all the e_i are 1 or 2. It follows that the conductors of the primitive cubic characters are the square-free polynomials $F \in \mathbb{F}_q[T]$ supported on primes of even degree, and for each such conductor, there are $2^{\omega(F)}$ characters, where $\omega(F)$ is the number of primes dividing F.

Lemma 2.10. *Suppose q is odd and $q \equiv 2 \pmod 3$, and let $N_{\mathrm{nK}}(g)$ be the number of even primitive cubic characters with conductor of genus g. Then, for all $\varepsilon > 0$,*

$$N_{\mathrm{nK}}(g) = \begin{cases} B_{\mathrm{nK}} q^g + O\left(q^{(1/2+\varepsilon)g}\right) & 2 \mid g, \\ 0 & \text{otherwise,} \end{cases}$$

where $B_{\mathrm{nK}} = q^2 \mathcal{F}_{\mathrm{nK}}(1/q)$ and $\mathcal{F}_{\mathrm{nK}}(u)$ is defined by (20), and the implicit constant in the error term depends on q and ε.

As we will see later, any cubic characer over $\mathbb{F}_q[T]$ is even when $q \equiv 2 \pmod 3$. We have added the condition that the character is even for clarity.

Proof. Let $a(F)$ be number of cubic primitive characters of conductor F. By the above discussion, the generating series for $a(F)$ is given by

$$\mathcal{G}_{\mathrm{nK}}(u) = \sum_{F \in \mathcal{M}_q} a(F) u^{\deg(F)} = \prod_{2 \mid \deg(P)} \left(1 + 2u^{\deg(P)}\right),$$

which is analytic for $|u| < 1/q$ with simple poles at $u = 1/q$ and $u = -1/q$. This follows from the fact that the primes of even degree in $\mathbb{F}_q[T]$ are exactly the primes splitting in the quadratic extension $\mathbb{F}_{q^2}(T)/\mathbb{F}_q(T)$. Recall that

$$\mathcal{Z}_{q^2}(u^2) = \prod_{2 \mid \deg(P)} \left(1 - u^{\deg(P)}\right)^{-2} \prod_{2 \nmid \deg(P)} \left(1 - u^{2\deg(P)}\right)^{-1},$$

where $u = q^{-s}$ and the product is over primes P of $\mathbb{F}_q[T]$. The analytic properties of $\mathcal{G}_{\mathrm{nK}}(u)$ then follow from the analytic properties of $\mathcal{Z}_{q^2}(u^2)$, which is analytic everywhere except for simple poles when $u^2 = q^{-2}$.

We write

$$\begin{aligned} \mathcal{F}_{\mathrm{nK}}(u) &= \mathcal{G}_{\mathrm{nK}}(u)(1-qu)(1+qu) \\ &= \prod_{2 \mid \deg(P)} \left(1 + 2u^{\deg(P)}\right)\left(1 - u^{\deg(P)}\right)^2 \prod_{2 \nmid \deg(P)} \left(1 - u^{\deg(P)}\right)\left(1 + u^{\deg(P)}\right) \\ &= \prod_{2 \mid \deg(P)} \left(1 - 3u^{2\deg(P)} + 2u^{3\deg(P)}\right) \prod_{2 \nmid \deg(P)} \left(1 - u^{2\deg(P)}\right), \end{aligned} \tag{20}$$

which is analytic for $|u| < q^{-1/2}$. We seek a formula for the coefficient of u^{g+2} in $\mathcal{G}_{\mathrm{K}}(u)$, since even characters have conductor of degree $g+2$. Then, using Perron's formula (Lemma 2.1), and moving the integral from $|u| = q^{-2}$ to $|u| = q^{-(1/2+\varepsilon)}$ while picking the poles at $u = \pm q^{-1}$, we have

$$\begin{aligned} N_{\mathrm{nK}}(g) &= \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\mathcal{F}_{\mathrm{nK}}(u)}{u^{g+2}(1-qu)(1+qu)} \frac{du}{u} \\ &= \left(\frac{\mathcal{F}_{\mathrm{nK}}(1/q)}{2} + (-1)^g \frac{\mathcal{F}_{\mathrm{nK}}(-1/q)}{2}\right) q^{g+2} + O\left(q^{(1/2+\varepsilon)g}\right). \end{aligned}$$

Notice that $\mathcal{F}_{\mathrm{nK}}(1/q) = \mathcal{F}_{\mathrm{nK}}(-1/q)$, so the main term is zero when g is odd. In this case, we already knew that there are no primitive cubic characters with conductor of odd degree as every prime which divides the conductor has even degree. For g even, this proves the result. $\square$

It is more natural to describe these characters as characters over $\mathbb{F}_{q^2}[T]$ restricting to characters over $\mathbb{F}_q[T]$ as in the work of Bary-Soroker and Meisner [BSM19] (generalizing the work of Baier and Young [BY10] from number fields to function fields) by counting characters of $\mathbb{F}_{q^2}[T]$ whose restrictions to $\mathbb{F}_q[T]$ are cubic characters over $\mathbb{F}_q[T]$. In what follows, for f in the quadratic extension $\mathbb{F}_{q^2}[T]$ over $\mathbb{F}_q[T]$, we will denote by $\tilde{f}$ the Galois conjugate of f.

Notice that $q^2 \equiv 1 \pmod 3$, and we have then described the primitive cubic characters of $\mathbb{F}_{q^2}[T]$ in the paragraph before Lemma 2.10. Suppose that π is a prime in $\mathbb{F}_{q^2}[T]$ lying over a prime $P \in \mathbb{F}_q[T]$ such that P splits as $\pi\tilde{\pi}$. Notice that P splits in $\mathbb{F}_{q^2}[T]$ if and only if the degree of P is even. It is easy to see that the restriction of χ_π to $\mathbb{F}_q[T]$ is the character χ_P, and the restriction of $\chi_{\tilde{\pi}}$ to $\mathbb{F}_q[T]$ is the character $\overline{\chi_P}$ (possibly exchanging π and $\tilde{\pi}$). Then by running over all the characters χ_F where $F \in \mathbb{F}_{q^2}[T]$ is square-free and not divisible by a prime P of $\mathbb{F}_q[T]$, we are counting exactly the characters over $\mathbb{F}_{q^2}[T]$ whose restrictions are cubic characters over $\mathbb{F}_q[T]$, and each character over $\mathbb{F}_q[T]$ is counted exactly once. For more details, we refer the reader to [BSM19].

We also remark that any cubic character over $\mathbb{F}_q[T]$ is even when $q \equiv 2 \pmod 3$. Indeed, by the classification above, such a character comes from χ_F with $F \in \mathbb{F}_{q^2}[T]$, and for $\alpha \in \mathbb{F}_q \subseteq \mathbb{F}_{q^2}$, we have

$$\chi_F(\alpha) = \Omega^{-1}\left(\alpha^{\frac{q^2-1}{3}\deg(F)}\right).$$

Since q is odd and $q \equiv 2 \pmod 3$, we have that $(q-1) \mid (q^2-1)/3$.

By (8), if $F \in \mathbb{F}_q[T]$ is the conductor of a cubic primitive character χ over $\mathbb{F}_q[T]$, it follows that $\deg(F) = g+2$. By the classification above, it follows that $F = P_1 \dots P_s$ for distinct primes of even degree, and the character $\chi \pmod F$ is the restriction of a character of conductor $\pi_1 \dots \pi_s$ over $\mathbb{F}_{q^2}[T]$, where π_i is one of the primes lying above P_i. Then the degree of the conductor of this character over $\mathbb{F}_{q^2}[T]$ is equal to $g/2+1$.

We have then proved the following result.

Lemma 2.11. *Suppose $q \equiv 2 \pmod 3$. Then,*

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L_q(1/2, \chi) = \sum_{\substack{F \in \mathcal{H}_{q^2, g/2+1} \\ P \mid F \Rightarrow P \not\in \mathbb{F}_q[T]}} L_q(1/2, \chi_F).$$

2.3. **Generalized cubic Gauss sums and the Poisson summation formula.** Let χ_f be the cubic residue symbol defined before for $f \in \mathbb{F}_q[T]$. This is a character of modulus f, but not necessarily primitive. We define the generalized cubic Gauss sum by

$$G_q(V, f) = \sum_{u \pmod f} \chi_f(u) e_q\left(\frac{uV}{f}\right), \tag{21}$$

with the exponential function defined in (14). We remark that if χ_f has conductor f' with $\deg(f') < \deg(f)$, then $G(\chi_f) \neq G_q(1, f)$.

If $(a,f)=1$, we have

$$G_q(aV,f)=\overline{\chi_f}(a)G_q(V,f). \tag{22}$$

The following lemma shows that the shifted Gauss sum is almost multiplicative as a function of f, and we can determine it on powers of primes. We have the following.

Lemma 2.12. *Suppose that $q\equiv 1 \pmod 6$.*

(i) *If $(f_1,f_2)=1$, then*

$$\begin{aligned} G_q(V,f_1f_2) &= \chi_{f_1}(f_2)^2 G_q(V,f_1)G_q(V,f_2)\\ &= G_q(Vf_2,f_1)G_q(V,f_2). \end{aligned}$$

(ii) *If $V=V_1P^\alpha$ where $P\nmid V_1$, then*

$$G_q(V,P^i)=\begin{cases} 0 & \text{if } i\le\alpha \text{ and } i\not\equiv 0 \pmod 3,\\ \phi(P^i) & \text{if } i\le\alpha \text{ and } i\equiv 0 \pmod 3,\\ -|P|_q^{i-1} & \text{if } i=\alpha+1 \text{ and } i\equiv 0 \pmod 3,\\ \epsilon(\chi_{P^i})\omega(\chi_{P^i})\chi_{P^i}(V_1^{-1})|P|_q^{i-\frac12} & \text{if } i=\alpha+1 \text{ and } i\not\equiv 0 \pmod 3,\\ 0 & \text{if } i\ge\alpha+2, \end{cases}$$

where ϕ is the Euler ϕ-function for polynomials. We recall that $\epsilon(\chi)=1$ when χ is even. For the case of χ_{P^i}, this happens if $3\mid\deg(P^i)$.

Proof. The proof of (i) is the same as in [Flo17c]. We write $u \pmod{f_1f_2}$ as $u=u_1f_1+u_2f_2$ for $u_1 \pmod{f_2}$ and $u_2 \pmod{f_1}$. Then,

$$\begin{aligned} G_q(V,f_1f_2)=&\chi_{f_2}(f_1)\chi_{f_1}(f_2)\sum_{u_1 \,(\mathrm{mod}\, f)_2}\sum_{u_2\,(\mathrm{mod}\, f)_1}\chi_{f_1}(u_2)\chi_{f_2}(u_1)e_q\left(\frac{u_1V}{f_2}\right)e_q\left(\frac{u_2V}{f_1}\right)\\ =&\overline{\chi_{f_1}}(f_2)G_q(V,f_1)G_q(V,f_2) \end{aligned}$$

by cubic reciprocity (see Lemma 2.7). The second line of (i) follows from (22).

Now we focus on the proof of (ii).

Assume that $i\le\alpha$. Then

$$G_q(V,P^i)=\sum_{u \,(\mathrm{mod}\, P^i)}\chi_{P^i}(u)e_q(uV_1P^{\alpha-i}).$$

The exponential above is equal to 1 since $uV_1P^{\alpha-i}\in\mathbb{F}_q[T]$, and if $i\equiv 0 \pmod 3$, then $\chi_{P^i}(u)=1$ when $(u,P)=1$. The conclusion easily follows in this case. If $i\not\equiv 0\pmod 3$, the conclusion also follows easily from orthogonality of characters.

Now assume that $i=\alpha+1$. Write $u \pmod{P^i}$ as $u=PA+C$, with $A \pmod{P^{i-1}}$ and $C \pmod P$. Then

$$G_q(V,P^i)=\sum_{A\,(\mathrm{mod}\, P^{i-1})}\sum_{C\,(\mathrm{mod}\, P)}\chi_{P^i}(C)e_q\left(\frac{CV_1}{P}\right)=|P|_q^{i-1}\chi_{P^i}(V_1^{-1})\sum_{C\,(\mathrm{mod}\, P)}\chi_{P^i}(C)e_q\left(\frac{C}{P}\right).$$

If $i\equiv 0 \pmod 3$, then $\chi_{P^i}(V_1^{-1})=1$ and

$$\sum_{C\,(\mathrm{mod}\, P)}\chi_{P^i}(C)e_q\left(\frac{C}{P}\right)=\sum_{\substack{C\,(\mathrm{mod}\, P)\\ C\neq 0}}e_q\left(\frac{C}{P}\right)=-1,$$

and the conclusion follows. So assume that $i \not\equiv 0 \pmod 3$. Then

$$\sum_{C \pmod P} \chi_{P^i}(C) e_q\left(\frac{C}{P}\right) = \begin{cases} -q \sum_{f \in \mathcal{M}_{q,\deg(P)-1}} \chi_{P^i}(f) & 3 \mid \deg(P), \\ \epsilon(\chi_{P^i})\sqrt{q} \sum_{f \in \mathcal{M}_{q,\deg(P)-1}} \chi_{P^i}(f) & 3 \nmid \deg(P), \end{cases}$$

and using Lemma 2.2, we can rewrite this as

$$\sum_{C \pmod P} \chi_{P^i}(C) e_q\left(\frac{C}{P}\right) = \begin{cases} \omega(\chi_{P^i}) q^{\deg(P)/2} & 3 \mid \deg(P), \\ \epsilon(\chi_{P^i})\omega(\chi_{P^i}) q^{\deg(P)/2} & 3 \nmid \deg(P). \end{cases}$$

Thus, we get

$$G_q(V, P^i) = \begin{cases} \omega(\chi_{P^i})\chi_{P^i}(V_1^{-1})|P|_q^{i-1/2} & 3 \mid \deg(P), \\ \\ \epsilon(\chi_{P^i})\omega(\chi_{P^i})\chi_{P^i}(V_1^{-1})|P|_q^{i-1/2} & 3 \nmid \deg(P). \end{cases}$$

If $i \geq \alpha + 2$, then again the proof goes through exactly as in [Flo17c]. □

Now we state the Poisson summation formula for cubic characters although we will not use it directly. Recall that for any non-principal character on $\mathbb{F}_q^*$, $\tau(\chi)$ is the standard Gauss sum defined over $\mathbb{F}_q$ by equation (4). Also recall that for χ odd, $|\tau(\chi)| = \sqrt{q}$, and $\tau(\chi) = \epsilon(\chi)\sqrt{q}$. For χ even, $\epsilon(\chi) = 1$.

Proposition 2.13. *Let f be a monic polynomial in $\mathbb{F}_q[x]$ with $\deg(f) = n$, and let m be a positive integer. If $\deg(f) \equiv 0 \pmod 3$, then*

$$\sum_{h \in \mathcal{M}_{q,m}} \chi_f(h) = \frac{q^m}{|f|_q}\left[G_q(0,f) + (q-1)\sum_{V \in \mathcal{M}_{q,\leq n-m-2}} G_q(V,f) - \sum_{V \in \mathcal{M}_{q,n-m-1}} G_q(V,f)\right].$$

If $\deg(f) \not\equiv 0 \pmod 3$, then

$$\sum_{h \in \mathcal{M}_{q,m}} \chi_f(h) = \frac{q^{m+\frac{1}{2}}}{|f|_q}\overline{\epsilon(\chi_f)} \sum_{V \in \mathcal{M}_{q,n-m-1}} G_q(V,f).$$

We remark that the sums above are zero when taken over $\mathcal{M}_{q,j}$ with $j < 0$.

Proof. As in [Flo17c], we have

$$\sum_{h \in \mathcal{M}_{q,m}} \chi_f(h) = \frac{q^m}{|f|_q} \sum_{\deg(V) \leq n-m-1} G_q(V,f) e_q\left(-\frac{Vx^m}{f}\right).$$

Using (22), we have

$$\begin{aligned}\sum_{h \in \mathcal{M}_{q,m}} \chi_f(h) =& \frac{q^m}{|f|_q}\Big[G_q(0,f) + \sum_{a=1}^{q-1} \overline{\chi_f}(a) \sum_{V \in \mathcal{M}_{q,\leq n-m-2}} G_q(V,f) \\ &+ \sum_{a=1}^{q-1} \overline{\chi_f}(a) e^{-2\pi i \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)/p} \sum_{V \in \mathcal{M}_{q,n-m-1}} G_q(V,f)\Big].\end{aligned}$$

Now if $\deg(f) \equiv 0 \pmod 3$ then χ_f is an even character, and

$$\sum_{a=1}^{q-1} \overline{\chi_f}(a) = q-1, \quad \sum_{a=1}^{q-1} \overline{\chi_f}(a) e^{-2\pi i \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)/p} = -1.$$

If $\deg(f) \not\equiv 0 \pmod 3$ then χ_f is an odd character, and

$$\sum_{a=1}^{q-1} \overline{\chi_f}(a) = 0, \quad \sum_{a=1}^{q-1} \overline{\chi_f}(a) e^{-2\pi i \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)/p} = \overline{\tau(\chi_f)}.$$

Also, if $\deg(f) \not\equiv 0 \pmod 3$, then f is not a cube, and the character χ_f is non-trivial, which implies that $G_q(0, f) = 0$ by the orthogonality relations. $\square$

3. Averages of cubic Gauss sums

In this section we prove several results concerning averages of cubic Gauss sums which will be needed later. Assume throughout that $q \equiv 1 \pmod 6$. For $a, n \in \mathbb{Z}$ and n positive, we denote by $[a]_n$ the residue of a modulo n such that $0 \leq [a]_n \leq n-1$.

We will prove the following.

Proposition 3.1. *Let $f = f_1 f_2^2 f_3^3$ with f_1 and f_2 square-free and coprime. We have, for $\varepsilon > 0$,*

$$\sum_{\substack{F \in \mathcal{M}_{q,d} \\ (F,f)=1}} G_q(f,F) = \delta_{f_2=1} \frac{q^{\frac{4d}{3} - \frac{4}{3}[d+\deg(f_1)]_3}}{\zeta_q(2) |f_1|_q^{2/3}} \overline{G_q(1,f_1)} \rho(1, [d+\deg(f_1)]_3) \prod_{P | f_1 f_3^*} \left(1 + \frac{1}{|P|_q}\right)^{-1}$$
$$+ O\left(\delta_{f_2=1} \frac{q^{\frac{d}{3}+\varepsilon d}}{|f_1|_q^{\frac{1}{6}}}\right) + \frac{1}{2\pi i} \oint_{|u|=q^{-\sigma}} \frac{\tilde{\Psi}_q(f,u)}{u^d} \frac{du}{u}$$

with $2/3 < \sigma < 4/3$ and where $\tilde{\Psi}_q(f,u)$ is given by (23) *and $\rho(1, [d+\deg(f_1)]_3)$ is given by* (28).

Moreover, we have

$$\frac{1}{2\pi i} \oint_{|u|=q^{-\sigma}} \frac{\tilde{\Psi}_q(f,u)}{u^d} \frac{du}{u} \ll q^{\sigma d} |f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)}.$$

To prove Proposition 3.1 we first need to understand the generating series of the Gauss sums. Let

$$\Psi_q(f,u) = \sum_{F \in \mathcal{M}_q} G_q(f,F) u^{\deg(F)}$$

and

$$\tilde{\Psi}_q(f,u) = \sum_{\substack{F \in \mathcal{M}_q \\ (F,f)=1}} G_q(f,F) u^{\deg(F)}. \tag{23}$$

The function $\Psi_q(f,u)$ was studied by Hoffstein [Hof92], and we will cite here the relevant results that we need, following the notation of Patterson [Pat07]. We postpone the proof of Proposition 3.1 to the next sections.

3.1. **Analytic properties of the generating series.** We first study the general Gauss sums associated to the n^{th} residue symbols as done in [Hof92, Pat07], and we specialize to $n = 3$ later. We always assume that $q \equiv 1 \pmod n$. Let $\eta \in (\mathbb{F}_q((1/T))^\times$ and define

$$\psi(f, \eta, u) = (1 - u^n q^n)^{-1} \sum_{\substack{F \in \mathcal{M}_q \\ F \sim \eta}} G_q(f, F) u^{\deg(F)},$$

where the equivalence relation is given by

$$F \sim \eta \Leftrightarrow F/\eta \in (\mathbb{F}_q((1/T))^\times)^n.$$

There is difference between our definition of $\psi(f, \eta, u)$ above, and the definition of $\psi(r, \eta, u)$ in [Pat07, p. 245]: we are summing over monic polynomials in $\mathbb{F}_q[T]$, and not all polynomials in $\mathbb{F}_q[T]$, as in [Hof92]. This explains the extra factors of the type $(q-1)/n$ which appear in [Pat07]. Because our polynomials are monic, it is enough to consider the equivalence classes that separate degrees, namely $\eta = \pi_\infty^{-i}$, where π_∞ is the uniformizer of the prime at infinity, i.e. T^{-1} in the completion $\mathbb{F}_q((1/T))$.

A little bit of basic algebra in $\mathbb{F}_q((1/T))$ shows that for any $i \in \mathbb{Z}$,

$$\psi(f, \pi_\infty^{-i}, u) = (1 - u^n q^n)^{-1} \sum_{\substack{F \in \mathcal{M}_q \\ \deg(F) \equiv i \pmod n}} G_q(f, F) u^{\deg(F)}.$$

Then $\psi(f, \pi_\infty^{-i}, u)$ depends only on the value of i modulo n.

We remark that since we have fixed the map between the n^{th} roots of unity in $\mathbb{F}_q^*$ and $\mu_n \subseteq \mathbb{C}^*$ at the beginning of this paper, we do not make this dependence explicit in our notation, as it is done in [Pat07].

Then we can write the generating series $\Psi_q(f, u)$ as

$$\Psi_q(f, u) = (1 - u^n q^n) \sum_{i=0}^{n-1} \psi(f, \pi_\infty^{-i}, u). \tag{24}$$

The main result of Hoffstein is a functional equation for $\psi(f, \pi_\infty^{-i}, u)$ [Hof92, Proposition 2.1], which we write below using the notation of Patterson.

Proposition 3.2. [Hof92, Proposition 2.1] *For $0 \leq i < n$ and $f \in \mathcal{M}_q$, we have*

$$q^{is}\psi(f, \pi_\infty^{-i}, q^{-s}) = q^{n(s-1)E} q^{(2-s)i} \psi(f, \pi_\infty^{-i}, q^{s-2}) \frac{1 - q^{-1}}{(1 - q^{ns-n-1})}$$
$$+ W_{f,i} q^{n(2-s)(B-2)} q^{2n - \deg(f) + 2i - 2} q^{(2-s)[1 + \deg(f) - i]_n} \psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2}) \frac{1 - q^{n-ns}}{1 - q^{ns-n-1}},$$

where $B = [(1 + \deg(f) - i)/n]$, $E = 1 - [(\deg(f) + 1 - 2i)/n]$, and $W_{f,i} = \tau(\chi_3^{2i-1}\overline{\chi_f})$ with χ_3 given by equation (3).

Remark 3.3. Note that we can rewrite the functional equation in the following form (for $n = 3$)

$$(1 - q^4 u^3)\psi(f, \pi_\infty^{-i}, u) = |f|_q u^{\deg(f)} \left[a_1(u) \psi\left(f, \pi_\infty^{-i}, \frac{1}{q^2 u}\right) + a_2(u) \psi\left(f, \pi_\infty^{i-1-\deg(f)}, \frac{1}{q^2 u}\right) \right], \tag{25}$$

where

$$a_1(u) = -(q^2u)(qu)^{-[\deg(f)+1-2i]_3}(1-q^{-1}),\ a_2(u) = -W_{f,i}(qu)^{-2}(1-q^3u^3),$$

with $W_{f,i}$ as above.

By setting $u = q^{-s}$ and letting $u \to \infty$ in the functional equation, Hoffstein showed that

$$\psi(f, \pi_\infty^{-i}, u) = \frac{u^i P(f,i,u^n)}{(1-q^{n+1}u^n)}, \tag{26}$$

where $P(f,i,x)$ is a polynomial of degree at most $[(1+\deg(f)-i)/n]$ in x. We remark that while $\psi(f, \pi_\infty^{-i}, u)$ depends only on the value of i modulo n, this is not the case for $P(f,i,u^n)$.

Remark 3.4. Note that, from (26), the left-hand side of equation (25) above has no pole at $u^3 = 1/q^2$, so neither does the right-hand side.

We let

$$C(f,i) = \sum_{F \in \mathcal{M}_{q,i}} G_q(f,F).$$

By setting $x = u^n = q^{-ns}$, we can write for $0 \le i \le n-1$,

$$P(f,i,x) = \frac{1-q^{n+1}x}{1-q^n x} \sum_{j \ge 0} C(f, i+nj)x^j.$$

If $j \ge [(1+\deg(f)-i)/n]$ with $0 \le i \le n-1$, then we have the recurrence relation

$$C(f, i+n(j+1)) = q^{n+1}C(f, i+nj).$$

Using that, we can rewrite, for any $B \ge [(1+\deg(f)-i)/n]$,

$$\begin{aligned} P(f,i,x) &= \frac{1-q^{n+1}x}{1-q^n x}\left(\sum_{0\le j<B} C(f,i+nj)x^j + \sum_{j \ge B} C(f,i+nB)(q^{n+1})^{j-B}x^j\right) \\ &= \frac{1-q^{n+1}x}{1-q^n x}\left(\sum_{0\le j<B} C(f,i+nj)x^j + C(f,i+nB)x^B \sum_{j\ge 0}(q^{n+1})^j x^j\right) \\ &= \frac{1-q^{n+1}x}{1-q^n x}\sum_{0\le j<B} C(f,i+nj)x^j + \frac{C(f,i+nB)}{1-q^n x}x^B. \end{aligned} \tag{27}$$

Let

$$\rho(f,i) = \lim_{s \to 1+\frac{1}{n}} (1-q^{n+1-ns})q^{is}\psi(f, \pi_\infty^{-i}, q^{-s}) = P(f,i,q^{-n-1}). \tag{28}$$

Using the formula above for $P(f,i,x)$, it follows that

$$\rho(f,i) = \frac{C(f,i')}{(1-q^{-1})q^{\frac{n+1}{n}(i'-i)}},$$

where $i' \equiv i \pmod n$, and $i' \ge \deg(f)$.

To prove Proposition 3.1 we need to obtain an explicit formula for the residue in equation (28) which we do in the next subsection.

3.2. **Explicit formula for the residue $\rho(f,i)$.** From now on, we will specialize to $n=3$. For π prime, following Patterson's notation, let

$$\psi_\pi(f,\pi_\infty^{-i},u)=(1-u^3q^3)^{-1}\sum_{\substack{F\in\mathcal{M}_q\\ \deg(F)\equiv i \,(\mathrm{mod}\,3)\\ (F,\pi)=1}} G_q(f,F)u^{\deg(F)}.$$

We will need the following result.

Lemma 3.5. *Let π be a prime such that $\pi\nmid f$. We have the following relations*

$$\psi_\pi(f,\pi_\infty^{-i},q^{-s}) = \psi(f,\pi_\infty^{-i},q^{-s})-G_q(f,\pi)|\pi|_q^{-s}\psi_\pi(f\pi,\pi_\infty^{-i+\deg(\pi)},q^{-s}), \tag{29}$$

$$\psi_\pi(f\pi,\pi_\infty^{-i},q^{-s}) = \psi(f\pi,\pi_\infty^{-i},q^{-s})-\overline{G_q(f,\pi)}|\pi|_q^{1-2s}\psi_\pi(f,\pi_\infty^{-i+2\deg(\pi)},q^{-s}), \tag{30}$$

$$\psi_\pi(f\pi^2,\pi_\infty^{-i},q^{-s}) = (1-|\pi|_q^{2-3s})^{-1}\psi(f\pi^2,\pi_\infty^{-i},q^{-s}). \tag{31}$$

Proof. These equations appear in page 249 of [Pat07] as part of the "Hecke theory" equations. For completeness we give here the details of the proof of (30). The proofs of the other two identities proceed in a similar fashion. Consider

$$\begin{aligned}\psi_\pi(f\pi,\pi_\infty^{-i},q^{-s}) =&(1-q^{3(1-s)})^{-1}\sum_{\substack{F\in\mathcal{M}_q\\ \deg(F)\equiv i \,(\mathrm{mod}\,3)\\ (F,\pi)=1}}\frac{G_q(f\pi,F)}{|F|_q^s}\\ =&\psi(f\pi,\pi_\infty^{-i},q^{-s})-(1-q^{3(1-s)})^{-1}\sum_{\substack{F_1\in\mathcal{M}_q\\ \deg(F_1)\equiv i-\deg(\pi)\,(\mathrm{mod}\,3)}}\frac{G_q(f\pi,\pi F_1)}{|\pi|_q^s|F_1|_q^s}.\end{aligned}$$

Note that in the second sum above, we need $\pi||F_1$, otherwise the Gauss sum will vanish by Lemma 2.12. We write $F_1=\pi F_2$ with $\pi\nmid F_2$. Part (i) of Lemma 2.12 implies that $G_q(f\pi,\pi^2F_2)=G_q(f\pi^3,F_2)G_q(f\pi,\pi^2)=G_q(f,F_2)G_q(f\pi,\pi^2)$. Moreover, part (ii) of Lemma 2.12 implies that $G_q(f\pi,\pi^2)=|\pi|_q\overline{G_q(f,\pi)}$, where we have used that $\chi_\pi(-1)=1$ since it is a cubic character. Putting all of this together yields (30). ☐

Lemma 3.6. *Let π be a prime such that $\pi\nmid f$. We have the following relation*

$$\psi(f\pi^{j+3},\pi_\infty^{-i},q^{-s})-|\pi|_q^{3-3s}\psi(f\pi^j,\pi_\infty^{-i},q^{-s})=(1-|\pi|_q^{2-3s})\psi_\pi(f\pi^j,\pi_\infty^{-i},q^{-s}). \tag{32}$$

Proof. We have

(33)

$$\begin{aligned}\sum_{\deg(F)\equiv i\,(\mathrm{mod}\,3)}\frac{G_q(f\pi^j,F)}{|F|_q^s}=&\sum_{\substack{\deg(F)\equiv i\,(\mathrm{mod}\,3)\\ (F,\pi)=1}}\frac{G_q(f\pi^j,F)}{|F|_q^s}+\sum_{\ell=1}^{[\frac{j}{3}]}|\pi|_q^{-3\ell s}\sum_{\substack{\deg(F)\equiv i\,(\mathrm{mod}\,3)\\ (F,\pi)=1}}\frac{G_q(f\pi^j,\pi^{3\ell}F)}{|F|_q^s}\\ &+|\pi|_q^{-(j+1)s}\sum_{\substack{\deg(F)\equiv i-(j+1)\deg(\pi)\,(\mathrm{mod}\,3)\\ (F,\pi)=1}}\frac{G_q(f\pi^j,\pi^{j+1}F)}{|F|_q^s}.\end{aligned}$$

Now when $(F,\pi)=1$, by (22), it follows that $G_q(f\pi^k,F)=G_q(f\pi^{[k]_3},F)$. We also have using Lemma 2.12 and (22),

$$\begin{aligned}
G_q(f\pi^j,\pi^{3\ell}F) =&G_q(f\pi^{j+3\ell},F)G_q(f\pi^j,\pi^{3\ell})=G_q(f\pi^j,F)\phi(\pi^{3\ell}),\\
G_q(f\pi^j,\pi^{j+1}F) =&G_q(f\pi^{2j+1},F)G_q(f\pi^j,\pi^{j+1})=G_q(f\pi^{2[j]_3+1},F)G_q(f\pi^{[j]_3},\pi^{[j]_3+1})|\pi|_q^{j-[j]_3}\\
=&G_q(f\pi^{[j]_3},\pi^{[j]_3+1}F)|\pi|_q^{j-[j]_3}.
\end{aligned}$$

Using the relations above in (33) and rearranging, we get that

$$\begin{aligned}
\sum_{\deg(F)\equiv i \,(\mathrm{mod}\,3)} \frac{G_q(f\pi^j,F)}{|F|_q^s} =& \left(1+\sum_{\ell=1}^{\left[\frac{j}{3}\right]}\frac{\phi(\pi^{3\ell})}{|\pi|_q^{3\ell s}}\right)\sum_{\substack{\deg(F)\equiv i \,(\mathrm{mod}\,3)\\(F,\pi)=1}}\frac{G_q(f\pi^j,F)}{|F|_q^s}\\
&+|\pi|_q^{(j-[j]_3)(1-s)}\sum_{\substack{\deg(F)\equiv i \,(\mathrm{mod}\,3)\\ \pi|F}}\frac{G_q(f\pi^{[j]_3},F)}{|F|_q^s}.
\end{aligned}$$

We now do the same with $j+3$ and take the difference. Then we have

$$\sum_{\deg(F)\equiv i \,(\mathrm{mod}\,3)}\frac{G_q(f\pi^{j+3},F)}{|F|_q^s}-|\pi|_q^{3-3s}\sum_{\deg(F)\equiv i \,(\mathrm{mod}\,3)}\frac{G_q(f\pi^j,F)}{|F|_q^s}=(1-|\pi|_q^{2-3s})\sum_{\substack{\deg(F)\equiv i \,(\mathrm{mod}\,3)\\(F,\pi)=1}}\frac{G_q(f\pi^j,F)}{|F|_q^s}.$$

Dividing by $(1-q^{3(1-s)})$, we obtain the result. □

We will also use the following periodicity result, which is stated in [Pat07] and in [KP84, p. 135].

Lemma 3.7 (The Periodicity Theorem)**.** *Let π be a prime such that $\pi\nmid f$. Then*

$$\rho(f\pi^{j+3},i)=\rho(f\pi^j,i).$$

We also need the following.

Lemma 3.8. *Let π be a prime such that $\pi\nmid f$. Then*

$$\lim_{s\to 4/3} q^{is}(1-q^{4-3s})\psi_\pi(f\pi^j,\pi_\infty^{-i},q^{-s})=\frac{\rho(f\pi^j,i)}{1+|\pi|_q^{-1}}.$$

Proof. We multiply relation (32) by $q^{is}(1-q^{4-3s})/(1-q^{3(1-s)})$ and take the limit as $s\to\frac{4}{3}$. This yields

$$\rho(f\pi^{j+3},i)-|\pi|_q^{-1}\rho(f\pi^j,i)=(1-|\pi|_q^{-2})\lim_{s\to 4/3}q^{is}(1-q^{4-3s})\psi_\pi(f\pi^j,\pi_\infty^{-i},q^{-s}).$$

Using Lemma 3.7 we obtain the result. □

We now explicitly compute the residue $\rho(f,i)$.

Lemma 3.9. *Let $f=f_1f_2^2f_3^3$ with f_1, f_2 square-free and coprime. For $n=3$, we have that $\rho(f,i)=0$ if $f_2\neq 1$ and*

(34) $$\rho(f,i)=\overline{G_q(1,f_1)}|f_1|_q^{-2/3}q^{\frac{4i}{3}-\frac{4}{3}[i-2\deg(f)]_3}\rho\left(1,[i-2\deg(f)]_3\right),$$

when $f_2=1$. Here

$$\rho(1,0)=1,\qquad \rho(1,1)=\tau(\chi_3)q,\qquad \rho(1,2)=0.$$

Proof. We start by computing $\rho(1,[i]_3)$. Recall by definition that

$$\begin{aligned}
G_q(1,F) = & \sum_{v \,(\mathrm{mod}\, F)} \chi_F(v) e_q\left(\frac{v}{F}\right) \\
= & \sum_{\deg(v)\le \deg(F)-2} \chi_F(v) + \sum_{\deg(v)=\deg(F)-1} \chi_F(v) e_q\left(\frac{v}{F}\right) \\
= & \sum_{c\in\mathbb{F}_q^*} \sum_{v\in\mathcal{M}_{q,\le\deg(F)-2}} \chi_F(c)\chi_F(v) + \sum_{c\in\mathbb{F}_q^*} \sum_{v\in\mathcal{M}_{q,\deg(F)-1}} \chi_F(c)\chi_F(v) e^{2\pi i \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(c)/p} \\
= & \sum_{c\in\mathbb{F}_q^*} \chi_F(c) \sum_{v\in\mathcal{M}_{q,\le\deg(F)-2}} \chi_F(v) + \tau(\chi_F) \sum_{v\in\mathcal{M}_{q,\deg(F)-1}} \chi_F(v).
\end{aligned}$$

First suppose that $[i]_3 = 0$, i.e., $\deg(F) \equiv 0 \,(\mathrm{mod}\, 3)$. Then, χ_F is even and

$$\sum_{v\in\mathcal{M}_{q,\le\deg(F)-2}} \chi_F(v) + \sum_{v\in\mathcal{M}_{q,\deg(F)-1}} \chi_F(v) = \begin{cases} 0 & F \neq \square, \\ \frac{\phi(F)}{q-1} & F = \square. \end{cases}$$

Then we write

$$G_q(1,F) = \phi(F)\delta_{\square}(F) + \left(\tau(\chi_F) - \sum_{c\in\mathbb{F}_q^*} \chi_F(c)\right) \sum_{v\in\mathcal{M}_{q,\deg(F)-1}} \chi_F(v),$$

where the term $\delta_{\square}(F) = 1$ is 1 if $F = \square$ and 0 otherwise.

Since $[i]_3 = 0$, $\tau(\chi_F) = -1$ and $\sum_{c\in\mathbb{F}_q^*} \chi_F(c) = q-1$, and we have

$$\begin{aligned}
\psi(1,\pi_\infty^0,u) = & (1-u^3q^3)^{-1} \sum_{\substack{F\in\mathcal{M}_q \\ 3|\deg(F)}} G_q(1,F) u^{\deg(F)} \\
= & (1-u^3q^3)^{-1} \sum_{\substack{F\in\mathcal{M}_q \\ \deg(F)=\square}} \phi(F) u^{\deg(F)} - q(1-u^3q^3)^{-1} \sum_{\substack{F\in\mathcal{M}_q \\ 3|\deg(F)}} \sum_{v\in\mathcal{M}_{q,\deg(F)-1}} \chi_F(v) u^{\deg(F)}.
\end{aligned}$$

Notice that

$$\sum_{F\in\mathcal{M}_{q,k}} \sum_{v\in\mathcal{M}_{q,k-1}} \chi_F(v) = \sum_{v\in\mathcal{M}_{q,k-1}} \sum_{F\in\mathcal{M}_{q,k}} \chi_v(F) = q^k \#\{v \in \mathcal{M}_{q,k-1} \,|\, v = \square\}, \tag{35}$$

and this gives zero when $k \not\equiv 1 \,(\mathrm{mod}\, 3)$.

This gives

$$\begin{aligned}\psi(1,\pi_\infty^0,u) =&(1-u^3q^3)^{-1}\sum_{F\in\mathcal{M}_q}\phi(F^3)u^{3\deg(F)}\\ =&(1-u^3q^3)^{-1}\sum_{k=0}^{\infty}\sum_{F\in\mathcal{M}_{q,k}}\phi(F)|F|_q^2u^{3\deg(F)}\\ =&(1-u^3q^3)^{-1}\sum_{k=0}^{\infty}q^{2k}u^{3k}\sum_{F\in\mathcal{M}_{q,k}}\phi(F)\\ =&(1-u^3q^3)^{-1}\sum_{k=0}^{\infty}q^{4k}u^{3k}(1-q^{-1}),\end{aligned}$$

where we have used Proposition 2.7 in [Ros02]. Finally, we get

$$\psi(1,\pi_\infty^0,u) = \frac{1-q^{-1}}{(1-u^3q^3)(1-u^3q^4)},$$

and taking the residue,

$$\rho(1,0) = 1.$$

When $[i]_3 \neq 0$, $\sum_{c\in\mathbb{F}_q^*}\chi_F(c) = 0$ and we obtain,

$$\begin{aligned}\psi(1,\pi_\infty^{-i},u) =&(1-u^3q^3)^{-1}\sum_{\substack{F\in\mathcal{M}_q\\ \deg(F)\equiv i \pmod 3}}G_q(1,F)u^{\deg(F)}\\ =&\frac{\tau(\chi_3^i)}{1-u^3q^3}\sum_{\substack{F\in\mathcal{M}_q\\ \deg(F)\equiv i \pmod 3}}\sum_{v\in\mathcal{M}_{q,\deg(F)-1}}\chi_F(v)u^{\deg(F)}.\end{aligned}$$

When $[i]_3 = 2$, from equation (35), we immediately get that the sum above is zero and

$$\rho(1,2) = 0.$$

On the other hand, if $[i]_3 = 1$ we have, by cubic reciprocity,

$$\begin{aligned}\psi(1,\pi_\infty^{-1},u) =&\frac{\tau(\chi_3)}{1-u^3q^3}\sum_{j=0}^{\infty}u^{3j+1}\sum_{v\in\mathcal{M}_{q,3j}}\sum_{F\in\mathcal{M}_{q,1+3j}}\chi_v(F)\\ =&\frac{\tau(\chi_3)}{(1-u^3q^3)}\sum_{j=0}^{\infty}u^{3j+1}\sum_{w\in\mathcal{M}_{q,j}}\frac{\phi(w^3)}{|w^3|_q}q^{3j+1}\\ =&\frac{\tau(\chi_3)}{(1-u^3q^3)}\sum_{j=0}^{\infty}u^{3j+1}q^j(1-q^{-1})q^{3j+1}\\ =&\frac{\tau(\chi_3)(q-1)u}{(1-u^3q^3)}\sum_{j=0}^{\infty}(q^4u^3)^j = \frac{\tau(\chi_3)(q-1)u}{(1-u^3q^3)(1-u^3q^4)},\end{aligned}$$

where we have used again Proposition 2.7 in [Ros02]. Taking the residue, we get

$$\rho(1,1) = \lim_{s\to 4/3} \frac{\tau(\chi_3)(q-1)}{(1-u^3q^3)} = \tau(\chi_3)q.$$

To obtain equation (34), we start by multiplying equation (30) by $q^{is}(1-q^{4-3s})$ and taking the limit as $s \to 4/3$. By Lemma 3.8 for $\pi \nmid f$ we get that

$$\rho(f\pi, i)\left[1 - \frac{1}{1+|\pi|_q^{-1}}\right] = \overline{G_q(f,\pi)}|\pi|_q^{-\frac{5}{3}} q^{\frac{8}{3}\deg(\pi)} \frac{\rho(f, i-2\deg(\pi))}{1+|\pi|_q^{-1}},$$

which simplifies to

$$\rho(f\pi, i) = \overline{G_q(f,\pi)}|\pi|_q^{-\frac{2}{3}} q^{\frac{8}{3}\deg(\pi)} \rho(f, i-2\deg(\pi)). \tag{36}$$

Multiplying equation (31) by $q^{is}(1-q^{4-3s})$, taking the limit as $s \to 4/3$, and applying Lemma 3.8 we get that

$$\rho(f\pi^2, i) = \rho(f\pi^2, i)\left[\frac{1-|\pi|_q^{-2}}{1+|\pi|_q^{-1}}\right],$$

which implies that

$$\rho(f\pi^2, i) = 0. \tag{37}$$

Notice that by the Periodicity Theorem (Lemma 3.7), $\rho(f, i)$ depends on the cubic-free part of f. From this and equation (37) we can suppose that $f = f_1$ with f_1 square-free. Write $f = \pi_1 \cdots \pi_k$. By (36), we have

$$\begin{aligned}
\rho(f,i) =& \overline{G_q(f/\pi_k, \pi_k)}|\pi_k|_q^{-\frac{2}{3}} q^{\frac{8}{3}\deg(\pi_k)} \rho(f/\pi_k, i-2\deg(\pi_k)) \\
=& \overline{G_q(f/\pi_k, \pi_k)}|\pi_k|_q^{-\frac{2}{3}} q^{\frac{4i}{3}-\frac{4}{3}[i-2\deg(\pi_k)]_3} \rho(f/\pi_k, [i-2\deg(\pi_k)]_3) \\
=& \cdots \\
=& \prod_{j=1}^{k} \overline{G_q\left(\prod_{\ell=1}^{j-1}\pi_\ell, \pi_j\right)} |f|_q^{-\frac{2}{3}} q^{\frac{4i}{3}-\frac{4}{3}\left[i-2\sum_{j=1}^{k}\deg(\pi_j)\right]_3} \rho\left(1, \left[i-2\sum_{j=1}^{k}\deg(\pi_j)\right]_3\right)
\end{aligned}$$

In the equation above, note that

$$\prod_{j=1}^{k} \overline{G_q\left(\prod_{\ell=1}^{j-1}\pi_\ell, \pi_j\right)} = \overline{G_q(1,f)},$$

which follows by induction on the number of prime divisors of f and part (i) of Lemma 2.12. This finishes the proof of Lemma 3.9. □

3.3. Upper bounds for $\Psi_q(f,u)$ and $\tilde{\Psi}_q(f,u)$.

We will first prove the following result which provides an upper bound for $\Psi_q(f,u)$.

Lemma 3.10. *For $1/2 \le \sigma \le 3/2$ and $|u^3 - q^{-4}| > \delta$ where $\delta > 0$, we have, for $\varepsilon > 0$,*

$$\Psi_q(f,u) \ll_{\delta,\varepsilon} |f|_q^{\frac{1}{2}\left(\frac{3}{2}-\sigma\right)+\varepsilon},$$

where $u = q^{-s}$ as usual, and $\sigma = \mathrm{Re}(s)$.

Proof. The bound for $\Psi_q(f, q^{-s})$ for $1/2 < \mathrm{Re}(s) \leq 3/2$ and $|u^3 - q^{-4}| > \delta$ follows from the functional equation and the Phragmén–Lindelöf principle. By (24), it suffices to show that the bound holds for $\psi(f, \pi_\infty^{-i}, q^{-s})$ for $i = 0, 1, 2$, which follows from the functional equation and the Phragmén–Lindelöf principle.

First, it follows from (26) and (27) that for $B = [(1 + \deg(f) - i)/3]$ we have

$$\begin{aligned}\psi(f, \pi_\infty^{-i}, u) =& \frac{u^i P(f, i, u^3)}{1 - q^4 u^3} \\ =& \frac{1}{1 - q^3 u^3} \sum_{0 \leq j < B} C(f, i + 3j) u^{i+3j} + \frac{C(f, i + 3B) u^{i+3B}}{(1 - q^4 u^3)(1 - q^3 u^3)}.\end{aligned}$$

We now bound $|C(f, k)|$. Write $F = F_1 F_2$ with $(F_1, f) = 1$ and $F_2 \mid f^\infty$ (by this we mean that the primes of F_2 divide f.) We use repeatedly that $|G_q(f, F_1 F_2)| = |G_q(f, F_1)||G_q(f, F_2)|$. By Lemma 2.12 we have for $F_2 \mid f^\infty$ that $|G_q(f, F_2)| = 0$ unless $F_2 \mid f^2$. We write

$$\begin{aligned}\sum_{F \in \mathcal{M}_{q,k}} |G_q(f, F)| =& \sum_{j=0}^{k} \sum_{\substack{F_1 \in \mathcal{M}_{q,j} \\ (F_1, f) = 1}} |G_q(f, F_1)| \sum_{\substack{F_2 \in \mathcal{M}_{q,k-j} \\ F_2 \mid f^2}} |G_q(f, F_2)| \\ \leq& \sum_{j=0}^{k} \sum_{\substack{F_1 \in \mathcal{M}_{q,j} \\ (F_1, f) = 1}} q^{j/2} \sum_{\substack{F_2 \in \mathcal{M}_{q,k-j} \\ F_2 \mid f^2}} q^{k-j} \\ \ll& \sum_{j=0}^{k} q^{3j/2} q^{k-j} |f|_q^\varepsilon \\ \ll& q^{3k/2} |f|_q^\varepsilon.\end{aligned}$$

Thus

$$|C(f, k)| \ll q^{3k/2} |f|_q^\varepsilon.$$

We get that for $\sigma \leq 3/2$

$$|\psi(f, \pi_\infty^{-i}, q^{-s})| \ll \sum_{k=0}^{3B+2} q^{(3/2-\sigma)k} |f|_q^\varepsilon \ll |f|_q^{3/2-\sigma+\varepsilon},$$

with an absolute constant in that region. In particular,

$$\psi(f, \pi_\infty^{-i}, q^{-s}) \ll |f|_q^\varepsilon \tag{38}$$

when $\mathrm{Re}(s) = 3/2$.

From the functional equation of Remark 3.3, we have for $1/2 \leq \mathrm{Re}(s) \leq 3/2$ and $|u^3 - q^{-4}| > \delta$ that

$$\psi(f, \pi_\infty^{-i}, q^{-s}) = a_1(s)\, |f|_q^{1-s} \psi(f, \pi_\infty^{-i}, q^{s-2}) + a_2(s)\, |f|_q^{1-s} \psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2}), \tag{39}$$

where $a_1(s)$ and $a_2(s)$ are absolutely bounded above and below in the region considered (independently of f).

Using the bound (38) and the functional equation gives that

$$\psi(f, \pi_\infty^{-i}, q^{-s}) \ll |f|_q^{1/2+\varepsilon}$$

when $\mathrm{Re}(s) = 1/2$.

We consider the function $\Phi(f, \pi_\infty^{-i}, s) = (1 - q^{4-3s})(1 - q^{3s-2})\psi(f, \pi_\infty^{-i}, q^{-s})\psi(f, \pi_\infty^{-i}, q^{s-2})$. Then $\Phi(f, \pi_\infty^{-i}, s)$ is holomorphic in the region $1/2 \leq \mathrm{Re}(s) \leq 3/2$, and $\Phi(f, \pi_\infty^{-i}, s) \ll |f|_q^{1/2+\varepsilon}$ for $\mathrm{Re}(s) = 3/2$ and $\mathrm{Re}(s) = 1/2$.

Using the Phragmén–Lindelöf principle, it follows that for $1/2 \leq \mathrm{Re}(s) \leq 3/2$, we have that

$$\Phi(f, \pi_\infty^{-i}, s) = (1 - q^{4-3s})(1 - q^{3s-2})\psi(f, \pi_\infty^{-i}, q^{-s})\psi(f, \pi_\infty^{-i}, q^{s-2}) \ll |f|_q^{1/2+\varepsilon}.$$

Using the functional equation (39), this gives

(40)
$$(1-q^{4-3s})(1-q^{3s-2}) \left[a_1(s)\psi(f, \pi_\infty^{-i}, q^{s-2})^2 + a_2(s)\psi(f, \pi_\infty^{-i}, q^{s-2})\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2})\right] \ll |f|_q^{\sigma-\frac{1}{2}+\varepsilon}$$

in the region $1/2 \leq \mathrm{Re}(s) \leq 3/2$ and $|u^3 - q^{-4}| > \delta$.

If $\deg(f) + 1 \equiv 2i \pmod 3$, then the formula above implies that

$$\psi(f, \pi_\infty^{-i}, q^{s-2}) + \psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2}) \ll |f|_q^{\frac{1}{2}\left(\sigma-\frac{1}{2}+\varepsilon\right)}. \tag{41}$$

Now assume that $\deg(f) + 1 \not\equiv 2i \pmod 3$. Similarly we consider the function $\tilde{\Phi}(s) = (1 - q^{4-3s})(1 - q^{3s-2})\psi(f, \pi_\infty^{-i}, q^{-s})\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2})$. Then, using the same arguments as above we get that

$$\begin{aligned}(1 - q^{4-3s})(1 - q^{3s-2}) &\left[a_1(s)\psi(f, \pi_\infty^{-i}, q^{s-2})\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2}) + a_2(s)\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2})^2\right] \\ &\ll |f|_q^{\sigma-\frac{1}{2}+\varepsilon}.\end{aligned} \tag{42}$$

Combining the two equations (40) and (42), it would follow that

$$\begin{aligned}&(1 - q^{4-3s})(1 - q^{3s-2}) \left[\psi(f, \pi_\infty^{-i}, q^{s-2}) + \psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2})\right] \\ &\times \left[a_1(s)\psi(f, \pi_\infty^{-i}, q^{s-2}) + a_2(s)\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2})\right] \ll |f|_q^{\sigma-\frac{1}{2}+\varepsilon}.\end{aligned} \tag{43}$$

Switching i with $\deg(f) + 1 - i$ (since $\deg(f) + 1 \not\equiv 2i \pmod 3$), we get that there exist absolutely bounded constants $b_1(s)$ and $b_2(s)$ such that

$$\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{-s}) = b_1(s)|f|_q^{1-s}\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2}) + b_2(s)|f|_q^{1-s}\psi(f, \pi_\infty^{-i}, q^{s-2}).$$

If (a_1, a_2) and (b_2, b_1) are not linearly independent, then from the equation above and (39) it follows that

$$\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{-s}) = \lambda(s)\psi(f, \pi_\infty^{-i}, q^{-s}),$$

for some $\lambda(s)$. Combining this with equation (43), we get that

$$\psi(f, \pi_\infty^{-i}, q^{s-2}) \ll |f|_q^{\frac{1}{2}\left(\sigma-\frac{1}{2}+\varepsilon\right)},$$

and the conclusion again follows by replacing $2 - s$ by s.

If (a_1, a_2) and (b_2, b_1) are linearly independent, then

$$\begin{aligned}&(1 - q^{4-3s})(1 - q^{3s-2}) \left[\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2}) + \psi(f, \pi_\infty^{-i}, q^{s-2})\right] \\ &\times \left[b_1(s)\psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2}) + b_2(s)\psi(f, \pi_\infty^{-i}, q^{s-2})\right] \ll |f|_q^{\sigma-\frac{1}{2}+\varepsilon}.\end{aligned}$$

From the equation above and (43), by the linear independence condition, we get that

$$\left[\psi(f, \pi_\infty^{-i}, q^{s-2}) + \psi(f, \pi_\infty^{i-1-\deg(f)}, q^{s-2})\right] \psi(f, \pi_\infty^{-i}, q^{s-2}) \ll |f|_q^{\sigma-\frac{1}{2}+\varepsilon}$$

and

$$\left[\psi(f,\pi_\infty^{-i},q^{s-2})+\psi(f,\pi_\infty^{i-1-\deg(f)},q^{s-2})\right]\psi(f,\pi_\infty^{i-1-\deg(f)},q^{s-2})\ll|f|_q^{\sigma-\frac{1}{2}+\varepsilon}.$$

By summing the two equations above, we recover equation (41) without any restrictions on i,

$$\psi(f,\pi_\infty^{-i},q^{s-2})+\psi(f,\pi_\infty^{i-1-\deg(f)},q^{s-2})\ll|f|_q^{\frac{1}{2}\left(\sigma-\frac{1}{2}+\varepsilon\right)}.$$

Summing over $i=0,1,2$ and replacing $2-s$ by s finishes the proof.

□

In order to obtain an upper bound for $\tilde{\Psi}_q(f,u)$ (recall its definition (23)) we first need to relate it to $\Psi_q(f,u)$ which we do in the next lemma.

Lemma 3.11. *Let $f=f_1f_2^2f_3^3$ with f_1,f_2 square-free and co-prime, and let f_3^* be the product of the primes dividing f_3 but not dividing f_1f_2. Then,*

$$\begin{aligned}\tilde{\Psi}_q(f,u) &= \prod_{P|f_1f_2}\left(1-(u^3q^2)^{\deg(P)}\right)^{-1}\sum_{a|f_3^*}\mu(a)G_q(f_1f_2^2,a)u^{\deg(a)}\prod_{P|a}(1-(u^3q^2)^{\deg(P)})^{-1}\\ &\quad\times\sum_{\ell|af_1}\mu(\ell)(u^2q)^{\deg(\ell)}\overline{G_q(1,\ell)}\chi_\ell(af_1f_2^2/\ell)\Psi_q(af_1f_2^2/\ell,u).\end{aligned}\tag{44}$$

If $1/2\le\sigma\le3/2$ and $|u^3-q^{-4}|,|u^3-q^{-2}|>\delta$, then, for $\varepsilon>0$,

$$\tilde{\Psi}_q(f,u)\ll_{\delta,\varepsilon}|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)+\varepsilon}.$$

Proof. We first show that the last assertion follows from the expression (44) for $\tilde{\Psi}_q(f,u)$.

Suppose that $1/2\le\sigma\le3/2$ and $|u^3-q^{-4}|,|u^3-q^{-2}|>\delta$. Then, for $\mathrm{Re}(s)=\sigma$,

$$\begin{aligned}\tilde{\Psi}_q(f,u) &\ll \sum_{a|f_3^*}|a|_q^{1/2-\sigma}\sum_{\ell|af_1}|\ell|_q^{3/2-2\sigma}\left|\frac{af_1f_2^2}{\ell}\right|_q^{\frac{1}{2}\left(\frac{3}{2}-\sigma\right)+\varepsilon}\\ &\ll \sum_{a|f_3^*}|a|_q^{\frac{5-6\sigma}{4}+\varepsilon}\sum_{\ell|af_1}|\ell|_q^{\frac{3-6\sigma}{4}}\left|f_1f_2^2\right|_q^{\frac{1}{2}\left(\frac{3}{2}-\sigma\right)+\varepsilon}\\ &\ll \sum_{a|f_3^*}|a|_q^{\frac{5-6\sigma}{4}+\varepsilon}\left|f_1f_2^2\right|_q^{\frac{1}{2}\left(\frac{3}{2}-\sigma\right)+\varepsilon}\\ &\ll \max\left\{|f_3^*|_q^{\varepsilon},|f_3|_q^{\frac{5-6\sigma}{4}+\varepsilon}\right\}\left|f_1f_2^2\right|_q^{\frac{1}{2}\left(\frac{3}{2}-\sigma\right)+\varepsilon}\ll|f|_q^{\frac{1}{2}\left(\frac{3}{2}-\sigma\right)+\varepsilon}.\end{aligned}$$

We now prove (44). We first remark that by definition of f_1,f_2,f_3^*, we have that $(f,F)=1\iff(f_1f_2,F)=1$ and $(f_3^*,F)=1$ with $(f_1f_2,f_3^*)=1$. If $(f_1f_2f_3,F)=1$, then $G_q(f_1f_2^2f_3^3,F)=\overline{\chi_F}(f_3^3)G_q(f_1f_2^2,F)=G_q(f_1f_2^2,F)$, and

$$\begin{aligned}\tilde{\Psi}_q(f,u) &= \sum_{(F,f_1f_2f_3^*)=1}G_q(f_1f_2^2,F)u^{\deg(F)}\\ &= \sum_{a|f_3^*}\mu(a)u^{\deg(a)}\sum_{(F,f_1f_2)=1}G_q(f_1f_2^2,aF)u^{\deg(F)}.\end{aligned}$$

If $(a,F)\neq 1$, then there is a prime P such that $P^2\mid aF$ and $P\nmid f_1f_2^2$, and then $G_q(f_1f_2^2,aF)=0$. We can then suppose that $(a,F)=1$, and then by Lemma 2.12 (i), we have that $G_q(f_1f_2^2,aF)=G_q(f_1f_2^2,a)G_q(af_1f_2^2,F)$, and

$$\tilde{\Psi}_q(f,u) = \sum_{a\mid f_3^*}\mu(a)G_q(f_1f_2^2,a)u^{\deg(a)}\sum_{(F,af_1f_2)=1}G_q(af_1f_2^2,F)u^{\deg(F)}. \tag{45}$$

Notice that af_1f_2 is square-free and that a, f_1 and f_2 are two-by-two co-prime.

Let P be a prime dividing f_2, and we write $f_2=Pf_2'$, and $F=P^iF'$ with $(F'f_2',P)=1$. Then, by Lemma 2.12,

$$G_q(af_1f_2'^2P^2,P^iF')=G_q(af_1f_2'^2P^2,P^i)G_q(af_1f_2'^2P^{2+i},F')=\begin{cases}G_q(af_1f_2'^2P^2,F') & i=0,\\ -|P|_q^2G_q(af_1f_2'^2P^2,F') & i=3,\\ 0 & \text{otherwise.}\end{cases}$$

We remark that we have used that $G_q(af_1f_2'^2P^5,F')=G_q(af_1f_2'^2P^2,F')$ for the second line, since $(P,F')=1$. This gives

$$\begin{aligned}
&\sum_{(F,af_1f_2)=1}G_q(af_1f_2^2,F)u^{\deg(F)}\\
=&\sum_{(F,af_1f_2')=1}G_q(af_1f_2^2,F)u^{\deg(F)}-\sum_{\substack{(F,af_1f_2')=1\\P\mid F}}G_q(af_1f_2'^2P^2,F)u^{\deg(F)}\\
=&\sum_{(F,af_1f_2')=1}G_q(af_1f_2^2,F)u^{\deg(F)}+\sum_{(F',af_1f_2)=1}G_q(af_1f_2'^2P^2,F')u^{\deg(F')+3\deg(P)}q^{2\deg(P)}\\
=&\sum_{(F,af_1f_2')=1}G_q(af_1f_2^2,F)u^{\deg(F)}+(u^3q^2)^{\deg(P)}\sum_{(F',af_1f_2)=1}G_q(af_1f_2^2,F')u^{\deg(F')},
\end{aligned}$$

or equivalently

$$\left(1-(u^3q^2)^{\deg(P)}\right)\sum_{(F,af_1f_2)=1}G_q(af_1f_2^2,F)u^{\deg(F)} = \sum_{(F,af_1f_2')=1}G_q(af_1f_2^2,F)u^{\deg(F)}.$$

By induction on the prime divisors of f_2, we get

$$\sum_{(F,af_1f_2)=1}G_q(af_1f_2^2,F)u^{\deg(F)}=\prod_{P\mid f_2}\left(1-(u^3q^2)^{\deg(P)}\right)^{-1}\sum_{(F,af_1)=1}G_q(af_1f_2^2,F)u^{\deg(F)},$$

and plugging in (45), we have

$$\tilde{\Psi}_q(f,u)=\prod_{P\mid f_2}\left(1-(u^3q^2)^{\deg(P)}\right)^{-1}\sum_{a\mid f_3^*}\mu(a)G_q(f_1f_2^2,a)u^{\deg(a)}\sum_{(F,af_1)=1}G_q(af_1f_2^2,F)u^{\deg(F)}. \tag{46}$$

We now do the same thing for $\sum_{(F,af_1)=1}G_q(af_1f_2^2,F)u^{\deg(F)}$, dealing with the primes dividing $f_1^*:=af_1$ one by one.

Let $f_1^*=Pf_1'$, and we write

$$\sum_{(F,f_1^*)=1}G_q(f_1^*f_2^2,F)u^{\deg(F)}=\sum_{(F,f_1')=1}G_q(f_1^*f_2^2,F)u^{\deg(F)}-\sum_{\substack{(F,f_1')=1\\F=P^iF',i\geq 1}}G_q(f_1'Pf_2^2,P^iF')u^{\deg(F')+i\deg(P)}.$$

Using Lemma 2.12, we compute that

$$
\begin{aligned}
G_q(f_1'Pf_2^2, P^iF') &= G_q(f_1'Pf_2^2, P^i)G_q(f_1'P^{i+1}f_2^2, F') \\
&= \begin{cases} G_q(f_1'Pf_2^2, F') & i=0, \\ G_q(f_1'P^3f_2^2, F')\epsilon(\chi_{P^2})\omega(\chi_{P^2})\chi_P(f_1'f_2^2)|P|_q^{3/2} & i=2, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

where we recall that $\epsilon(\chi_{P^2}) = 1$ when $3 \mid \deg(P)$.

Then,

$$
\begin{aligned}
&\sum_{(F,f_1^*)=1} G_q(f_1^*f_2^2, F)u^{\deg(F)} \\
&= \sum_{(F,f_1')=1} G_q(f_1^*f_2^2, F)u^{\deg(F)} - \sum_{(F',f_1^*)=1} G_q(f_1'f_2^2, F')u^{\deg(F')+2\deg(P)}q^{3\deg(P)/2}\epsilon(\chi_{P^2})\omega(\chi_{P^2})\chi_P(f_1'f_2^2) \\
&= \sum_{(F,f_1')=1} G_q(f_1^*f_2^2, F)u^{\deg(F)} - (u^2q^{3/2})^{\deg(P)}\epsilon(\chi_{P^2})\omega(\chi_{P^2})\chi_P(f_1'f_2^2) \sum_{(F',f_1^*)=1} G_q\left(\frac{f_1^*f_2^2}{P}, F'\right)u^{\deg(F')}.
\end{aligned} \tag{47}
$$

Now we focus on

$$
\sum_{(F,f_1'P)=1} G_q(f_1'f_2^2, F)u^{\deg(F)} = \sum_{(F,f_1')=1} G_q(f_1'f_2^2, F)u^{\deg(F)} - \sum_{\substack{(F,f_1')=1 \\ P\mid F}} G_q(f_1'f_2^2, F)u^{\deg(F)}.
$$

As before, write $F = P^iF'$. By Lemma 2.12 as always,

$$
\begin{aligned}
G_q(f_1'f_2^2, P^iF') &= G_q(f_1'f_2^2, P^i)G_q(f_1'f_2^2P^i, F') \\
&= \begin{cases} G_q(f_1'f_2^2, F') & i=0, \\ G_q(f_1'f_2^2P, F')\epsilon(\chi_P)\omega(\chi_P)\chi_{P^2}(f_1'f_2^2)|P|_q^{1/2} & i=1, \\ 0 & i \geq 2, \end{cases}
\end{aligned}
$$

and we get

$$
\begin{aligned}
&\sum_{(F,f_1'P)=1} G_q(f_1'f_2^2, F)u^{\deg(F)} \\
&= \sum_{(F,f_1')=1} G_q(f_1'f_2^2, F)u^{\deg(F)} - \sum_{(F',Pf_1')=1} G_q(f_1'f_2^2, F'P)u^{\deg(F')+\deg(P)} \\
&= \sum_{(F,f_1')=1} G_q(f_1'f_2^2, F)u^{\deg(F)} - (uq^{1/2})^{\deg(P)}\epsilon(\chi_P)\omega(\chi_P)\chi_{P^2}(f_1'f_2^2) \sum_{(F',Pf_1')=1} G_q(f_1'f_2^2P, F')u^{\deg(F')} \\
&= \sum_{(F,f_1')=1} G_q(f_1'f_2^2, F)u^{\deg(F)} - (uq^{1/2})^{\deg(P)}\epsilon(\chi_P)\omega(\chi_P)\chi_{P^2}(f_1'f_2^2) \sum_{(F,f_1^*)=1} G_q(f_1^*f_2^2, F)u^{\deg(F)}.
\end{aligned}
$$

Now we incorporate the equation above into equation (47).

$$
\begin{aligned}
&\sum_{(F,f_1^*)=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} \\
&= \sum_{(F,f_1')=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} - (u^2 q^{3/2})^{\deg(P)} \epsilon(\chi_{P^2}) \omega(\chi_{P^2}) \chi_P(f_1' f_2^2) \sum_{(F',f_1^*)=1} G_q(f_1' f_2^2, F') u^{\deg(F')} \\
&= \sum_{(F,f_1')=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} - (u^2 q^{3/2})^{\deg(P)} \epsilon(\chi_{P^2}) \omega(\chi_{P^2}) \chi_P(f_1' f_2^2) \sum_{(F,f_1')=1} G_q(f_1' f_2^2, F) u^{\deg(F)} \\
&\quad + (u^3 q^2)^{\deg(P)} \sum_{(F,f_1^*)=1} G_q(f_1^* f_2^2, F) u^{\deg(F)}.
\end{aligned}
$$

Rearranging, we write

$$
\begin{aligned}
&(1 - (u^3 q^2)^{\deg(P)}) \sum_{(F,f_1^*)=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} \\
&= \sum_{(F,f_1')=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} - (u^2 q^{3/2})^{\deg(P)} \chi_P(f_1' f_2^2) \epsilon(\chi_{P^2}) \omega(\chi_{P^2}) \sum_{(F,f_1')=1} G_q(f_1' f_2^2, F) u^{\deg(F)}
\end{aligned}
$$

or

$$
\begin{aligned}
\sum_{(F,f_1^*)=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} &= (1 - (u^3 q^2)^{\deg(P)})^{-1} \sum_{(F,f_1')=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} \\
&- (1 - (u^3 q^2)^{\deg(P)})^{-1} (u^2 q^{3/2})^{\deg(P)} \chi_P(f_1' f_2^2) \epsilon(\chi_{P^2}) \omega(\chi_{P^2}) \sum_{(F,f_1')=1} G_q(f_1' f_2^2, F) u^{\deg(F)}.
\end{aligned}
$$

By applying this idea to each of the primes in the factorization of the square-free polynomial f_1^*, we obtain

$$
\begin{aligned}
\sum_{(F,f_1^*)=1} G_q(f_1^* f_2^2, F) u^{\deg(F)} &= \prod_{P | f_1^*} (1 - (u^3 q^2)^{\deg(P)})^{-1} \\
&\times \sum_{\ell | f_1^*} \mu(\ell) (u^2 q^{3/2})^{\deg(\ell)} \overline{\left(\prod_{P|\ell} \chi_P \left(\frac{\ell}{P} \right) \right)} \chi_\ell \left(\frac{f_1^* f_2^2}{\ell} \right) \left(\prod_{P|\ell} \epsilon(\chi_{P^2}) \omega(\chi_{P^2}) \right) \\
&\times \Psi_q \left(\frac{f_1^* f_2^2}{\ell}, u \right).
\end{aligned}
$$

Putting everything together in (46), we get

$$\tilde{\Psi}_q(f,u) = \prod_{P|f_2} \left(1-(u^3q^2)^{\deg(P)}\right)^{-1} \sum_{a|f_3^*} \mu(a)G_q(f_1f_2^2,a)u^{\deg(a)} \sum_{(F,af_1)=1} G_q(af_1f_2^2,F)u^{\deg(F)}$$

$$= \prod_{P|f_2} \left(1-(u^3q^2)^{\deg(P)}\right)^{-1} \sum_{a|f_3^*} \mu(a)G_q(f_1f_2^2,a)u^{\deg(a)} \prod_{P|af_1} (1-(u^3q^2)^{\deg(P)})^{-1}$$

$$\times \sum_{\ell|af_1} \mu(\ell)(u^2q^{3/2})^{\deg(\ell)} \overline{\left(\prod_{P|\ell} \chi_P\left(\frac{\ell}{P}\right)\right)} \chi_\ell\left(\frac{af_1f_2^2}{\ell}\right) \left(\prod_{P|\ell} \epsilon(\chi_{P^2})\omega(\chi_{P^2})\right) \Psi_q\left(\frac{af_1f_2^2}{\ell},u\right)$$

$$= \prod_{P|f_1f_2} \left(1-(u^3q^2)^{\deg(P)}\right)^{-1} \sum_{a|f_3^*} \mu(a)G_q(f_1f_2^2,a)u^{\deg(a)} \prod_{P|a}(1-(u^3q^2)^{\deg(P)})^{-1}$$

$$\times \sum_{\ell|af_1} \mu(\ell)(u^2q^{3/2})^{\deg(\ell)} \overline{\left(\prod_{P|\ell} \chi_P\left(\frac{\ell}{P}\right)\right)} \chi_\ell\left(\frac{af_1f_2^2}{\ell}\right) \left(\prod_{P|\ell} \epsilon(\chi_{P^2})\omega(\chi_{P^2})\right) \Psi_q\left(\frac{af_1f_2^2}{\ell},u\right).$$

Now note that

$$\prod_{P|\ell}\left(\overline{\chi_P}\left(\frac{\ell}{P}\right)\epsilon(\chi_{P^2})\omega(\chi_{P^2})\right) = \frac{\overline{G_q(1,\ell)}}{|\ell|_q^{1/2}},$$

which finishes the proof of the lemma. □

3.4. **Proof of Proposition 3.1.** We are now ready to prove Proposition 3.1.

Proof. By applying Perron's formula (Lemma 2.1) for a small circle C around the origin and using expression (44), we have

$$\sum_{\substack{F\in\mathcal{M}_{q,d}\\(F,f)=1}} G_q(f,F) = \frac{1}{2\pi i}\oint_C \frac{\tilde{\Psi}_q(f,u)}{u^d}\frac{du}{u} = \frac{1}{2\pi i}\oint_C \prod_{P|f_1f_2}\left(1-(u^3q^2)^{\deg(P)}\right)^{-1}\sum_{a|f_3^*}\mu(a)G_q(f_1f_2^2,a)$$

$$\times \prod_{P|a}(1-(u^3q^2)^{\deg(P)})^{-1}\sum_{\ell|af_1}\mu(\ell)|\ell|_q\overline{G_q(1,\ell)}\chi_\ell\left(\frac{af_1f_2^2}{\ell}\right)$$

$$\times \frac{\Psi_q\left(\frac{af_1f_2^2}{\ell},u\right)u^{\deg(a)+2\deg(\ell)}}{u^d}\frac{du}{u}. \tag{48}$$

Now we write

$$\Psi_q\left(\frac{af_1f_2^2}{\ell},u\right) = (1-u^3q^3)\left[\psi\left(\frac{af_1f_2^2}{\ell},\pi_\infty^0,u\right)+\psi\left(\frac{af_1f_2^2}{\ell},\pi_\infty^{-1},u\right)+\psi\left(\frac{af_1f_2^2}{\ell},\pi_\infty^{-2},u\right)\right].$$

Each ψ has three poles, at $q^{-4/3}\xi_3^k, k=0,1,2$, where $\xi_3=e^{2\pi i/3}$. We compute the residues of the poles in the integral above. We recall that formula (26) gives

$$\psi(f,\pi_\infty^{-j},u) = \frac{u^jP(f,j,u^3)}{(1-q^4u^3)},$$

where $\frac{u^jP(f,j,u^3)}{1-q^4u^3}$ is a power series whose nonzero coefficients correspond to monomials with $\deg \equiv j \pmod 3$, and then the only ψ which gives a non-zero integral in equation (48) comes from $\psi(af_1f_2^2/\ell, \pi_\infty^{-j}, u)$ with j such that $j + \deg(a) + 2\deg(\ell) \equiv d \,(\mathrm{mod}\, 3)$. Note that if $j + \deg(a) + 2\deg(\ell) \geq d+1$, the integral in (48) is zero because the integrand has no poles inside C. Hence we assume that $j + \deg(a) + 2\deg(\ell) \leq d$.

In (48) we shift the contour of integration to $|u| = q^{-\sigma}$, where $2/3 < \sigma < 4/3$ and we encounter the poles when $u^3 = q^{-4}$. With j as before, we compute the residue of the integrand at $u^3 = q^{-4}$ and this gives

$$\mathrm{Res}_{u=\xi_3^k q^{-4/3}}\, \psi\left(\frac{af_1f_2^2}{\ell}, \pi_\infty^{-j}, u\right) u^{\deg(a)+2\deg(\ell)-d-1} = \frac{1}{3}(q^{\frac{4}{3}}\xi_3^{-k})^{d-\deg(a)-2\deg(l)-j}\rho\left(\frac{af_1f_2^2}{\ell}, j\right).$$

We get that

$$\sum_{\substack{F\in\mathcal{M}_{q,d}\\(F,f)=1}} G_q(f,F) = \frac{q^{\frac{4}{3}(d-j)}}{\zeta_q(2)} \sum_{\substack{a|f_3^*\\ \deg(a)\leq d-j}} \frac{\mu(a)G_q(f_1f_2^2,a)}{|a|_q^{\frac{4}{3}}} \prod_{P|af_1}\left(1-\frac{1}{|P|_q^2}\right)^{-1}$$
$$\times \sum_{\substack{\ell|af_1\\ 2\deg(\ell)\leq d-j-\deg(a)}} \frac{\mu(\ell)\overline{G_q(1,\ell)}}{|\ell|_q^{\frac{5}{3}}}\chi_\ell\left(\frac{af_1f_2^2}{\ell}\right)\rho\left(\frac{af_1f_2^2}{\ell}, j\right) + \frac{1}{2\pi i}\oint_{|u|=q^{-\sigma}} \frac{\tilde{\Psi}_q(f,u)}{u^d}\frac{du}{u}.$$

Using Lemma 3.9 and since af_1/ℓ is square-free and co-prime to f_2 it follows that

$$\rho\left(\frac{af_1f_2^2}{\ell}, j\right) = \delta_{f_2=1}\overline{G_q\left(1, \frac{af_1}{\ell}\right)}\left|\frac{\ell}{af_1}\right|_q^{2/3} q^{\frac{4j}{3}-\frac{4}{3}\left[j+\deg\left(\frac{af_1}{\ell}\right)\right]_3}\rho\left(1, \left[j+\deg\left(\frac{af_1}{\ell}\right)\right]_3\right).$$

Note that $j + \deg\left(\frac{af_1}{\ell}\right) \equiv d + \deg(f_1) \,(\mathrm{mod}\, 3)$, and

$$G_q(f_1,a)\overline{G_q(1,\ell)}\chi_\ell\left(\frac{af_1}{\ell}\right)\overline{G_q\left(1,\frac{af_1}{\ell}\right)} = G_q(f_1,a)\overline{G_q(1,af_1)} = |a|_q\overline{G_q(1,f_1)},$$

where we used Lemma 2.12. Combining the three equations above it follows that

$$\sum_{\substack{F\in\mathcal{M}_{q,d}\\(F,f)=1}} G_q(f,F) = \delta_{f_2=1}\frac{q^{\frac{4}{3}(d-[d+\deg(f_1)]_3)}\overline{G_q(1,f_1)}}{\zeta_q(2)|f_1|_q^{\frac{2}{3}}}\rho(1,[d+\deg(f_1)]_3) \sum_{\substack{a|f_3^*\\ \deg(a)\leq d-j}} \frac{\mu(a)}{|a|_q}$$

$$\times \prod_{P|af_1}\left(1-\frac{1}{|P|_q^2}\right)^{-1} \sum_{\substack{\ell|af_1\\ 2\deg(\ell)\leq d-j-\deg(a)}} \frac{\mu(\ell)}{|\ell|_q} + O(q^{\sigma d}|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)+\varepsilon}), \tag{49}$$

where we have used Lemma 3.11 to bound the integral.

Now using Perron's formula (Lemma 2.1) for the sum over ℓ we have

$$\sum_{\substack{\ell|af_1\\ 2\deg(\ell)\leq d-j-\deg(a)}} \frac{\mu(\ell)}{|\ell|_q} = \frac{1}{2\pi i}\oint \frac{\prod_{P|af_1}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)}{(1-x)x^{\left[\frac{d-j-\deg(a)}{2}\right]}}\frac{dx}{x}, \tag{50}$$

where we are integrating along a small circle around the origin. Let $\alpha(a) = 0$ if $\deg(a) \equiv d-j \,(\mathrm{mod}\, 2)$ and $\alpha(a) = 1$ otherwise. Introducing the sum over a and using Perron's formula, it follows that

$$\sum_{\substack{a|f_3^*\\ \deg(a)\le d-j}} \frac{\mu(a)}{|a|_q} x^{\frac{\deg(a)+\alpha(a)}{2}} \prod_{P|a}\left(1-\frac{1}{|P|_q^2}\right)^{-1}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)$$

$$=\frac{1+x^{\frac12}}{2}\sum_{\substack{a|f_3^*\\ \deg(a)\le d-j}} \frac{\mu(a)}{|a|_q} x^{\frac{\deg(a)}{2}} \prod_{P|a}\left(1-\frac{1}{|P|_q^2}\right)^{-1}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)$$

$$+(-1)^{d-j}\frac{1-x^{\frac12}}{2}\sum_{\substack{a|f_3^*\\ \deg(a)\le d-j}} \frac{\mu(a)}{|a|_q} x^{\frac{\deg(a)}{2}}(-1)^{\deg(a)} \prod_{P|a}\left(1-\frac{1}{|P|_q^2}\right)^{-1}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)$$

$$=\frac{1+x^{\frac12}}{2}\frac{1}{2\pi i}\oint \frac{\prod_{P|f_3^*}\left(1-\frac{(x^{\frac12}w)^{\deg(P)}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)}{|P|_q\left(1-\frac{1}{|P|_q^2}\right)}\right)}{(1-w)w^{d-j}}\frac{dw}{w}$$

$$+(-1)^{d-j}\frac{1-x^{\frac12}}{2}\frac{1}{2\pi i}\oint \frac{\prod_{P|f_3^*}\left(1-\frac{(-x^{\frac12}w)^{\deg(P)}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)}{|P|_q\left(1-\frac{1}{|P|_q^2}\right)}\right)}{(1-w)w^{d-j}}\frac{dw}{w}$$

$$=\frac{1}{2\pi i}\oint \frac{\prod_{P|f_3^*}\left(1-\frac{(x^{\frac12}w)^{\deg(P)}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)}{|P|_q\left(1-\frac{1}{|P|_q^2}\right)}\right)}{(1-w^2)w^{d-j}}(1+x^{\frac12}w)\frac{dw}{w}, \tag{51}$$

where again we are integrating along a small circle around the origin and we did the change of variables $w\to -w$ to the second integral to reach the last line. Let $\mathcal{R}(x,w)$ denote the Euler product above. Using equations (50) and (51) it follows that

$$\sum_{\substack{a|f_3^*\\ \deg(a)\le d-j}} \frac{\mu(a)}{|a|_q}\prod_{P|af_1}\left(1-\frac{1}{|P|_q^2}\right)^{-1}\sum_{\substack{\ell|af_1\\ 2\deg(\ell)\le d-j-\deg(a)}}\frac{\mu(\ell)}{|\ell|_q}$$

$$=\frac{1}{(2\pi i)^2}\oint\oint\prod_{P|f_1}\left(1-\frac{1}{|P|_q^2}\right)^{-1}\left(1-\frac{x^{\deg(P)}}{|P|_q}\right)\frac{\mathcal{R}(x,w)}{(1-x)(1-w^2)(x^{\frac12}w)^{d-j}}(1+x^{\frac12}w)\frac{dx}{x}\frac{dw}{w}.$$

We first shift the contour in the integral over x to $|x|=q^{1-\varepsilon}$ and we encounter a pole at $x=1$. We then shift the contour over w to $|w|=q^{\frac12-\varepsilon}$ and encounter a pole at $w=1$. Then

$$\sum_{\substack{a|f_3^*\\ \deg(a)\le d-j}} \frac{\mu(a)}{|a|_q}\prod_{P|af_1}\left(1-\frac{1}{|P|_q^2}\right)^{-1}\sum_{\substack{\ell|af_1\\ 2\deg(\ell)\le d-j-\deg(a)}}\frac{\mu(\ell)}{|\ell|_q}=\prod_{P|f_1f_3^*}\left(1+\frac{1}{|P|_q}\right)^{-1}+O(q^{\varepsilon d-d}).$$

Using the formula above in (49) and the fact that $|G_q(1,f_1)|=|f_1|_q^{\frac12}$ finishes the proof of the first statement of Proposition 3.1.

□

4. The non-Kummer setting

We now assume that q is odd with $q \equiv 2 \pmod 3$. We will prove Theorem 1.1.

4.1. **Setup and sieving.** Using Proposition 2.4 and Lemma 2.11, we have to compute

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L_q(1/2,\chi) \quad = \quad S_{1,\text{principal}} + S_{1,\text{dual}},$$

where
(52)

$$S_{1,\text{principal}} = \sum_{f \in \mathcal{M}_{q,\leq A}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,g/2+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} \chi_F(f) + \frac{1}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q,A+1}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,g/2+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} \chi_F(f)$$

and

$$\begin{aligned} S_{1,\text{dual}} = & \sum_{f \in \mathcal{M}_{q,\leq g-A-1}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,g/2+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} \omega(\chi_F)\overline{\chi_F}(f) \\ & + \frac{1}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q,g-A}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,g/2+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} \omega(\chi_F)\overline{\chi_F}(f). \end{aligned} \tag{53}$$

We will choose $A \equiv 0 \pmod 3$, and we recall that $0 \leq A \leq g-1$. For the principal term, we will compute the contribution from cube polynomials f and bound the contribution from non-cubes. We write

$$S_{1,\text{principal}} = S_{1,\cube} + S_{1,\neq\cube},$$

where $S_{1,\cube}$ corresponds to the sum with f a cube in equation (52) and $S_{1,\neq\cube}$ corresponds to the sum with f not a cube, namely,

$$S_{1,\cube} = \sum_{\substack{f \in \mathcal{M}_{q,\leq A} \\ f=\cube}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,g/2+1} \\ (F,f)=1 \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} 1, \tag{54}$$

and

$$S_{1,\neq\cube} = \sum_{\substack{f \in \mathcal{M}_{q,\leq A} \\ f\neq\cube}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,\frac{g}{2}+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} \chi_F(f) + \frac{1}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q,A+1}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,\frac{g}{2}+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} \chi_F(f).$$

Since $A \equiv 0 \pmod 3$, note that the second term in (52) does not contribute to the expression (54) for $S_{1,\cube}$.

The main results used to prove Theorem 1.1 are summarized in the following lemmas whose proofs we postpone to the next sections.

Lemma 4.1. *The main term $S_{1,\cube}$ is given by the following asymptotic formula*

$$S_{1,\cube} = \frac{q^{g+2}\zeta_q(3/2)}{\zeta_q(3)} \mathcal{A}_{\text{nK}}\left(\frac{1}{q^2}, \frac{1}{q^{3/2}}\right) + \frac{q^{g+2-\frac{A}{6}}\zeta_q(1/2)}{\zeta_q(3)} \mathcal{A}_{\text{nK}}\left(\frac{1}{q^2}, \frac{1}{q}\right) + O(q^{g-\frac{A}{2}+\varepsilon g}),$$

with $\mathcal{A}_{\mathrm{nK}}(x,u)$ *given by equation* (59)*. In particular,*

$$\mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2},\frac{1}{q^{3/2}}\right)=\prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{odd}}}\left(1-\frac{1}{|R|_q^2+1}\right)\prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{even}}}\left(1-\frac{1}{(|R|_q+1)^2}-\frac{2}{|R|_q^{\frac{1}{2}}(|R|_q+1)^2}\right),$$

and

$$\mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2},\frac{1}{q}\right)=\prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{odd}}}\left(1-\frac{1}{|R|_q^2+1}\right)\prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{even}}}\left(1-\frac{3}{(|R|_q+1)^2}\right).$$

In combination with the dual term $S_{1,\mathrm{dual}}$ this gives the following result.

Lemma 4.2. *We have*

$$S_{1,\square}+S_{1,\mathrm{dual}}=\frac{q^{g+2}\zeta_q(3/2)}{\zeta_q(3)}\mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2},\frac{1}{q^{3/2}}\right)+O\left(q^{g-\frac{A}{2}+\varepsilon g}+q^{\frac{5g}{6}+\varepsilon g}+q^{\frac{3g}{2}-(2-\sigma)A}\right),$$

where $7/6\le\sigma<4/3$*.*

We also have the following upper bound for $S_{1,\neq\square}$.

Lemma 4.3. *We have that*

$$S_{1,\neq\square}\ll q^{\frac{g+A}{2}+\varepsilon g}.$$

4.2. **The main term.** Here we will prove Lemma 4.1. In equation (54), write $f=k^3$. Recall that $A\equiv 0\,(\mathrm{mod}\,3)$. Then $S_{1,\square}$ can be rewritten as

$$S_{1,\square}=\sum_{k\in\mathcal{M}_{q,\le\frac{A}{3}}}\frac{1}{q^{3\deg(k)/2}}\sum_{\substack{F\in\mathcal{H}_{q^2,g/2+1}\\ (F,k)=1\\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}}1.$$

We first look at the generating series of the sum over F. We use the fact that

$$\sum_{\substack{D\in\mathbb{F}_q[T]\\ D|F}}\mu(D)=\begin{cases}1 & \text{if } F \text{ has no prime divisor in } \mathbb{F}_q[T],\\ 0 & \text{otherwise,}\end{cases}\tag{55}$$

where we have taken μ over $\mathbb{F}_q[T]$. Then

$$\sum_{\substack{F\in\mathcal{H}_{q^2}\\ (F,k)=1\\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}}x^{\deg(F)}=\sum_{\substack{F\in\mathcal{H}_{q^2}\\ (F,k)=1}}x^{\deg(F)}\sum_{\substack{D\in\mathbb{F}_q[T]\\ D|F}}\mu(D)=\sum_{\substack{D\in\mathbb{F}_q[T]\\ (D,k)=1}}\mu(D)x^{\deg(D)}\sum_{\substack{F\in\mathcal{H}_{q^2}\\ (F,Dk)=1}}x^{\deg(F)}.\tag{56}$$

We evaluate the sum over F in the equation above and we have that

$$\sum_{\substack{F\in\mathcal{H}_{q^2}\\ (F,kD)=1}}x^{\deg(F)}=\prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\ P\nmid Dk}}(1+x^{\deg(P)})=\frac{\mathcal{Z}_{q^2}(x)}{\mathcal{Z}_{q^2}(x^2)\prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\ P|Dk}}(1+x^{\deg(P)})},$$

so from equation (56) and the above it follows that

$$\sum_{\substack{F\in\mathcal{H}_{q^2}\\(F,k)=1\\P|F\Rightarrow P\notin\mathbb{F}_q[T]}} x^{\deg(F)} = \frac{\mathcal{Z}_{q^2}(x)}{\mathcal{Z}_{q^2}(x^2)\prod\limits_{\substack{P\in\mathbb{F}_{q^2}[T]\\P|k}}(1+x^{\deg(P)})}\sum_{\substack{D\in\mathbb{F}_q[T]\\(D,k)=1}}\frac{\mu(D)x^{\deg(D)}}{\prod\limits_{\substack{P\in\mathbb{F}_{q^2}[T]\\P|D}}(1+x^{\deg(P)})}.$$

Now we write down an Euler product for the sum over D and we have that

(57)

$$\sum_{\substack{D\in\mathbb{F}_q[T]\\(D,k)=1}}\frac{\mu(D)x^{\deg(D)}}{\prod\limits_{\substack{P\in\mathbb{F}_{q^2}[T]\\P|D}}(1+x^{\deg(P)})} = \prod_{\substack{R\in\mathbb{F}_q[T]\\(R,k)=1\\\deg(R)\text{ odd}}}\left(1-\frac{x^{\deg(R)}}{1+x^{\deg(R)}}\right)\prod_{\substack{R\in\mathbb{F}_q[T]\\(R,k)=1\\\deg(R)\text{ even}}}\left(1-\frac{x^{\deg(R)}}{(1+x^{\frac{\deg(R)}{2}})^2}\right),$$

where the product over R is over monic, irreducible polynomials. Let $A_R(x)$ denote the first Euler factor above and $B_R(x)$ the second. Then we rewrite the sum over D as

$$(57) = \frac{\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\\deg(R)\text{ odd}}}A_R(x)\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\\deg(R)\text{ even}}}B_R(x)}{\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\R|k\\\deg(R)\text{ odd}}}A_R(x)\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\R|k\\\deg(R)\text{ even}}}B_R(x)},$$

and putting everything together, it follows that

$$(58)\qquad \sum_{\substack{F\in\mathcal{H}_{q^2}\\(F,k)=1\\P|F\Rightarrow P\notin\mathbb{F}_q[T]}} x^{\deg(F)} = \frac{\mathcal{Z}_{q^2}(x)\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\\deg(R)\text{ odd}}}A_R(x)\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\\deg(R)\text{ even}}}B_R(x)}{\mathcal{Z}_{q^2}(x^2)\prod\limits_{\substack{P\in\mathbb{F}_{q^2}[T]\\P|k}}(1+x^{\deg(P)})\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\R|k\\\deg(R)\text{ odd}}}A_R(x)\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\R|k\\\deg(R)\text{ even}}}B_R(x)}.$$

We now introduce the sum over k and we have

$$\sum_{k\in\mathcal{M}_q}\frac{u^{\deg(k)}}{\prod\limits_{\substack{P\in\mathbb{F}_{q^2}[T]\\P|k}}(1+x^{\deg(P)})\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\R|k\\\deg(R)\text{ odd}}}A_R(x)\prod\limits_{\substack{R\in\mathbb{F}_q[T]\\R|k\\\deg(R)\text{ even}}}B_R(x)}$$

$$=\prod_{\substack{R\in\mathbb{F}_q[T]\\\deg(R)\text{ odd}}}\left[1+\frac{u^{\deg(R)}}{(1+x^{\deg(R)})A_R(x)(1-u^{\deg(R)})}\right]\prod_{\substack{R\in\mathbb{F}_q[T]\\\deg(R)\text{ even}}}\left[1+\frac{u^{\deg(R)}}{(1+x^{\frac{\deg(R)}{2}})^2B_R(x)(1-u^{\deg(R)})}\right],$$

where R denotes a monic irreducible polynomial in $\mathbb{F}_q[T]$. Combining the equation above and (58) we get that the generating series for the double sum over F and k is equal to

$$\begin{aligned}\sum_{k\in\mathcal{M}_q} u^{\deg(k)} \sum_{\substack{F\in\mathcal{H}_{q^2}\\(F,k)=1\\P|F\Rightarrow P\notin\mathbb{F}_q[T]}} x^{\deg(F)} =& \frac{\mathcal{Z}_{q^2}(x)}{\mathcal{Z}_{q^2}(x^2)} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ odd}}} \frac{1}{(1+x^{\deg(R)})(1-u^{\deg(R)})}\\ &\times \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ even}}} \frac{1}{(1+x^{\frac{\deg(R)}{2}})^2}\left(1+2x^{\frac{\deg(R)}{2}}+\frac{u^{\deg(R)}}{1-u^{\deg(R)}}\right)\\ =&\mathcal{Z}_q(u)\frac{\mathcal{Z}_{q^2}(x)}{\mathcal{Z}_{q^2}(x^2)}\mathcal{A}_{\mathrm{nK}}(x,u),\end{aligned}$$

where

$$\mathcal{A}_{\mathrm{nK}}(x,u) = \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ odd}}} \frac{1}{1+x^{\deg(R)}} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ even}}} \frac{1}{(1+x^{\frac{\deg(R)}{2}})^2}\left(1+2x^{\frac{\deg(R)}{2}}(1-u^{\deg(R)})\right). \tag{59}$$

Using Perron's formula (Lemma 2.1) twice in (54) and the expression of the generating series above, we get that

$$S_{1,\square} = \frac{1}{(2\pi i)^2}\oint\oint \frac{\mathcal{A}_{\mathrm{nK}}(x,u)(1-q^2x^2)}{(1-qu)(1-q^2x)(1-q^{3/2}u)x^{\frac{g}{2}+1}(q^{3/2}u)^{\frac{A}{3}}}\,\frac{dx}{x}\,\frac{du}{u},$$

where we are integrating along circles of radii $|u| < \frac{1}{q^{3/2}}$ and $|x| < \frac{1}{q^2}$. First note that $\mathcal{A}_{\mathrm{nK}}(x,u)$ is analytic for $|x| < 1/q, |xu| < 1/q, |xu^2| < 1/q^2$. We initially pick $|u| = 1/q^{\frac{3}{2}+\varepsilon}$ and $|x| = 1/q^{2+\varepsilon}$. We shift the contour over x to $|x| = 1/q^{1+\varepsilon}$ and we encounter a pole at $x = 1/q^2$. Note that the new double integral will be bounded by $O(q^{\frac{g}{2}+\varepsilon g})$. Then

$$S_{1,\square} = \frac{q^{g+2}}{\zeta_q(3)}\frac{1}{2\pi i}\oint \frac{\mathcal{A}_{\mathrm{nK}}(\frac{1}{q^2},u)}{(1-qu)(1-q^{3/2}u)(q^{3/2}u)^{\frac{A}{3}}}\,\frac{du}{u} + O(q^{\frac{g}{2}+\varepsilon g}).$$

Now we shift the contour of integration to $|u| = q^{-\varepsilon}$ and we encounter two simple poles: one at $u = 1/q^{\frac{3}{2}}$ and one at $u = 1/q$. We evaluate the residues and then

$$S_{1,\square} = \frac{q^{g+2}\zeta_q(3/2)}{\zeta_q(3)}\mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2},\frac{1}{q^{3/2}}\right) + \frac{q^{g+2-\frac{A}{6}}\zeta_q(1/2)}{\zeta_q(3)}\mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2},\frac{1}{q}\right) + O(q^{g-\frac{A}{2}+\varepsilon g}),$$

which finishes the proof of Lemma 4.1.

4.3. **The contribution from non-cubes.** Recall that $S_{1,\neq\square}$ is the term with f not a cube in $S_{1,\mathrm{principal}}$ of (52). Since $A \equiv 0 \,(\mathrm{mod}\, 3)$, the term we want to bound is equal to

$$S_{1,\neq\square} = \sum_{\substack{f\in\mathcal{M}_{q,\leq A}\\ f\neq\square}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F\in\mathcal{H}_{q^2,\frac{g}{2}+1}\\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}} \chi_F(f) + \frac{1}{1-\sqrt{q}} \sum_{f\in\mathcal{M}_{q,A+1}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F\in\mathcal{H}_{q^2,\frac{g}{2}+1}\\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}} \chi_F(f).$$

Let S_{11} be the first term above and S_{12} the second. Note that it is enough to bound S_{11}, since bounding S_{12} will follow in a similar way. We use equation (55) again for the sum over F and we have

$$S_{11} = \sum_{\substack{f \in \mathcal{M}_{q, \leq A} \\ f \neq \square}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D \in \mathcal{M}_{q, \leq \frac{g}{2}+1} \\ (D,f)=1}} \mu(D) \sum_{\substack{F \in \mathcal{H}_{q^2, \frac{g}{2}+1-\deg(D)} \\ (F,D)=1}} \chi_F(f). \tag{60}$$

Note that we used the fact that $\chi_D(f) = 1$ since $D, f \in \mathbb{F}_q[T]$. Now we look at the generating series for the sum over F. We have the following.

$$\sum_{\substack{F \in \mathcal{H}_{q^2} \\ (F,D)=1}} \chi_F(f) u^{\deg(F)} = \prod_{\substack{P \in \mathbb{F}_{q^2}[T] \\ P \nmid Df}} \left(1 + \chi_P(f) u^{\deg(P)}\right) = \frac{\mathcal{L}_{q^2}\left(u, \chi_f\right)}{\mathcal{L}_{q^2}(u^2, \overline{\chi_f})} \prod_{\substack{P \in \mathbb{F}_{q^2}[T] \\ P \nmid f \\ P \mid D}} \frac{1 - \chi_P(f) u^{\deg(P)}}{1 - \overline{\chi_P}(f) u^{2\deg(P)}}.$$

Using Perron's formula (Lemma 2.1) and the generating series above, we have

$$\sum_{\substack{F \in \mathcal{H}_{q^2, \frac{g}{2}+1-\deg(D)} \\ (F,D)=1}} \chi_F(f) = \frac{1}{2\pi i} \oint \frac{\mathcal{L}_{q^2}\left(u, \chi_f\right)}{\mathcal{L}_{q^2}(u^2, \overline{\chi_f}) u^{\frac{g}{2}+1-\deg(D)}} \prod_{\substack{P \in \mathbb{F}_{q^2}[T] \\ P \nmid f \\ P \mid D}} \frac{1 - \chi_P(f) u^{\deg(P)}}{1 - \overline{\chi_P}(f) u^{2\deg(P)}} \frac{du}{u},$$

where we are integrating along a circle of radius $|u| = \frac{1}{q}$ around the origin. Now we use the Lindelöf bound for the L–function in the numerator and a lower bound for the L–function in the denominator. We have, by Lemmas 2.5 and 2.6,

$$\left|\mathcal{L}_{q^2}\left(u, \chi_f\right)\right| \ll q^{2\varepsilon \deg(f)}, \quad \left|\mathcal{L}_{q^2}(u^2, \overline{\chi_f})\right| \gg q^{-2\varepsilon \deg(f)}.$$

Then

$$\sum_{\substack{F \in \mathcal{H}_{q^2, \frac{g}{2}+1-\deg(D)} \\ (F,D)=1}} \chi_F(f) \ll q^{\frac{g}{2}-\deg(D)} q^{4\varepsilon \deg(f) + 2\varepsilon \deg(D)}.$$

Trivially bounding the sums over D and f in (60) gives a total upper bound of

$$S_{11} \ll q^{\frac{A+g}{2}+\varepsilon g},$$

and similarly for S_{12}. This finishes the proof of Lemma 4.3.

4.4. **The dual term.** Here we will evaluate $S_{1,\mathrm{dual}}$ and prove Lemma 4.2. Recall the expression (53) for $S_{1,\mathrm{dual}}$. We further write $S_{1,\mathrm{dual}} = S_{11,\mathrm{dual}} + S_{12,\mathrm{dual}}$.

For F as in the expression (53), we have that χ_F is an even primitive character over $\mathbb{F}_q[T]$ of modulus $F\tilde{F}$ (recall that $\tilde{F}$ is the Galois conjugate of F). The modulus has degree $2\deg(F) = g+2$ and by Corollary 2.3 the sign of the functional equation is

$$\omega(\chi_F) = q^{-\frac{g}{2}-1} G(\chi_F),$$

where the Gauss sum is

$$G(\chi_F) = \sum_{\alpha \in \mathbb{F}_q[T]/(F\tilde{F})} \chi_F(\alpha)\, e_q\left(\frac{\alpha}{F\tilde{F}}\right).$$

By the Chinese Remainder Theorem, since F and $\tilde{F}$ are co-prime, if β runs over the classes in $\mathbb{F}_{q^2}[T]/(F)$ then $\beta\tilde{F} + \tilde{\beta}F$ runs over the classes in $\mathbb{F}_q[T]/(F\tilde{F})$. Then

$$\begin{aligned} G(\chi_F) = & \sum_{\beta \in \mathbb{F}_{q^2}[T]/(F)} \chi_F(\beta\tilde{F})\ e_q\left(\frac{\beta\tilde{F} + \tilde{\beta}F}{F\tilde{F}}\right) \\ = & \sum_{\beta \in \mathbb{F}_{q^2}[T]/(F)} \chi_F(\beta)\ e_{q^2}\left(\frac{\beta}{F}\right) \\ = & G_{q^2}(1, F), \end{aligned}$$

where we have used that $\chi_F(\tilde{F}) = 1$ due to cubic reciprocity.

Using the fact that $G_{q^2}(1, F)\overline{\chi_F}(f) = G_{q^2}(f, F)$ when $(f, F) = 1$ and $\overline{\chi_F}(f) = 0$ otherwise, we get

$$S_{11,\text{dual}} = q^{-\frac{g}{2}-1} \sum_{f \in \mathcal{M}_{q, \leq g-A-1}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2, \frac{g}{2}+1} \\ (F,f)=1 \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} G_{q^2}(f, F), \tag{61}$$

and

$$S_{12,\text{dual}} = \frac{q^{-\frac{g}{2}-1}}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q,g-A}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathcal{H}_{q^2, \frac{g}{2}+1} \\ (F,f)=1 \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} G_{q^2}(f, F). \tag{62}$$

We first prove the following important feature of $G_{q^2}(1, f)$.

Lemma 4.4. *Let $f \in \mathbb{F}_q[T]$ be square-free. Then*

$$G_{q^2}(1, f) = q^{\deg(f)}.$$

Proof. As usual, we denote by $\tilde{\alpha}$ the Galois conjugate of α. We have

$$\begin{aligned} \overline{G_{q^2}(1, f)} = & \sum_{\alpha \in \mathbb{F}_{q^2}[T]/(f)} \overline{\chi_f}(\alpha) e_{q^2}\left(\frac{-\alpha}{f}\right) = \sum_{\alpha \in \mathbb{F}_{q^2}[T]/(f)} \chi_f(\tilde{\alpha}) e_{q^2}\left(\frac{-\tilde{\alpha}}{f}\right) \\ = & \sum_{\alpha \in \mathbb{F}_{q^2}[T]/(f)} \chi_f(\alpha) e_{q^2}\left(\frac{-\alpha}{f}\right) = \chi_f(-1) \sum_{\alpha \in \mathbb{F}_{q^2}[T]/(f)} \chi_f(\alpha) e_{q^2}\left(\frac{\alpha}{f}\right) \\ = & G_{q^2}(1, f). \end{aligned}$$

In the first line we used the fact that $e_{q^2}(-\alpha/f) = e_{q^2}(-\tilde{\alpha}/f)$ which follows because $\operatorname{tr}(\alpha) = \operatorname{tr}(\tilde{\alpha})$. In the second line we used that $\chi_f(-1) = \Omega^{-1}((-1)^{\frac{q^2-1}{3}\deg(f)}) = 1$.

Notice that for $f, g \in \mathbb{F}_q[T]$, $(f, g) = 1$, $\chi_f(g) = 1$ because

$$\overline{\chi_f}(g) = \chi_{\tilde{f}}(\tilde{g}) = \chi_f(g),$$

which implies that $\chi_f(g) \in \mathbb{R}$, hence it has to be equal to 1.

Then by Lemma 2.12, we have that

$$G_{q^2}(1, fg) = G_{q^2}(1, f)G_{q^2}(1, g).$$

Now if $P \in \mathbb{F}_q[T]$, then

$$G_{q^2}(1,P)^2 = \epsilon(\chi_P)^2\omega(\chi_P)^2|P|_{q^2} = \overline{\epsilon(\chi_P)\omega(\chi_P)|P|_{q^2}^{1/2}}|P|_{q^2}^{1/2} = \overline{G_{q^2}(1,P)}q^{\deg(P)}$$

and from this we conclude that

$$G_{q^2}(1,P) = q^{\deg(P)}.$$

By multiplicativity, since f is square-free,

$$G_{q^2}(1,f) = q^{\deg(f)}.$$

□

Now we go back to (61) and (62). Using the sieve (55), we get that

$$\begin{aligned}
\sum_{\substack{F\in\mathcal{H}_{q^2,\frac{g}{2}+1}\\(F,f)=1\\P|F\Rightarrow P\notin\mathbb{F}_q[T]}} G_{q^2}(f,F) &= \sum_{\substack{D\in\mathbb{F}_q[T]\\\deg(D)\leq g/2+1\\(D,f)=1}} \mu(D) \sum_{\substack{F\in\mathcal{M}_{q^2,g/2+1-\deg(D)}\\(F,f)=1}} G_{q^2}(f,DF)\\
&= \sum_{\substack{D\in\mathbb{F}_q[T]\\\deg(D)\leq g/2+1\\(D,f)=1}} \mu(D)G_{q^2}(f,D) \sum_{\substack{F\in\mathcal{M}_{q^2,g/2+1-\deg(D)}\\(F,Df)=1}} \chi_F^2(D)G_{q^2}(f,F)\\
&= \sum_{\substack{D\in\mathbb{F}_q[T]\\\deg(D)\leq g/2+1\\(D,f)=1}} \mu(D)G_{q^2}(f,D) \sum_{\substack{F\in\mathcal{M}_{q^2,g/2+1-\deg(D)}\\(F,Df)=1}} G_{q^2}(fD,F), \qquad (63)
\end{aligned}$$

where we have used that $G_{q^2}(f,DF)=0$ if $(D,F)\neq 1$, since $(f,DF)=1$ due to the last case of Lemma 2.12 (ii) for $\alpha=0$.

Using Proposition 3.1 (recall that we are working in $\mathbb{F}_{q^2}[T]$ and that $f=f_1f_2^2f_3^3$ with f_1, f_2 square-free and coprime) we get that

$$\begin{aligned}
\sum_{\substack{F\in\mathcal{M}_{q^2,g/2+1-\deg(D)}\\(F,fD)=1}} G_{q^2}(fD,F) &= \delta_{f_2=1}\frac{q^{\frac{4g}{3}+\frac{8}{3}-4\deg(D)-\frac{4}{3}\deg(f_1)-\frac{8}{3}[g/2+1+\deg(f_1)]_3}}{\zeta_{q^2}(2)}\overline{G_{q^2}(1,f_1D)}\\
&\quad\times\rho(1,[g/2+1+\deg(f_1)]_3)\prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\P|fD}}\left(1+\frac{1}{|P|_{q^2}}\right)^{-1}\\
&\quad+O\left(\delta_{f_2=1}q^{\frac{g}{3}+\varepsilon g-\deg(D)(1+2\varepsilon)-\frac{\deg(f)_1}{3}}\right)+\frac{1}{2\pi i}\oint_{|u|=q^{-2\sigma}}\frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}}\frac{du}{u}, \qquad (64)
\end{aligned}$$

with $\delta_{f_2=1}=1$ if $f_2=1$ and $\delta_{f_2=1}=0$ otherwise. Combining equations (61), (63), (64) and Lemma 4.4, we write

$$S_{11,\mathrm{dual}} = M_1 + E_1, \qquad (65)$$

where M_1 corresponds to the main term in (64) and E_1 corresponds to the two error terms in (64).

We will obtain an asymptotic formula for M_1 and keep E_1 in its integral form, postponing bounding it for the moment. We will then do the same for the term $S_{12,\mathrm{dual}}$, obtaining a main term and an integral form error term. We will combine the two expressions to obtain a main term of size $q^{g-\frac{A}{6}}$ for $S_{1,\mathrm{dual}}$, together with the integral form errors. We will then note that the term of size $q^{g-\frac{A}{6}}$ cancels out the corresponding term from Theorem 4.1. Only at this point will we bound the integrals. Note that the order in which we perform these calculations matters: bounding the error terms wouldn't allow us to detect the cancellation with the term of size $q^{g-\frac{A}{6}}$ in Theorem 4.1, as we do not a priori know the sizes of the various terms involved.

We have

$$\begin{aligned}M_1 =&\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)} \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{\delta_{f_2=1}q^{-\frac{8}{3}[g/2+1+\deg(f_1)]_3}}{q^{\deg(f)/2+\deg(f_1)/3}} \sum_{\substack{D\in\mathbb{F}_q[T]\\ \deg(D)\leq g/2+1\\ (D,f)=1}} \mu(D)q^{-4\deg(D)}|G_{q^2}(1,D)|^2\\ &\times \rho(1,[g/2+1+\deg(f_1)]_3) \prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\ P|fD}} \left(1+\frac{1}{|P|_{q^2}}\right)^{-1}\\ =&\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)} \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{\delta_{f_2=1}q^{-\frac{8}{3}[g/2+1+\deg(f_1)]_3}}{q^{\deg(f)/2+\deg(f_1)/3}}\rho(1,[g/2+1+\deg(f_1)]_3)\\ &\times \prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\ P|f}} \left(1+\frac{1}{|P|_{q^2}}\right)^{-1} \sum_{\substack{D\in\mathbb{F}_q[T]\\ \deg(D)\leq g/2+1\\ (D,f)=1}} \mu(D)q^{-2\deg(D)} \prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\ P|D}} \left(1+\frac{1}{|P|_{q^2}}\right)^{-1}.\end{aligned}$$

We first treat the sum over D. We consider the generating series of the sum over D. We have that

$$\begin{aligned}&\sum_{\substack{D\in\mathbb{F}_q[T]\\ (D,f)=1}} \frac{\mu(D)}{q^{2\deg(D)}} \prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\ P|D}} \left(1+\frac{1}{|P|_{q^2}}\right)^{-1} w^{\deg(D)}\\ &= \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{odd}\\ R\nmid f}} \left[1-\frac{w^{\deg(R)}}{q^{2\deg(R)}(1+\frac{1}{q^{2\deg(R)}})}\right] \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{even}\\ R\nmid f}} \left[1-\frac{w^{\deg(R)}}{q^{2\deg(R)}(1+\frac{1}{q^{\deg(R)}})^2}\right],\end{aligned}$$

where we have counted the primes in $\mathbb{F}_{q^2}[T]$ by counting the primes of $\mathbb{F}_q[T]$ lying under them. Recall from Section 2.2 that $P\in\mathbb{F}_q[T]$ splits in $\mathbb{F}_{q^2}[T]$ if and only if $\deg(P)$ is even.

Let $A_{\mathrm{dual},R}(w)$ denote the first factor above and $B_{\mathrm{dual},R}(w)$ the second factor, for any R (not restricted to those $R\nmid f$). Define

$$\mathcal{J}_{\mathrm{nK}}(w) = \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{odd}}} A_{\mathrm{dual},R}(w) \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{even}}} B_{\mathrm{dual},R}(w),$$

which is absolutely convergent for $|w|<q$.

Then by Perron's formula (Lemma 2.1) we have

$$\sum_{\substack{D\in\mathbb{F}_q[T]\\ \deg(D)\le g/2+1\\ (D,f)=1}} \mu(D)q^{-2\deg(D)} \prod_{\substack{P\in\mathbb{F}_{q^2}[T]\\ P|D}} \left(1+\frac{1}{|P|_{q^2}}\right)^{-1} = \frac{1}{2\pi i}\oint \frac{\mathcal{J}_{\mathrm{nK}}(w)}{w^{g/2+1}(1-w)}$$
$$\times \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ odd}\\ R|f}} A_{\mathrm{dual},R}(w)^{-1} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ even}\\ R|f}} B_{\mathrm{dual},R}(w)^{-1}\,\frac{dw}{w}.$$

Now we introduce the sum over f. Using the expression for the sum over D above, we get that

$$M_1 = \frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)} \sum_{f\in\mathcal{M}_{q,\le g-A-1}} \frac{\delta_{f_2=1}\rho(1,[g/2+1+\deg(f_1)]_3)}{q^{\frac{8}{3}[g/2+1+\deg(f_1)]_3}q^{\deg(f)/2+\deg(f_1)/3}} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ odd}\\ R|f}} \left(1+\frac{1}{q^{2\deg(R)}}\right)^{-1}$$

(66)

$$\times \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ even}\\ R|f}} \left(1+\frac{1}{q^{\deg(R)}}\right)^{-2} \frac{1}{2\pi i}\oint \frac{\mathcal{J}_{\mathrm{nK}}(w)}{w^{g/2+1}(1-w)} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ odd}\\ R|f}} A_{\mathrm{dual},R}(w)^{-1} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ even}\\ R|f}} B_{\mathrm{dual},R}(w)^{-1}\,\frac{dw}{w}.$$

Let

$$\mathcal{H}_{\mathrm{nK}}(u,w) = \sum_f \frac{\delta_{f_2=1}}{q^{\deg(f)/2+\deg(f_1)/3}} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ odd}\\ R|f}} C_R(w)^{-1} \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ even}\\ R|f}} D_R(w)^{-1}u^{\deg(f)},$$

where

$$C_R(w) = 1+\frac{1}{q^{2\deg(R)}} - \frac{w^{\deg(R)}}{q^{2\deg(R)}}, \qquad D_R(w) = \left(1+\frac{1}{q^{\deg(R)}}\right)^2 - \frac{w^{\deg(R)}}{q^{2\deg(R)}}.$$

Then we can write down an Euler product for $\mathcal{H}_{\mathrm{nK}}(u,w)$ and we have that

$$\mathcal{H}_{\mathrm{nK}}(u,w) = \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ odd}}} \left[1+C_R(w)^{-1}\left(\frac{1}{q^{\deg(R)/3}}\sum_{j=0}^{\infty}\frac{u^{(3j+1)\deg(R)}}{q^{(3j+1)\deg(R)/2}} + \sum_{j=1}^{\infty}\frac{u^{3j\deg(R)}}{q^{3j\deg(R)/2}}\right)\right]$$
$$\times \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\text{ even}}} \left[1+D_R(w)^{-1}\left(\frac{1}{q^{\deg(R)/3}}\sum_{j=0}^{\infty}\frac{u^{(3j+1)\deg(R)}}{q^{(3j+1)\deg(R)/2}} + \sum_{j=1}^{\infty}\frac{u^{3j\deg(R)}}{q^{3j\deg(R)/2}}\right)\right].$$

After simplifying, we have

$$\mathcal{H}_{\mathrm{nK}}(u,w) = \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{odd}}} \left[1 + C_R(w)^{-1}\left(\frac{u^{\deg(R)}}{|R|_q^{5/6}(1-\frac{u^{3\deg(R)}}{|R|_q^{3/2}})} + \frac{u^{3\deg(R)}}{|R|_q^{3/2} - u^{3\deg(R)}}\right)\right]$$

$$\times \prod_{\substack{R\in\mathbb{F}_q[T]\\ \deg(R)\,\mathrm{even}}} \left[1 + D_R(w)^{-1}\left(\frac{u^{\deg(R)}}{|R|_q^{5/6}(1-\frac{u^{3\deg(R)}}{|R|_q^{3/2}})} + \frac{u^{3\deg(R)}}{|R|_q^{3/2} - u^{3\deg(R)}}\right)\right]$$

$$(67) \qquad = \mathcal{Z}\left(\frac{u}{q^{5/6}}\right)\mathcal{B}_{\mathrm{nK}}(u,w),$$

with $\mathcal{B}_{\mathrm{nK}}(u,w)$ analytic in a wider region (for example, $\mathcal{B}_{\mathrm{nK}}(u,w)$ is absolutely convergent for $|u| < q^{\frac{11}{6}}$ and $|uw| < q^{\frac{11}{6}}$).

We will use Perron's formula (Lemma 2.1) for the sum over f in equation (66) which involves $\rho(1,[g/2+1+\deg(f_1)]_3$. Note that by Lemma 3.9, this depends on $g/2+1+\deg(f_1)\,(\mathrm{mod}\,3)$, and we treat each case in turn. Recall that $\deg f \equiv \deg f_1 \mod 3$, since $f_2 = 1$.

If $g/2+1+\deg(f_1) \equiv 0\,(\mathrm{mod}\,3)$, then $\deg(f_1) \equiv g-1\,(\mathrm{mod}\,3)$. In this case by Lemma 3.9, $\rho(1,0) = 1$. Applying Perron's formula (Lemma 2.1) for the sum over f with $\deg(f) \equiv g-1\,(\mathrm{mod}\,3)$ (recall that $A \equiv 0\,(\mathrm{mod}\,3)$), we get that this contribution is equal to

$$\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)} \oint\oint \frac{\mathcal{H}_{\mathrm{nK}}(u,w)\mathcal{J}_{\mathrm{nK}}(w)}{w^{g/2+1}(1-w)u^{g-A-1}(1-u^3)}\,\frac{dw}{w}\,\frac{du}{u},$$

where we are integrating along small circles around the origin.

If $g/2+1+\deg(f_1) \equiv 1\,(\mathrm{mod}\,3)$, then $\deg(f_1) \equiv g\,(\mathrm{mod}\,3)$, and by Lemma 3.9 again we have $\rho(1,1) = \tau(\chi_3)q^2$. Note that $\tau(\chi_3) = q\epsilon(\chi_3)$ and $\epsilon(\chi_3) = (-1)^{\frac{q^2-1}{3}} = 1$. Since q is odd, we have $\rho(1,1) = q^3$. We use Perron's formula for the sum over f with $\deg(f) \equiv g\,(\mathrm{mod}\,3)$, and since $g-A-1 \equiv g-1\,(\mathrm{mod}\,3)$, we have that $\deg(f) \le g-A-3$. Then we get that the contribution from $\deg(f) \equiv g\,(\mathrm{mod}\,3)$ in (66) is

$$\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)} \oint\oint \frac{\mathcal{H}_{\mathrm{nK}}(u,w)\mathcal{J}_{\mathrm{nK}}(w)q^{1/3}}{w^{g/2+1}(1-w)u^{g-A-3}(1-u^3)}\,\frac{dw}{w}\,\frac{du}{u},$$

where we are again integrating along small circles around the origin.

It is clear from Lemma 3.9 that there is no contribution when $g/2+1+\deg(f_1) \equiv 2\,(\mathrm{mod}\,3)$.

Combining the two equations above, we get that

$$M_1 = \frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)} \oint\oint \frac{\mathcal{H}_{\mathrm{nK}}(u,w)\mathcal{J}_{\mathrm{nK}}(w)}{w^{g/2+1}(1-w)}\left[\frac{1}{u^{g-A-1}(1-u^3)} + \frac{q^{1/3}}{u^{g-A-3}(1-u^3)}\right]\frac{dw}{w}\,\frac{du}{u},$$

where we integrate along small circles around the origin.

We first shift the contour over w to $|w| = q^{1-\epsilon}$ (since $\mathcal{J}_{\mathrm{nK}}(w)$ is absolutely convergent for $|w| < q$) and encounter the pole at $w = 1$. Note that $\mathcal{H}_{\mathrm{nK}}(u,1)$ has a pole at $u = q^{-1/6}$. Let

$$(68) \qquad \mathcal{K}_{\mathrm{nK}}(u) = \mathcal{B}_{\mathrm{nK}}(u,1)\mathcal{J}_{\mathrm{nK}}(1).$$

Then

$$\begin{aligned}M_1 =&\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)}\oint \frac{\mathcal{K}_{\mathrm{nK}}(u)}{(1-uq^{1/6})(1-u^3)u^{g-A-1}}(1+q^{1/3}u^2)\,\frac{du}{u}\\ &+\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)}\oint_{|u|=q^{-1/6-\varepsilon}}\oint_{|w|=q^{1-\varepsilon}}\frac{\mathcal{H}_{\mathrm{nK}}(u,w)\mathcal{J}_{\mathrm{nK}}(w)}{w^{g/2+1}(1-w)}\left[\frac{1}{u^{g-A-1}(1-u^3)}+\frac{q^{1/3}}{u^{g-A-3}(1-u^3)}\right]\frac{dw}{w}\,\frac{du}{u}\\ =&\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)}\oint \frac{\mathcal{K}_{\mathrm{nK}}(u)}{(1-uq^{1/6})(1-u^3)u^{g-A-1}}(1+q^{1/3}u^2)\,\frac{du}{u}+O\left(q^{\frac{g}{2}-\frac{A}{6}+\varepsilon g}\right).\end{aligned}$$

Note that $\mathcal{K}_{\mathrm{nK}}(u)$ is absolutely convergent for $|u|<q^{\frac{1}{6}}$. We shift the contour of integration to $|u|=q^{-\varepsilon}$, we compute the residue at $u=q^{-1/6}$ and we get that

$$\begin{aligned}M_1 =&2q^{g-\frac{A}{6}+2}\frac{\mathcal{K}_{\mathrm{nK}}(q^{-1/6})}{\zeta_{q^2}(2)(\sqrt{q}-1)}+O\left(q^{\frac{g}{2}-\frac{A}{6}+\varepsilon g}\right)\\ &+\frac{q^{5g/6+5/3}}{\zeta_{q^2}(2)}\oint_{|u|=q^{-\varepsilon}}\frac{\mathcal{K}_{\mathrm{nK}}(u)}{(1-uq^{1/6})(1-u^3)u^{g-A-1}}(1+q^{1/3}u^2)\,\frac{du}{u}\\ =&2q^{g-\frac{A}{6}+2}\frac{\mathcal{K}_{\mathrm{nK}}(q^{-1/6})}{\zeta_{q^2}(2)(\sqrt{q}-1)}+O\left(q^{\frac{5g}{6}+\varepsilon g}\right).\end{aligned} \tag{69}$$

Now we consider the error term E_1 from equation (65). The first term coming from the first error in equation (64) will be bounded by

$$\ll q^{-\frac{g}{2}}\sum_{f\in\mathcal{M}_{q,\leq g-A-1}}\frac{1}{q^{\deg(f)/2}}\sum_{\deg(D)\leq g/2+1}q^{\deg(D)}q^{\frac{g}{3}+\varepsilon g-\deg(D)(1+2\varepsilon)-\frac{\deg(f_1)}{3}}\ll q^{\left(\frac{1}{2}+\varepsilon\right)g-\frac{A}{6}}.$$

Then we get that

$$\begin{aligned}E_1 =&q^{-g/2-1}\frac{1}{2\pi i}\oint_{|u|=q^{-2\sigma}}\sum_{f\in\mathcal{M}_{q,\leq g-A-1}}\frac{1}{q^{\deg(f)/2}}\sum_{\substack{D\in\mathbb{F}_q[T]\\ \deg(D)\leq g/2+1\\ (D,f)=1}}\mu(D)G_{q^2}(f,D)\frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}}\,\frac{du}{u}\\ &+O(q^{\frac{g}{2}-\frac{A}{6}+\varepsilon g}),\end{aligned}$$

where recall that $2/3<\sigma<4/3$.

Combining the expressions for M_1 and E_1 it follows that

$$\begin{aligned}S_{11,\mathrm{dual}} =&2q^{g-\frac{A}{6}+2}\frac{\mathcal{K}_{\mathrm{nK}}(q^{-1/6})}{\zeta_{q^2}(2)(\sqrt{q}-1)}+O\left(q^{\frac{5g}{6}+\varepsilon g}\right)\\ &+q^{-g/2-1}\frac{1}{2\pi i}\oint_{|u|=q^{-2\sigma}}\sum_{f\in\mathcal{M}_{q,\leq g-A-1}}\frac{1}{q^{\deg(f)/2}}\sum_{\substack{D\in\mathbb{F}_q[T]\\ \deg(D)\leq g/2+1\\ (D,f)=1}}\mu(D)G_{q^2}(f,D)\frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}}\,\frac{du}{u}.\end{aligned}$$

We treat $S_{12,\text{dual}}$ similarly and since $\deg(f) = g - A$ we have $[g/2 + 1 + \deg(f_1)]_3 = 1$. Then as before $\rho(1,1) = \tau(\chi_3) = q^3$, and we get that

$$\begin{aligned} S_{12,\text{dual}} =& q^{g-\frac{A}{6}+2} \frac{\mathcal{K}_{\text{nK}}(q^{-1/6})}{\zeta_{q^2}(2)(1-\sqrt{q})} + O\left(q^{\frac{5g}{6}+\varepsilon g}\right) \\ &+ \frac{q^{-\frac{g}{2}-1}}{1-\sqrt{q}} \frac{1}{2\pi i} \oint_{|u|=q^{-2\sigma}} \sum_{f \in \mathcal{M}_{q,g-A}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D \in \mathbb{F}_q[T] \\ \deg(D) \leq g/2+1 \\ (D,f)=1}} \mu(D) G_{q^2}(f,D) \frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}} \frac{du}{u}. \end{aligned}$$

Combining the two equations above, we get that

(70)

$$\begin{aligned} S_{1,\text{dual}} =& -\frac{q^{g-\frac{A}{6}+2}\mathcal{K}_{\text{nK}}(q^{-1/6})\zeta_q(1/2)}{\zeta_{q^2}(2)} + O\left(q^{\frac{5g}{6}+\varepsilon g}\right) \\ &+ q^{-g/2-1} \frac{1}{2\pi i} \oint_{|u|=q^{-2\sigma}} \sum_{f \in \mathcal{M}_{q,\leq g-A-1}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D \in \mathbb{F}_q[T] \\ \deg(D) \leq g/2+1 \\ (D,f)=1}} \mu(D) G_{q^2}(f,D) \frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}} \frac{du}{u} \\ &+ \frac{q^{-\frac{g}{2}-1}}{1-\sqrt{q}} \frac{1}{2\pi i} \oint_{|u|=q^{-2\sigma}} \sum_{f \in \mathcal{M}_{q,g-A}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D \in \mathbb{F}_q[T] \\ \deg(D) \leq g/2+1 \\ (D,f)=1}} \mu(D) G_{q^2}(f,D) \frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}} \frac{du}{u}. \end{aligned}$$

We have

$$\mathcal{J}_{\text{nK}}(1) = \prod_{\substack{R \in \mathbb{F}_q[T] \\ \deg(R) \text{ odd}}} \left[1 - \frac{1}{|R|_q^2+1}\right] \prod_{\substack{R \in \mathbb{F}_q[T] \\ \deg(R) \text{ even}}} \left[1 - \frac{1}{(|R|_q+1)^2}\right],$$

and using the definition (67) for $\mathcal{B}_{\text{nK}}(u,w)$

$$\begin{aligned} \mathcal{B}_{\text{nK}}(q^{-1/6},1) =& \prod_{\substack{R \in \mathbb{F}_q[T] \\ \deg(R) \text{ odd}}} \left[1 + \frac{1}{|R|_q(1-\frac{1}{|R|_q^2})} + \frac{1}{|R|_q^{1/2}(|R|_q^{3/2} - \frac{1}{|R|_q^{1/2}})}\right] \left[1 - \frac{1}{|R|_q}\right] \\ &\times \prod_{\substack{R \in \mathbb{F}_q[T] \\ \deg(R) \text{ even}}} \left[1 + \frac{1}{1+\frac{2}{|R|_q}} \left(\frac{1}{|R|_q(1-\frac{1}{|R|_q^2})} + \frac{1}{|R|_q^{1/2}(|R|_q^{3/2} - \frac{1}{|R|_q^{1/2}})}\right)\right] \left[1 - \frac{1}{|R|_q}\right] \\ =& \prod_{\substack{R \in \mathbb{F}_q[T] \\ \deg(R) \text{ even}}} \left[1 + \frac{1}{(1+\frac{2}{|R|_q})(|R|_q - 1)}\right] \left[1 - \frac{1}{|R|_q}\right]. \end{aligned}$$

By (68),

$$\mathcal{K}_{\text{nK}}(q^{-1/6}) = \prod_{\substack{R \in \mathbb{F}_q[T] \\ \deg(R) \text{ odd}}} \left(1 - \frac{1}{|R|_q^2+1}\right) \prod_{\substack{R \in \mathbb{F}_q[T] \\ \deg(R) \text{ even}}} \left(1 - \frac{3}{(|R|_q+1)^2}\right),$$

and we have that $\mathcal{K}_{\mathrm{nK}}(q^{-1/6}) = \mathcal{A}_{\mathrm{nK}}(1/q^2, 1/q)$. Since $\zeta_q(3) = \zeta_{q^2}(2)$, by using equation (70) and Lemma 4.1 we note that the corresponding terms of size $q^{g-\frac{A}{6}}$ in the expressions for $S_{1,\square}$ and $S_{1,\mathrm{dual}}$ cancel out. Hence

$$\begin{aligned} S_{1,\square} + S_{1,\mathrm{dual}} =& \frac{q^{g+2}\zeta_q(3/2)}{\zeta_q(3)} \mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2}, \frac{1}{q^{3/2}}\right) + O(q^{g-\frac{A}{2}+\varepsilon g} + q^{\frac{5g}{6}+\varepsilon g}) \\ &+ q^{-\frac{g}{2}-1} \frac{1}{2\pi i} \oint_{|u|=q^{-2\sigma}} \sum_{f \in \mathcal{M}_{q,\leq g-A-1}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D \in \mathbb{F}_q[T] \\ \deg(D) \leq g/2+1 \\ (D,f)=1}} \mu(D) G_{q^2}(f,D) \frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}} \frac{du}{u} \\ &+ \frac{q^{-\frac{g}{2}-1}}{1-\sqrt{q}} \frac{1}{2\pi i} \oint_{|u|=q^{-2\sigma}} \sum_{f \in \mathcal{M}_{q,g-A}} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D \in \mathbb{F}_q[T] \\ \deg(D) \leq g/2+1 \\ (D,f)=1}} \mu(D) G_{q^2}(f,D) \frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{g/2+1-\deg(D)}} \frac{du}{u}. \end{aligned}$$

Now we consider the integral terms above. Note that it is enough to bound the first one. Using Lemma 3.11, the term in the second line above is bounded by

$$\begin{aligned} &\ll q^{-\frac{g}{2}} \sum_{f \in \mathcal{M}_{q,\leq g-A-1}} \frac{1}{q^{\deg(f)/2}} \sum_{\deg(D) \leq g/2+1} q^{\deg(D)} q^{\sigma g - 3\sigma \deg(D) + \frac{3}{2}\deg(D) + \deg(f)(\frac{3}{2}-\sigma)} \\ &\ll g q^{\frac{3}{2}g-(2-\sigma)A} \end{aligned}$$

as long as $\sigma \geq 7/6$. Then

$$S_{1,\square} + S_{1,\mathrm{dual}} = \frac{q^{g+2}\zeta_q(3/2)}{\zeta_q(3)} \mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2}, \frac{1}{q^{3/2}}\right) + O\left(q^{g-\frac{A}{2}+\varepsilon g} + q^{\frac{5g}{6}+\varepsilon g} + q^{\frac{3g}{2}-(2-\sigma)A+\varepsilon g}\right), \tag{71}$$

which finishes the proof of Lemma 4.2.

Remark 4.5. Note that the error term of size $q^{\frac{5g}{6}}$ can be computed explicitly from equation (69) by evaluating the residue when $u^3 = 1$. The other error terms will eventually dominate the term of size $q^{\frac{5g}{6}}$, so we do not carry out the computation. However, we believe this term will persist in the asymptotic formula.

4.5. **The proof of Theorem 1.1.** Using Lemmas 4.2 and 4.3, we get that

$$\sum_{\substack{F \in \mathcal{H}_{q^2,\frac{g}{2}+1} \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} L_q\left(\frac{1}{2}, \chi_F\right) = \frac{q^{g+2}\zeta_q(3/2)}{\zeta_q(3)} \mathcal{A}_{\mathrm{nK}}\left(\frac{1}{q^2}, \frac{1}{q^{3/2}}\right) + O\left(q^{\frac{g+A}{2}+\varepsilon g} + q^{\frac{5g}{6}+\varepsilon g} + q^{\frac{3g}{2}-(2-\sigma)A+\varepsilon g}\right),$$

where $7/6 \leq \sigma < 4/3$. Picking $\sigma = 7/6$ and $A = 3[g/4]$ finishes the proof of Theorem 1.1.

5. The Kummer setting

We now assume that q is odd with $q \equiv 1 \pmod 3$. We will prove Theorem 1.2.

5.1. **Setup and sieving.** By Lemma 2.9, we want to compute

$$\sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ F_2\in\mathcal{H}_{q,d_2} \\ (F_1,F_2)=1}} L_q\left(\frac{1}{2}, \chi_{F_1F_2^2}\right) = S_{2,\mathrm{principal}} + S_{2,\mathrm{dual}}, \tag{72}$$

where we have from Proposition 2.4 and Lemma 2.7 (cubic reciprocity)

$$S_{2,\mathrm{principal}} = \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ F_2\in\mathcal{H}_{q,d_2} \\ (F_1,F_2)=1}} \sum_{f\in\mathcal{M}_{q,\leq A}} \frac{\chi_f(F_1)\overline{\chi_f}(F_2)}{|f|_q^{1/2}}, \tag{73}$$

$$S_{2,\mathrm{dual}} = \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ F_2\in\mathcal{H}_{q,d_2} \\ (F_1,F_2)=1}} \omega(\chi_{F_1}\overline{\chi_{F_2}}) \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{\overline{\chi_f}(F_1)\chi_f(F_2)}{|f|_q^{1/2}}. \tag{74}$$

We will choose $A \equiv 0 \,(\mathrm{mod}\, 3)$. For the principal term, we will compute the contribution from cube polynomials f and bound the contribution from non-cubes. We write

$$S_{2,\mathrm{principal}} = S_{2,\square} + S_{2,\neq\square},$$

where

$$S_{2,\square} = \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ F_2\in\mathcal{H}_{q,d_2} \\ (F_1,F_2)=1}} \sum_{\substack{f\in\mathcal{M}_{q,\leq A} \\ f=\square}} \frac{\chi_f(F_1)\overline{\chi_f}(F_2)}{|f|_q^{1/2}}, \tag{75}$$

and

$$S_{2,\neq\square} = \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ F_2\in\mathcal{H}_{q,d_2} \\ (F_1,F_2)=1}} \sum_{\substack{f\in\mathcal{M}_{q,\leq A} \\ f\neq\square}} \frac{\chi_f(F_1)\overline{\chi_f}(F_2)}{|f|_q^{1/2}}. \tag{76}$$

The main results used to prove Theorem 1.2 are summarized in the following lemmas whose proofs we postpone to the next sections.

Lemma 5.1. *The main term $S_{2,\square}$ is given by the following asymptotic formula*

$$S_{2,\square} = C_{\mathrm{K},1}gq^{g+1} + C_{\mathrm{K},2}q^{g+1} + D_{\mathrm{K},1}gq^{g+1-\frac{A}{6}} + D_{\mathrm{K},2}q^{g+1-\frac{A}{6}} + O\left(q^{\frac{g}{3}+\varepsilon g} + q^{g-\frac{5A}{6}+\varepsilon g}\right),$$

for some explicit constants $C_{\mathrm{K},1}, C_{\mathrm{K},2}, D_{\mathrm{K},1}, D_{\mathrm{K},2}$ (see formula (88)).

We also have the following upper bounds for $S_{2,\neq\square}$ and $S_{2,\mathrm{dual}}$.

Lemma 5.2. *We have that*

$$S_{2,\neq\square} \ll q^{\frac{A+g}{2}+\varepsilon g}.$$

Lemma 5.3. *The dual term is bounded by*

$$S_{2,\mathrm{dual}} \ll q^{(1+\varepsilon)g-\frac{A}{6}} + q^{\left(\frac{23}{12}-\frac{\sigma}{2}+\varepsilon\right)g-\left(\frac{13}{12}-\frac{\sigma}{2}\right)A} + q^{\frac{3g}{2}-A(2-\sigma)+\varepsilon g},$$

for $7/6 \leq \sigma < 4/3$.

We finish the section by sieving out the values of F_1 and F_2.

Lemma 5.4. *For f a monic polynomial in $\mathbb{F}_q[T]$ the following holds.*

$$\sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} \chi_f(F_1)\overline{\chi_f}(F_2) = \sum_{\substack{D_1\in\mathcal{M}_{q,\le d_1/2}\\ D_2\in\mathcal{M}_{q\le d_2/2}}} \mu(D_1)\mu(D_2)\chi_f(D_1^2D_2)$$
$$\times \sum_{\substack{\deg(H)\le\min\{d_1-\deg(D_1),d_2-\deg(D_2)\}\\ \deg(H)-\deg(D_1,H)\le d_1-2\deg(D_1)\\ \deg(H)-\deg(D_2,H)\le d_2-2\deg(D_2)\\ (H,f)=1}} \mu(H)\chi_f((D_1,H)^2(D_2,H))$$
$$\times \sum_{\substack{L_1\in\mathcal{M}_{q,d_1-2\deg(D_1)-\deg(H)+\deg(D_1,H)}\\ L_2\in\mathcal{M}_{q,d_2-2\deg(D_2)-\deg(H)+\deg(D_2,H)}}} \chi_f(L_1)\overline{\chi_f}(L_2).$$

Proof. We have that

$$\sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} \chi_f(F_1)\overline{\chi_f}(F_2) = \sum_{\substack{D_1\in\mathcal{M}_{q,\le d_1/2}\\ D_2\in\mathcal{M}_{q,\le d_2/2}}} \mu(D_1)\mu(D_2)\chi_f(D_1^2D_2) \sum_{\substack{F_1'\in\mathcal{M}_{q,d_1-2\deg(D_1)}\\ F_2'\in\mathcal{M}_{q,d_2-2\deg(D_2)}\\ (D_1F_1',D_2F_2')=1}} \chi_f(F_1')\overline{\chi_f}(F_2')$$
$$= \sum_{\substack{D_1\in\mathcal{M}_{q,\le d_1/2}\\ D_2\in\mathcal{M}_{q,\le d_2/2}}} \mu(D_1)\mu(D_2)\chi_f(D_1^2D_2) \sum_{H\in\mathcal{M}_{q,\le\min\{d_1-\deg(D_1),d_2-\deg(D_2)\}}} \mu(H)$$
$$\times \sum_{\substack{F_1'\in\mathcal{M}_{q,d_1-2\deg(D_1)}\\ F_2'\in\mathcal{M}_{q,d_2-2\deg(D_2)}\\ H|(D_1F_1',D_2F_2')}} \chi_f(F_1')\overline{\chi_f}(F_2').$$

We remark that $H \mid (D_1F_1', D_2F_2')$ is equivalent to $H_1 = \frac{H}{(D_1,H)} \mid F_1'$ and $H_2 = \frac{H}{(D_2,H)} \mid F_2'$. This gives

$$\sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} \chi_f(F_1)\overline{\chi_f}(F_2) = \sum_{\substack{D_1\in\mathcal{M}_{q,\le d_1/2}\\ D_2\in\mathcal{M}_{q,\le d_2/2}}} \mu(D_1)\mu(D_2)\chi_f(D_1^2D_2)$$
$$\times \sum_{\substack{\deg(H)\le\min\{d_1-\deg(D_1),d_2-\deg(D_2)\}\\ \deg(H_1)\le d_1-2\deg(D_1)\\ \deg(H_2)\le d_2-2\deg(D_2)}} \mu(H)\chi_f(H_1H_2^2) \sum_{\substack{F_1''\in\mathcal{M}_{q,d_1-2\deg(D_1)-\deg(H_1)}\\ F_2''\in\mathcal{M}_{q,d_2-2\deg(D_2)-\deg(H_2)}}} \chi_f(F_1'')\overline{\chi_f}(F_2'').$$

∎

We rewrite Lemma 5.4 in the following form.

Corollary 5.5. *For f a monic polynomial in $\mathbb{F}_q[T]$ the following holds.*

$$\sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} \chi_f(F_1)\overline{\chi_f}(F_2) = \sum_{\substack{H\in\mathcal{M}_{q,\le\min\{d_1,d_2\}}\\ (H,f)=1}} \mu(H) \sum_{\substack{R_1\in\mathcal{M}_{q,\le d_1-\deg(H)}\\ R_1\mid H}} \mu(R_1)\chi_f(R_1)$$
$$\times \sum_{\substack{R_2\in\mathcal{M}_{q,\le d_2-\deg(H)}\\ R_2\mid H}} \mu(R_2)\chi_f(R_2)^2 \sum_{\substack{D_1\in\mathcal{M}_{q,\le\frac{d_1-\deg(H)-\deg(R_1)}{2}}\\ (D_1,H)=1}} \mu(D_1)\chi_f(D_1)^2$$
$$\times \sum_{\substack{D_2\in\mathcal{M}_{q,\le\frac{d_2-\deg(H)-\deg(R_2)}{2}}\\ (D_2,H)=1}} \mu(D_2)\chi_f(D_2) \sum_{\substack{L_1\in\mathcal{M}_{q,d_1-2\deg(D_1)-\deg(H)-\deg(R_1)}\\ L_2\in\mathcal{M}_{q,d_2-2\deg(D_2)-\deg(H)-\deg(R_2)}}} \chi_f(L_1)\overline{\chi_f}(L_2).$$

Proof. This follows by taking $R_i = (D_i, H)$ in Lemma 5.4. $\square$

5.2. **The main term.** Here we will obtain an asymptotic formula for the main term (75) by proving Lemma 5.1. Recall that

$$S_{2,\square} = \sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} \sum_{\substack{f\in\mathcal{M}_{q,\le A}\\ f=\square}} \frac{\chi_f(F_1)\overline{\chi_f}(F_2)}{|f|_q^{1/2}}.$$

Let $2g+1 \equiv a \,(\mathrm{mod}\, 3)$ and $g \equiv b \,(\mathrm{mod}\, 3)$ with $a, b \in \{0,1,2\}$. Notice that then $1+2a \equiv b \,(\mathrm{mod}\, 3)$. Recall that $A \equiv 0 \,(\mathrm{mod}\, 3)$. Since $d_1 + d_2 = g+1$ and $d_1 + 2d_2 \equiv 1 \,(\mathrm{mod}\, 3)$, it follows that $d_1 \equiv a \,(\mathrm{mod}\, 3)$. In the equation above, write $f = k^3$. Then the main term $S_{2,\square}$ can be rewritten as

$$S_{2,\square} = \sum_{\substack{d_1+d_2=g+1\\ d_1\equiv a \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} \sum_{\substack{k\in\mathcal{M}_{q,\le\frac{A}{3}}\\ (k,F_1F_2)=1}} \frac{1}{|k|_q^{3/2}}. \tag{77}$$

We consider the generating series

$$\mathcal{C}_{\mathrm{K}}(x,y,u) = \sum_{\substack{F_1,F_2\in\mathcal{H}_q\\ (F_1,F_2)=1}} \sum_{\substack{k\in\mathcal{M}_q\\ (k,F_1F_2)=1}} x^{\deg(F_1)} y^{\deg(F_2)} \frac{u^{\deg(k)}}{|k|_q^{3/2}}. \tag{78}$$

Note that

$$\sum_{\substack{k\in\mathcal{M}_q\\ (k,F_1F_2)=1}} \frac{u^{\deg(k)}}{|k|_q^{3/2}} = \prod_{P\nmid F_1F_2} \left(1 - \frac{u^{\deg(P)}}{|P|_q^{3/2}}\right)^{-1} = \mathcal{Z}_q\left(\frac{u}{q^{3/2}}\right) \prod_{P\mid F_1F_2} \left(1 - \frac{u^{\deg(P)}}{|P|_q^{3/2}}\right). \tag{79}$$

Let $C_{P,K}(u)$ denote the Euler factor above. Now we introduce the sum over F_2 and we have that

$$\sum_{\substack{F_2\in\mathcal{H}_q\\ (F_2,F_1)=1}} y^{\deg(F_2)} \prod_{P\mid F_2} C_{P,K}(u) = \prod_{P} \left(1 + y^{\deg(P)} C_{P,K}(u)\right) \prod_{P\mid F_1} \left(1 + y^{\deg(P)} C_{P,K}(u)\right)^{-1}. \tag{80}$$

Let $B_{P,K}(y,u)$ be the P-factor when $P \mid F_1$. Finally, introducing the sum over F_1 and combining equations (79) and (80), we have that

$$\sum_{F_1 \in \mathcal{H}_q} x^{\deg(F_1)} \prod_{P | F_1} C_{P,K}(u) B_{P,K}(y,u) = \prod_{P} \left(1 + x^{\deg(P)} C_{P,K}(u) B_{P,K}(y,u)\right). \tag{81}$$

Combining equations (78), (79), (80) and (81) and simplifying, we get that

$$\begin{aligned} \mathcal{C}_{\mathrm{K}}(x,y,u) =& \mathcal{Z}_q\left(\frac{u}{q^{3/2}}\right) \prod_P \left(1 + (x^{\deg(P)} + y^{\deg(P)}) \left(1 - \frac{u^{\deg(P)}}{|P|_q^{3/2}}\right)\right) \\ =& \mathcal{Z}_q\left(\frac{u}{q^{3/2}}\right) \mathcal{Z}_q(x)\mathcal{Z}_q(y)\mathcal{D}_{\mathrm{K}}(x,y,u), \end{aligned} \tag{82}$$

where

$$\begin{aligned} \mathcal{D}_{\mathrm{K}}(x,y,u) = \prod_P \Big(& 1 - x^{2\deg(P)} - y^{2\deg(P)} - (xy)^{\deg(P)} + (x^2y)^{\deg(P)} + (y^2x)^{\deg(P)} \\ & - \frac{(ux)^{\deg(P)}}{|P|_q^{3/2}} - \frac{(uy)^{\deg(P)}}{|P|_q^{3/2}} + \frac{(x^2u)^{\deg(P)}}{|P|_q^{3/2}} + \frac{(y^2u)^{\deg(P)}}{|P|_q^{3/2}} + \frac{2(xyu)^{\deg(P)}}{|P|_q^{3/2}} \\ & - \frac{(x^2yu)^{\deg(P)}}{|P|_q^{3/2}} - \frac{(y^2xu)^{\deg(P)}}{|P|_q^{3/2}} \Big). \end{aligned} \tag{83}$$

Note that $\mathcal{D}_{\mathrm{K}}(x,y,u)$ has an analytic continuation when $|x| < 1, |y| < 1, |x^2y| < \frac{1}{q}, |y^2x| < \frac{1}{q}, |xu| < q^{3/2}, |yu| < q^{3/2}, |x^2u| < \sqrt{q}, |y^2u| < \sqrt{q}, |xyu| < \sqrt{q}$. Using equation (82) and Perron's formula (Lemma 2.1) three times in equation (77), we get that

$$S_{2,\square} = \sum_{\substack{d_1+d_2=g+1 \\ d_1 \equiv a \,(\mathrm{mod}\, 3)}} \frac{1}{(2\pi i)^3} \oint\oint\oint \frac{\mathcal{D}_{\mathrm{K}}(x,y,u)}{(1-\frac{u}{\sqrt{q}})(1-qx)(1-qy)(1-u)u^{A/3}y^{d_2}x^{d_1}} \frac{du}{u}\frac{dy}{y}\frac{dx}{x},$$

where we initially integrate along circles around the origin of radii $|u| = \frac{1}{q^{\varepsilon}}, |x| = |y| = \frac{1}{q^{1+\varepsilon}}$.

We first shift the contour over u to $|u| = q^{5/2}$, and encounter two poles: one at $u = 1$ and another at $u = \sqrt{q}$. We compute the residues of the poles and then

$$\begin{aligned} \oint_{|u|=\frac{1}{q^{\varepsilon}}} \frac{\mathcal{D}_{\mathrm{K}}(x,y,u)}{(1-\frac{u}{\sqrt{q}})(1-u)u^{A/3}} \frac{du}{u} =& \zeta_q(3/2)\mathcal{D}_{\mathrm{K}}(x,y,1) + q^{-\frac{A}{6}}\zeta_q(1/2)\mathcal{D}_{\mathrm{K}}(x,y,\sqrt{q}) \\ &+ \oint_{|u|=q^{5/2}} \frac{\mathcal{D}_{\mathrm{K}}(x,y,u)}{(1-\frac{u}{\sqrt{q}})(1-u)u^{A/3}} \frac{du}{u}. \end{aligned}$$

Plugging this into the expression for $S_{2,\square}$ and bounding the new triple integral by $q^{g-\frac{5A}{6}+\varepsilon g}$ give

$$
\begin{aligned}
(84)\quad S_{2,\square} =& \zeta_q(3/2) \sum_{\substack{d_1+d_2=g+1\\ d_1\equiv a \,(\mathrm{mod}\, 3)}} \frac{1}{(2\pi i)^2} \oint_{|x|=\frac{1}{q^{1+\varepsilon}}} \oint_{|y|=\frac{1}{q^{1+\varepsilon}}} \frac{\mathcal{D}_{\mathrm{K}}(x,y,1)}{(1-qx)(1-qy)y^{d_2}x^{d_1}} \frac{dy}{y}\frac{dx}{x}\\
(85)\quad & + q^{-\frac{A}{6}}\zeta_q(1/2) \sum_{\substack{d_1+d_2=g+1\\ d_1\equiv a \,(\mathrm{mod}\, 3)}} \frac{1}{(2\pi i)^2} \oint_{|x|=\frac{1}{q^{1+\varepsilon}}} \oint_{|y|=\frac{1}{q^{1+\varepsilon}}} \frac{\mathcal{D}_{\mathrm{K}}(x,y,\sqrt{q})}{(1-qx)(1-qy)y^{d_2}x^{d_1}} \frac{dy}{y}\frac{dx}{x}\\
& + O(q^{g-\frac{5A}{6}+\varepsilon g}).
\end{aligned}
$$

We first focus on the first term (84). Note that $\mathcal{D}_{\mathrm{K}}(x,y,1)$ has an analytic continuation for $|x|<1, |y|<1, |x^2y|<\frac{1}{q}, |y^2x|<\frac{1}{q}$.

We remark that in (84) we can shift the contours of integration to the smaller circles $|x|=q^{-3}$ and $|y|=q^{-2}$ without changing the value of the integral as we are not crossing any pole.

We write $d_1=3k+a$ and compute the sum over d_1. Note that $k\le[(g+1-a)/3]=[g/3]$. Then

$$
(84) = \zeta_q(3/2)\frac{1}{(2\pi i)^2}\oint_{|x|=q^{-3}}\oint_{|y|=q^{-2}} \frac{\mathcal{D}_{\mathrm{K}}(x,y,1)}{(1-qx)(1-qy)(y^3-x^3)}\left[\frac{y^{2+a-b}}{x^{g+a-b}}-\frac{x^{3-a}}{y^{g+1-a}}\right]\frac{dy}{y}\frac{dx}{x}.
$$

We write the integral above as a difference of two integrals. Note that the second double integral vanishes, because the integrand for the integral over x has no poles inside the circle $|x|=q^{-3}$.

Hence

$$
(84) = \zeta_q(3/2)\frac{1}{(2\pi i)^2}\oint_{|x|=q^{-3}}\oint_{|y|=q^{-2}} \frac{\mathcal{D}_{\mathrm{K}}(x,y,1)y^{2+a-b}}{(1-qx)(1-qy)(y^3-x^3)x^{g+a-b}}\frac{dy}{y}\frac{dx}{x}.
$$

Note that for the integral over y, the only poles of the integrand inside the circle $|y|=q^{-2}$ are at $y^3=x^3$, so when $y=x\xi_3^i$ for $i\in\{0,1,2\}$ and $\xi_3=e^{2\pi i/3}$. Hence

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{|y|=q^{-2}} \frac{\mathcal{D}_{\mathrm{K}}(x,y,1)y^{2+a-b}}{x^{g+a-b}(1-qy)(y^3-x^3)}\frac{dy}{y} =& \frac{1}{3x^{g+1}}\left[\frac{\mathcal{D}_{\mathrm{K}}(x,x,1)}{1-qx}+\frac{\mathcal{D}_{\mathrm{K}}(x,\xi_3 x,1)\xi_3^{2+a-b}}{1-q\xi_3 x}\right.\\
&\left.+\frac{\mathcal{D}_{\mathrm{K}}(x,\xi_3^2 x,1)\xi_3^{2(2+a-b)}}{1-q\xi_3^2 x}\right].
\end{aligned}
$$

To compute the integral over x, we shift the the contour of integration to $|x|=q^{-1/3+\varepsilon}$, evaluating the residues at $x=q^{-1}$ corresponding to each of the three functions above.

Notice that the first integral has a double pole at $s = 1/q$. This gives

$$\begin{aligned}(84) =&\zeta_q(3/2)\frac{1}{2\pi i}\oint_{|x|=q^{-3}} \frac{1}{3(1-qx)x^{g+1}}\left[\frac{\mathcal{D}_{\mathrm{K}}(x,x,1)}{1-qx}+\frac{\mathcal{D}_{\mathrm{K}}(x,\xi_3 x,1)\xi_3^{2+a-b}}{1-q\xi_3 x}\right.\\ &\left.+\frac{\mathcal{D}_{\mathrm{K}}(x,\xi_3^2 x,1)\xi_3^{2(2+a-b)}}{1-q\xi_3^2 x}\right]\frac{dx}{x}\\ =&\frac{\zeta_q(3/2)}{3}\left[(g+2)q^{g+1}\mathcal{D}_{\mathrm{K}}(\tfrac{1}{q},\tfrac{1}{q},1)-q^g\frac{d}{dx}\mathcal{D}_{\mathrm{K}}(x,x,1)|_{x=1/q}\right.\\ &+q^{g+1}\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{\xi_3}{q},1)\xi_3^{1+2a}}{1-\xi_3}+q^{g+1}\frac{\mathcal{D}_{\mathrm{K}}(\frac{\xi_3^2}{q},\frac{1}{q},1)\xi_3^{a}}{1-\xi_3^2}\\ &\left.+q^{g+1}\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{\xi_3^2}{q},1)\xi_3^{2+a}}{1-\xi_3^2}+q^{g+1}\frac{\mathcal{D}_{\mathrm{K}}(\frac{\xi_3}{q},\frac{1}{q},1)\xi_3^{2a}}{1-\xi_3}\right]+O(q^{\frac{g}{3}+\varepsilon g}),\end{aligned}$$

where we have used the fact that $1 + 2a \equiv b \pmod 3$.

Since $\mathcal{D}_{\mathrm{K}}(x, y, 1) = \mathcal{D}_{\mathrm{K}}(y, x, 1)$, we further simplify (84) to

$$\begin{aligned}(84) =&\zeta_q(3/2)\frac{q^{g+1}}{3}\left[(g+2)\mathcal{D}_{\mathrm{K}}(\tfrac{1}{q},\tfrac{1}{q},1)-\frac{1}{q}\frac{d}{dx}\mathcal{D}_{\mathrm{K}}(x,x,1)|_{x=1/q}\right.\\ &\left.-\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{\xi_3}{q},1)\xi_3^{2a+2}}{1-\xi_3}-\frac{\mathcal{D}_{\mathrm{K}}(\frac{\xi_3^2}{q},\frac{1}{q},1)\xi_3^{a+1}}{1-\xi_3^2}\right]+O(q^{\frac{g}{3}+\varepsilon g})\\ =&C_{\mathrm{K},1}gq^{g+1}+C_{\mathrm{K},2}q^{g+1}+O(q^{\frac{g}{3}+\varepsilon g}),\end{aligned}$$

where

(86)

$$C_{\mathrm{K},1} =\zeta_q(3/2)\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{1}{q},1)}{3},$$

(87)

$$C_{\mathrm{K},2} =\zeta_q(3/2)\left[\frac{2\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{1}{q},1)}{3}-\frac{1}{3q}\frac{d}{dx}\mathcal{D}_{\mathrm{K}}(x,x,1)|_{x=1/q}-\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{\xi_3}{q},1)\xi_3^{g+1}}{3(1-\xi_3)}-\frac{\mathcal{D}_{\mathrm{K}}(\frac{\xi_3^2}{q},\frac{1}{q},1)\xi_3^{2g+2}}{3(1-\xi_3^2)}\right],$$

and where we used the fact that $2g + 1 \equiv a \pmod 3$. We remark that the constants above are real, which reflects the fact that the sum is a real number.

We similarly compute the term (85) and we get that

$$\begin{aligned}(85) =&\frac{\zeta_q(1/2)q^{g+1-\frac{A}{6}}}{3}\left[(g+2)\mathcal{D}_{\mathrm{K}}(\tfrac{1}{q},\tfrac{1}{q},\sqrt{q})-\frac{1}{q}\frac{d}{dx}\mathcal{D}_{\mathrm{K}}(x,x,\sqrt{q})|_{x=1/q}-\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{\xi_3}{q},\sqrt{q})\xi_3^{2a+2}}{1-\xi_3}\right.\\ &\left.-\frac{\mathcal{D}_{\mathrm{K}}(\frac{\xi_3^2}{q},\frac{1}{q},\sqrt{q})\xi_3^{a+1}}{1-\xi_3^2}\right]+O(q^{\frac{g}{3}-\frac{A}{6}+\varepsilon g}).\end{aligned}$$

Putting everything together, we get that

$$
\begin{aligned}
S_{2,\square} ={}& \zeta_q(3/2)\frac{q^{g+1}}{3}\Bigg[(g+2)\mathcal{D}_{\mathrm{K}}(\tfrac{1}{q},\tfrac{1}{q},1)-\frac{1}{q}\frac{d}{dx}\mathcal{D}_{\mathrm{K}}(x,x,1)|_{x=1/q}-\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{\xi_3}{q},1)\xi_3^{g+1}}{1-\xi_3}\\
&-\frac{\mathcal{D}_{\mathrm{K}}(\frac{\xi_3^2}{q},\frac{1}{q},1)\xi_3^{2g+2}}{1-\xi_3^2}\Bigg]+\frac{\zeta_q(1/2)q^{g+1-\frac{A}{6}}}{3}\Bigg[(g+2)\mathcal{D}_{\mathrm{K}}(\tfrac{1}{q},\tfrac{1}{q},\sqrt{q})-\frac{1}{q}\frac{d}{dx}\mathcal{D}_{\mathrm{K}}(x,x,\sqrt{q})|_{x=1/q}\\
&-\frac{\mathcal{D}_{\mathrm{K}}(\frac{1}{q},\frac{\xi_3}{q},\sqrt{q})\xi_3^{g+1}}{1-\xi_3}-\frac{\mathcal{D}_{\mathrm{K}}(\frac{\xi_3^2}{q},\frac{1}{q},\sqrt{q})\xi_3^{2g+2}}{1-\xi_3^2}\Bigg]+O\Big(q^{\frac{g}{3}+\varepsilon g}\Big)+O\Big(q^{g-\frac{5A}{6}+\varepsilon g}\Big).
\end{aligned}
\tag{88}
$$

5.3. **The contribution from non-cubes.** Here we will prove Lemma 5.2. Recall the definition (76) of $S_{2,\neq\square}$, the term coming from the contribution of non-cube polynomials.

Using the sieve of Corollary 5.5, we rewrite $S_{2,\neq\square}$ as

$$
\begin{aligned}
S_{2,\neq\square} ={}& \sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}}\sum_{\substack{f\in\mathcal{M}_{q,\leq A}\\ f\neq\square}}\frac{1}{|f|_q^{1/2}}\sum_{\substack{H\in\mathcal{M}_{q,\leq\min\{d_1,d_2\}}\\ (H,f)=1}}\mu(H)\sum_{\substack{R_1|H\\ \deg(R_1)\leq d_1-\deg(H)}}\mu(R_1)\chi_f(R_1)\\
&\times\sum_{\substack{R_2|H\\ \deg(R_2)\leq d_2-\deg(H)}}\mu(R_2)\chi_f(R_2)^2\sum_{\substack{D_1\in\mathcal{M}_{q,\leq\frac{d_1-\deg(H)-\deg(R_1)}{2}}\\ (D_1,R_1)=1}}\mu(D_1)\chi_f(D_1)^2\\
&\times\sum_{\substack{D_2\in\mathcal{M}_{q,\leq\frac{d_2-\deg(H)-\deg(R_2)}{2}}\\ (D_2,R_2)=1}}\mu(D_2)\chi_f(D_2)\sum_{\substack{L_1\in\mathcal{M}_{q,d_1-2\deg(D_1)-\deg(H)-\deg(R_1)}\\ L_2\in\mathcal{M}_{q,d_2-2\deg(D_2)-\deg(H)-\deg(R_2)}}}\chi_f(L_1)\overline{\chi_f}(L_2).
\end{aligned}
$$

Using Perron's formula for the sums over L_1 and L_2, we have that

$$\sum_{L_1\in\mathcal{M}_{q,d_1-2\deg(D_1)-\deg(H)-\deg(R_1)}}\chi_f(L_1)=\frac{1}{2\pi i}\oint_{|u|=q^{-1/2}}\frac{\mathcal{L}(u,\chi_f)}{u^{d_1-2\deg(D_1)-\deg(H)-\deg(R_1)}}\frac{du}{u}$$

and

$$\sum_{L_2\in\mathcal{M}_{q,d_2-2\deg(D_2)-\deg(H)-\deg(R_2)}}\overline{\chi_f}(L_2)=\frac{1}{2\pi i}\oint_{|u|=q^{-1/2}}\frac{\mathcal{L}(u,\overline{\chi_f})}{u^{d_2-2\deg(D_2)-\deg(H)-\deg(R_2)}}\frac{du}{u}.$$

Note that since f is not a cube, the numerators in the expressions above have no poles, so we can integrate over the circle of radius $|u|=q^{-1/2}$. Using the Lindelöf hypothesis (Lemma 2.5) for $\mathcal{L}(u,\chi_f)$ we have that

$$\sum_{L_1\in\mathcal{M}_{q,d_1-2\deg(D_1)-\deg(H)-\deg(R_1)}}\chi_f(L_1)\ll|f|_q^{\varepsilon}\frac{q^{d_1/2}}{|D_1|_q\sqrt{|HR_1|_q}},$$

and

$$\sum_{L_2\in\mathcal{M}_{q,d_2-2\deg(D_2)-\deg(H)-\deg(R_2)}}\overline{\chi_f}(L_2)\ll|f|_q^{\varepsilon}\frac{q^{d_2/2}}{|D_2|_q\sqrt{|HR_2|_q}}.$$

Now the sums over $D_1$and D_2 are both bounded by $q^{\varepsilon g}$, while the sums over R_1 and R_2 are both bounded by $\tau(H)$, where $\tau(H)$ denotes the divisor function. Introducing the sum over f, we trivially bound it by $q^{\frac{A}{2}+\varepsilon g}$, obtaining that

$$S_{2,\neq\square} \ll q^{\frac{A+g}{2}+\varepsilon g}.$$

5.4. **The dual term.** We now treat the dual term by proving Lemma 5.3. Recall from equation (74) that

$$S_{2,\mathrm{dual}} = \sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} \omega(\chi_{F_1}\overline{\chi_{F_2}}) \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{\overline{\chi_f}(F_1)\chi_f(F_2)}{|f|_q^{1/2}}.$$

Since $d_1+2d_2 \equiv 1 \,(\mathrm{mod}\, 3)$, by Corollary 2.3 and formula (5), the sign of the functional equation is

$$\begin{aligned}\omega(\chi_{F_1}\overline{\chi_{F_2}}) =&\overline{\epsilon(\chi_3)}q^{-(d_1+d_2)/2}G_q(\chi_{F_1}\overline{\chi_{F_2}})\\ =&\overline{\epsilon(\chi_3)}q^{-(d_1+d_2)/2}G_q(1,F_1)\overline{G_q(1,F_2)},\end{aligned}$$

where χ_3 is defined by (3). We rewrite the dual sum as

(89)
$$S_{2,\mathrm{dual}} =\overline{\epsilon(\chi_3)}q^{-(g+1)/2} \sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{1}{|f|_q^{1/2}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=(F_1F_2,f)=1}} G_q(f,F_1)\overline{G_q(f,F_2)},$$

where we have used the fact that

$$\overline{\chi_f}(F_1)G_q(1,F_1) = \begin{cases}\overline{\chi_{F_1}}(f)G_q(1,F_1) = G_q(f,F_1) & (f,F_1)=1,\\ 0 & \text{otherwise,}\end{cases}$$

and similarly for F_2. We first notice that if F_1 or F_2 are not square-free, then since $(F_1F_2,f) = 1$, we have by Lemma 2.12 that $G_q(f,F_1)=0$ or $G_q(f,F_2)=0$. Therefore, we can write

$$\begin{aligned}\sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=(F_1F_2,f)=1}} G_q(f,F_1)\overline{G_q(f,F_2)} &= \sum_{\substack{F_1\in\mathcal{M}_{q,d_1}\\ F_2\in\mathcal{M}_{q,d_2}\\ (F_1,F_2)=(F_1F_2,f)=1}} G_q(f,F_1)\overline{G_q(f,F_2)}\\ &= \sum_{\substack{F_1\in\mathcal{M}_{q,d_1}\\ F_2\in\mathcal{M}_{q,d_2}\\ (F_1F_2,f)=1}} \sum_{H|(F_1,F_2)} \mu(H)G_q(f,F_1)\overline{G_q(f,F_2)}\\ &= \sum_{\deg(H)\leq\min(d_1,d_2)} \mu(H) \sum_{\substack{F_1\in\mathcal{M}_{q,d_1-\deg(H)}\\ F_2\in\mathcal{M}_{q,d_2-\deg(H)}\\ (HF_1F_2,f)=1}} G_q(f,HF_1)\overline{G_q(f,HF_2)}.\end{aligned}$$

Again, if $(H,F_1)\neq 1$ or $(H,F_2)\neq 1$, then $G_q(f,HF_1)=0$ or $G_q(f,HF_2)=0$. If $(H,F_1F_2)=1$, we can apply Lemma 2.12 and write

$$\sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=(F_1F_2,f)=1}} G_q(f,F_1)\overline{G_q(f,F_2)} = \sum_{\substack{\deg(H)\leq \min(d_1,d_2)\\ (H,f)=1}} \mu(H)|H|_q \times \sum_{\substack{F_1\in\mathcal{M}_{q,d_1-\deg(H)}\\ (F_1,f)=1\\ (F_1,H)=1}} G_q(fH,F_1) \sum_{\substack{F_2\in\mathcal{M}_{q,d_2-\deg(H)}\\ (F_2,f)=1\\ (F_2,H)=1}} \overline{G_q(fH,F_2)}, \tag{90}$$

where we have used the fact that $G_q(f,H)\overline{G_q(f,H)}=|H|_q$. Using equation (89) it follows that

$$S_{2,\mathrm{dual}} =\overline{\epsilon(\chi_3)}q^{-(g+1)/2} \sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1\,(\mathrm{mod}\,3)}} \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{1}{|f|_q^{1/2}} \sum_{\substack{\deg(H)\leq \min(d_1,d_2)\\ (H,f)=1}} \mu(H)|H|_q \times \sum_{\substack{F_1\in\mathcal{M}_{q,d_1-\deg(H)}\\ (F_1,fH)=1}} G_q(fH,F_1) \sum_{\substack{F_2\in\mathcal{M}_{q,d_2-\deg(H)}\\ (F_2,fH)=1}} \overline{G_q(fH,F_2)}. \tag{91}$$

Using Proposition 3.1 we have that

$$\sum_{\substack{F_1\in\mathcal{M}_{q,d_1-\deg(H)}\\ (F_1,fH)=1}} G_q(fH,F_1) = \delta_{f_2=1}\frac{q^{\frac{4}{3}(d_1-\deg(H))-\frac{4}{3}[d_1+\deg(f_1)]_3}}{\zeta_q(2)|f_1H|_q^{\frac{2}{3}}}\overline{G_q(1,f_1H)}\rho(1,[d_1+\deg(f_1)]_3)$$
$$\times\prod_{P|fH}\left(1+\frac{1}{|P|_q}\right)^{-1} + O\left(\delta_{f_2=1}\frac{q^{\frac{d_1}{3}-\frac{\deg(H)}{2}+\varepsilon(d_1-\deg(H))}}{|f_1|_q^{\frac{1}{6}}} + q^{\sigma d_1+\left(\frac{3}{4}-\frac{3}{2}\sigma\right)\deg(H)}|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)}\right),$$

and a similar formula holds for the sum over F_2. Note that the second error term dominates the first error term. Then we have

$$\sum_{\substack{F_1\in\mathcal{M}_{q,d_1-\deg(H)}\\ (F_1,fH)=1}} G_q(fH,F_1) \sum_{\substack{F_2\in\mathcal{M}_{q,d_2-\deg(H)}\\ (F_2,fH)=1}} \overline{G_q(fH,F_2)}$$
$$=\delta_{f_2=1}\frac{q^{\frac{4(g+1)}{3}-3\deg(H)-\frac{4}{3}([d_1+\deg(f_1)]_3+[d_2+\deg(f_1)]_3)}}{\zeta_q(2)^2|f_1|_q^{\frac{1}{3}}}\rho(1,[d_1+\deg(f_1)]_3)\overline{\rho(1,[d_2+\deg(f_1)]_3)}$$
$$\times\prod_{P|fH}\left(1+\frac{1}{|P|_q}\right)^{-2}$$
$$+O\left(\frac{q^{\frac{4d_1}{3}-\frac{3}{2}\deg(H)}}{|f_1|_q^{\frac{1}{6}}}q^{\sigma d_2+\left(\frac{3}{4}-\frac{3}{2}\sigma\right)\deg(H)}|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)}\right)+O\left(\frac{q^{\frac{4d_2}{3}-\frac{3}{2}\deg(H)}}{|f_1|_q^{\frac{1}{6}}}q^{\sigma d_1+\left(\frac{3}{4}-\frac{3}{2}\sigma\right)\deg(H)}|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)}\right) \tag{92}$$
$$+O\left(q^{\sigma d_2+\left(\frac{3}{4}-\frac{3}{2}\sigma\right)\deg(H)}|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)}q^{\sigma d_1+\left(\frac{3}{4}-\frac{3}{2}\sigma\right)\deg(H)}|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)}\right). \tag{93}$$

Then the main term of $S_{2,\mathrm{dual}}$ is equal to

$$M_{\mathrm{dual}} = \overline{\epsilon(\chi_3)} \frac{q^{\frac{5}{6}(g+1)}}{\zeta_q(2)^2} \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2 \equiv 1 \, (\mathrm{mod}\, 3)}} \sum_{f \in \mathcal{M}_{q,\leq g-A-1}} \delta_{f_2=1} \frac{q^{-\frac{4}{3}([d_1+\deg(f_1)]_3+[d_2+\deg(f_1)]_3)}}{|f|_q^{1/2}|f_1|_q^{1/3}}$$
$$\times \rho(1,[d_1+\deg(f_1)]_3)\overline{\rho(1,[d_2+\deg(f_1)]_3)}$$
$$\times \sum_{\substack{\deg(H)\leq \min(d_1,d_2) \\ (H,f)=1}} \frac{\mu(H)}{|H|_q^2} \prod_{P|fH} \left(1+\frac{1}{|P|_q}\right)^{-2}.$$

Notice that the product of the terms involving ρ is nonzero only when $d_1+\deg(f_1) \equiv 1 \, (\mathrm{mod}\, 3)$ (and therefore $d_2+\deg(f_1) \equiv 0 \, (\mathrm{mod}\, 3)$). By Lemma 3.9,

$$M_{\mathrm{dual}} = \frac{q^{\frac{5}{6}g+1}}{\zeta_q(2)^2} \sum_{\substack{d_1+d_2=g+1 \\ d_1+2d_2 \equiv 1 \, (\mathrm{mod}\, 3)}} \sum_{f \in \mathcal{M}_{q,\leq g-A-1}} \delta_{f_2=1} \frac{1}{|f|_q^{1/2}|f_1|_q^{1/3}}$$
$$\times \sum_{\substack{\deg(H)\leq \min(d_1,d_2) \\ (H,f)=1}} \frac{\mu(H)}{|H|_q^2} \prod_{P|fH} \left(1+\frac{1}{|P|_q}\right)^{-2},$$

where we have also used that $\tau(\chi_3)=\epsilon(\chi_3)\sqrt{q}$.

We look at the generating series of the sum over H. We have

$$\sum_{(H,f)=1} \frac{\mu(H)}{|H|_q^2} \prod_{P|H} \left(1+\frac{1}{|P|_q}\right)^{-2} w^{\deg(H)} = \prod_{P\nmid f} \left(1-\frac{w^{\deg(P)}}{(|P|_q+1)^2}\right).$$

Let $R_P(w)$ denote the P–factor above and let $\mathcal{R}_{\mathrm{K}}(w)=\prod_P R_P(w)$. By Perron's formula, we get that

$$\sum_{\substack{\deg(H)\leq \min\{d_1,d_2\} \\ (H,f)=1}} \frac{\mu(H)}{|H|_q^2} \prod_{P|H} \left(1+\frac{1}{|P|_q}\right)^{-2} = \frac{1}{2\pi i} \oint \frac{\mathcal{R}_{\mathrm{K}}(w)\prod_{P|f} R_P(w)^{-1}}{(1-w)w^{\min\{d_1,d_2\}}} \frac{dw}{w}.$$

Recall from Section 5.2 that $d_1 \equiv a \, (\mathrm{mod}\, 3), 2g+1 \equiv a \, (\mathrm{mod}\, 3), g \equiv b \, (\mathrm{mod}\, 3)$ and $A \equiv 0 \, (\mathrm{mod}\, 3)$. Then we need $\deg(f) \equiv b \, (\mathrm{mod}\, 3)$. Now we look at the sum over f. The generating series is

$$\sum_f \delta_{f_2=1} \frac{u^{\deg(f)}}{|f|_q^{1/2}|f_1|_q^{1/3}} \prod_{P|f} \left(1+\frac{1}{|P|_q}\right)^{-2} R_P(w)^{-1}$$
$$= \prod_P \left[1+\frac{1}{R_P(w)(1+\frac{1}{|P|_q})^2}\left(\frac{1}{|P|_q^{1/3}} \sum_{j=0}^{\infty} \frac{u^{(3j+1)\deg(P)}}{|P|_q^{(3j+1)/2}} + \sum_{j=1}^{\infty} \frac{u^{3j\deg(P)}}{|P|_q^{3j/2}}\right)\right]$$
$$= \prod_P \left[1+\frac{1}{R_P(w)(1+\frac{1}{|P|_q})^2} \frac{u^{\deg(P)}(1+\frac{u^{2\deg(P)}}{|P|_q^{2/3}})}{|P|_q^{5/6}(1-\frac{u^{3\deg(P)}}{|P|_q^{3/2}})}\right].$$

Let

$$\mathcal{E}_{\mathrm{K}}(u,w)=\prod_P R_P(w)\left[1+\frac{1}{R_P(w)(1+\frac{1}{|P|_q})^2}\frac{u^{\deg(P)}(1+\frac{u^{2\deg(P)}}{|P|_q^{2/3}})}{|P|_q^{5/6}(1-\frac{u^{3\deg(P)}}{|P|_q^{3/2}})}\right]=\mathcal{Z}_q\left(\frac{u}{q^{5/6}}\right)\mathcal{U}_{\mathrm{K}}(u,w),$$

and

$$\mathcal{U}_{\mathrm{K}}(u,w)=\prod_P\left(1-\frac{u^{\deg(P)}}{|P|_q^{5/6}}\right)\left(1-\frac{w^{\deg(P)}}{(|P|_q+1)^2}+\frac{|P|_q^{7/6}u^{\deg(P)}(1+\frac{u^{2\deg(P)}}{|P|_q^{2/3}})}{(|P|_q+1)^2(1-\frac{u^{3\deg(P)}}{|P|_q^{3/2}})}\right).$$

Write $\deg(f)=3k+b$. Since $\deg(f)\le g-A-1$ and $g-A-1\equiv b-1\,(\mathrm{mod}\,3)$, we have by Perron's formula

$$\begin{aligned}&\mathcal{R}_{\mathrm{K}}(w)\sum_{\substack{f\in\mathcal{M}_{q,\le g-A-1}\\ \deg(f)\equiv b\,(\mathrm{mod}\,3)}}\delta_{f_2=1}\frac{1}{|f_1|_q^{1/3}|f|_q^{1/2}}\prod_{P|f}\left(1+\frac{1}{|P|_q}\right)^{-2}R_P(w)^{-1}\\ =&\frac{1}{2\pi i}\oint\sum_{k=0}^{(g-A-3-b)/3}\frac{1}{u^{3k+b}}\mathcal{E}_{\mathrm{K}}(u,w)\,\frac{du}{u}\\ =&\frac{1}{2\pi i}\oint\frac{\mathcal{E}_{\mathrm{K}}(u,w)}{(1-u^3)u^{g-A-3}}\,\frac{du}{u},\end{aligned}\tag{94}$$

where we are integrating along a small circle around the origin.

Introducing the sum over d_1, we have

$$M_{\mathrm{dual}}=\frac{q^{\frac{5}{6}g+1}}{\zeta_q(2)^2}\frac{1}{(2\pi i)^2}\oint\oint\frac{\mathcal{U}_{\mathrm{K}}(u,w)}{(1-uq^{1/6})(1-u^3)u^{g-A-3}(1-w)}\sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv1\,(\mathrm{mod}\,3)}}w^{-\min\{d_1,d_2\}}\,\frac{dw}{w}\,\frac{du}{u},\tag{95}$$

where the integral is taken over small circles of radii $|u|<q^{-1/6}$ and $|w|<1$. Note that since $d_1\equiv a\,(\mathrm{mod}\,3)$, we have that $d_2\equiv a-1\,(\mathrm{mod}\,3)$. For simplicity of notation, let $\alpha=[a-1]_3$. We rewrite the sum over d_1,d_2 as

$$\sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv1\,(\mathrm{mod}\,3)}}w^{-\min\{d_1,d_2\}}=\sum_{k=0}^{[(g+1-2a)/6]}\frac{1}{w^{3k+a}}+\sum_{k=0}^{[(g-1-2\alpha)/6]}\frac{1}{w^{3k+\alpha}}.$$

Assume that g is odd. We have

$$[(g+1-2a)/6]=\frac{g-1-2a}{6},\ \ [(g-1-2\alpha)/6]=\frac{g-3-2\alpha}{6}.$$

Then using the above in (95) we get that

$$M_{\mathrm{dual}}=\frac{q^{\frac{5}{6}g+1}}{\zeta_q(2)^2}\frac{1}{(2\pi i)^2}\oint\oint\frac{\mathcal{U}_{\mathrm{K}}(u,w)(1+w)}{(1-uq^{1/6})(1-u^3)u^{g-A-3}(1-w)(1-w^3)w^{\frac{g-1}{2}}}\,\frac{dw}{w}\,\frac{du}{u},$$

where the integral is taken over small circles of radii $|u|<q^{-1/6}$ and $|w|<1$. Note that we have a pole at $u=q^{-1/6}$.

We compute the residue at $u = q^{-1/6}$ while moving the integral just before the poles at $u^3 = 1$ and obtain

$$\begin{aligned}
M_{\text{dual}} &= -q^{g-\frac{A}{6}+1}\frac{\zeta_q(1/2)}{\zeta_q(2)^2}\frac{1}{2\pi i}\oint \frac{\mathcal{U}_{\mathrm{K}}(q^{-1/6},w)(1+w)}{(1-w)(1-w^3)w^{\frac{g-1}{2}}}\frac{dw}{w} \\
&\quad + \frac{q^{\frac{5}{6}g+1}}{\zeta_q(2)^2}\frac{1}{(2\pi i)^2}\oint_{|u|=q^{-\varepsilon}}\oint_{|w|=q^{-\varepsilon}} \frac{\mathcal{U}_{\mathrm{K}}(u,w)(1+w)}{(1-uq^{1/6})(1-u^3)u^{g-A-3}(1-w)(1-w^3)w^{\frac{g-1}{2}}}\frac{dw}{w}\frac{du}{u}
\end{aligned}$$

(96)

$$= -q^{g-\frac{A}{6}+1}\frac{\zeta_q(1/2)}{\zeta_q(2)^2}\frac{1}{2\pi i}\oint \frac{\mathcal{U}_{\mathrm{K}}(q^{-1/6},w)(1+w)}{(1-w)(1-w^3)w^{\frac{g-1}{2}}}\frac{dw}{w} + O\left(q^{\frac{5g}{6}+\varepsilon g}\right),$$

where $|w| < 1$.

In the integral above we have a double pole at $w = 1$ and simple poles at $w = \xi_3, w = \xi_3^2$. We have

$$\mathcal{U}_{\mathrm{K}}(q^{-1/6},w) = \prod_P \left(1 - \frac{1}{|P|_q}\right)\left(1 - \frac{w^{\deg(P)}}{(|P|_q+1)^2} + \frac{1}{(|P|_q+1)(1-\frac{1}{|P|_q^2})}\right) := \mathcal{H}_{\mathrm{K}}(w).$$

We compute the residue of the double pole at $w = 1$ and get that it is equal to

$$-\frac{g+2}{3}\mathcal{H}_{\mathrm{K}}(1) + \frac{2\mathcal{H}_{\mathrm{K}}'(1)}{3}.$$

Note that

$$\frac{\mathcal{H}_{\mathrm{K}}(1)}{\zeta_q(2)^2} = \prod_P \frac{(|P|_q^2+2|P|_q-2)(|P|_q-1)^2}{|P|_q^4} = \mathcal{D}_{\mathrm{K}}(1/q,1/q,\sqrt{q}),$$

where recall that $\mathcal{D}_{\mathrm{K}}(x,y,u)$ is defined by (83).

Now we compute the residue of the pole at $w = \xi_3^{-1}$ which is equal to

$$\frac{\mathcal{H}_{\mathrm{K}}(\xi_3^2)(1+\xi_3^2)}{(1-\xi_3^2)^2(1-\xi_3)}\xi_3^{\frac{g-1}{2}} = -\frac{\mathcal{H}_{\mathrm{K}}(\xi_3^2)}{3(1-\xi_3^2)}\xi_3^{2g+2}.$$

The residue at $w = \xi_3$ is equal to

$$\frac{\mathcal{H}_{\mathrm{K}}(\xi_3)(1+\xi_3)}{(1-\xi_3)^2(1-\xi_3^2)}\xi_3^{g-1} = -\frac{\mathcal{H}_{\mathrm{K}}(\xi_3)}{3(1-\xi_3)}\xi_3^{g+1}.$$

Putting everything together, we have

$$\begin{aligned}
M_{\text{dual}} =& q^{g-\frac{A}{6}+1}\frac{\zeta_q(1/2)}{\zeta_q(2)^2}\left(-\frac{g+2}{3}\mathcal{H}_{\mathrm{K}}(1) + \frac{2\mathcal{H}_{\mathrm{K}}'(1)}{3} - \frac{\mathcal{H}_{\mathrm{K}}(\xi_3^2)}{3(1-\xi_3^2)}\xi_3^{2g+2} - \frac{\mathcal{H}_{\mathrm{K}}(\xi_3)}{3(1-\xi_3)}\xi_3^{g+1}\right) \\
&+ q^{g-\frac{A}{6}+1}\frac{\zeta_q(1/2)}{\zeta_q(2)^2}\frac{1}{2\pi i}\oint_{|w|=q^{1-\varepsilon}} \frac{\mathcal{H}_{\mathrm{K}}(w)(1+w)}{(1-w)(1-w^3)w^{\frac{g-1}{2}}}\frac{dw}{w} + O\left(q^{\left(\frac{5}{6}+\varepsilon\right)g}\right) \\
=& q^{g-\frac{A}{6}+1}\frac{\zeta_q(1/2)}{\zeta_q(2)^2}\left(-\frac{g+2}{3}\mathcal{H}_{\mathrm{K}}(1) + \frac{2\mathcal{H}_{\mathrm{K}}'(1)}{3} - \frac{\mathcal{H}_{\mathrm{K}}(\xi_3^2)}{3(1-\xi_3^2)}\xi_3^{2g+2} - \frac{\mathcal{H}_{\mathrm{K}}(\xi_3)}{3(1-\xi_3)}\xi_3^{g+1}\right) \\
&+ O\left(q^{\frac{5g}{6}+\varepsilon g}\right).
\end{aligned}$$

Remark 5.6. As in Remark 4.5, the error term of size $q^{\frac{5g}{6}}$ can be computed explicitly by evaluating the residue when $u^3 = 1$ in (96). The other error terms will eventually dominate the term of size $q^{\frac{5g}{6}}$, so we do not carry out the computation. However, we believe this term will persist in the asymptotic formula.

Now assume that g is even. Then

$$[(g+1-2a)/6] = \frac{g-4-2a}{6}, \quad [(g+1-2\alpha)/6] = \frac{g-2\alpha}{6}.$$

Similarly as before, we get that

$$\begin{aligned} M_{\text{dual}} =& \frac{q^{\frac{5}{6}g+1}}{\zeta_q(2)^2}\frac{1}{(2\pi i)^2}\oint\oint \frac{\mathcal{U}_{\text{K}}(u,w)(1+w^2)}{(1-uq^{1/6})(1-u^3)u^{g-A-3}(1-w)(1-w^3)w^{\frac{g}{2}}}\frac{dw}{w}\frac{du}{u} \\ =& -q^{g-\frac{A}{6}+1}\frac{\zeta_q(1/2)}{\zeta_q(2)^2}\frac{1}{2\pi i}\oint \frac{\mathcal{U}_{\text{K}}(q^{-1/6},w)(1+w^2)}{(1-w)(1-w^3)w^{\frac{g}{2}}}\frac{dw}{w} + O\left(q^{\frac{5g}{6}+\varepsilon g}\right). \end{aligned}$$

Then the residues give

$$\frac{\mathcal{H}_{\text{K}}(\xi_3^2)(1+\xi_3)}{(1-\xi_3^2)^2(1-\xi_3)}\xi_3^{\frac{g}{2}} = -\frac{\mathcal{H}_{\text{K}}(\xi_3^2)}{3(1-\xi_3^2)}\xi_3^{2g+2},$$

and

$$\frac{\mathcal{H}_{\text{K}}(\xi_3)(1+\xi_3^2)}{(1-\xi_3)^2(1-\xi_3^2)}\xi_3^{g} = -\frac{\mathcal{H}_{\text{K}}(\xi_3)}{3(1-\xi_3)}\xi_3^{g+1},$$

so

$$\begin{aligned} M_{\text{dual}} =& q^{g-\frac{A}{6}+1}\frac{\zeta_q(1/2)}{\zeta_q(2)^2}\left(-\frac{g+2}{3}\mathcal{H}_{\text{K}}(1) + \frac{2\mathcal{H}_{\text{K}}'(1)}{3} - \frac{\mathcal{H}_{\text{K}}(\xi_3^2)}{3(1-\xi_3^2)}\xi_3^{2g+2} - \frac{\mathcal{H}_{\text{K}}(\xi_3)}{3(1-\xi_3)}\xi_3^{g+1}\right) \\ &+ O\left(q^{\frac{5g}{6}+\varepsilon g}\right). \end{aligned}$$

We remark that assuming g even leads to the same asymptotic formula as before.

We now bound the mixed terms (92) and (93) in $S_{2,\text{dual}}$. For the terms of the type (92) we have

$$\begin{aligned} &\ll q^{-\frac{g}{2}} \sum_{d_1+d_2=g+1} q^{\frac{4d_1}{3}+\sigma d_2} \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{|f|_q^{\frac{1}{2}(\frac{3}{2}-\sigma)}}{|f|_q^{\frac{1}{2}}|f_1|_q^{\frac{1}{6}}} \sum_{\substack{\deg(H)\leq \min(d_1,d_2)\\(H,f)=1}} q^{\left(-\frac{3}{2}+\frac{3}{4}-\frac{3}{2}\sigma+1\right)\deg(H)} \\ &\ll q^{-\frac{g}{2}} \sum_{d_1+d_2=g+1} q^{\frac{4d_1}{3}+\sigma d_2} \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} \frac{1}{|f|_q^{\frac{\sigma}{2}-\frac{1}{4}}|f_1|_q^{\frac{1}{6}}} \sum_{\substack{\deg(H)\leq \min(d_1,d_2)\\(H,f)=1}} q^{\left(\frac{1}{4}-\frac{3}{2}\sigma\right)\deg(H)}. \end{aligned}$$

Setting $\sigma \geq 5/6$, and bounding trivially the sum over H by $\ll q^{\varepsilon g}$, it follows that these terms are bounded by

$$\ll gq^{\frac{5}{6}g+\left(\frac{13}{12}-\frac{\sigma}{2}\right)(g-A)+\varepsilon g} \ll q^{\left(\frac{23}{12}-\frac{\sigma}{2}+\varepsilon\right)g-\left(\frac{13}{12}-\frac{\sigma}{2}\right)A}.$$

We now bound the error term coming from (93). This term will be bounded by

$$\ll q^{\sigma g-\frac{g}{2}} \sum_{d_1+d_2=g+1} \sum_{f\in\mathcal{M}_{q,\leq g-A-1}} |f|_q^{1-\sigma} \sum_{\substack{\deg(H)\leq \min(d_1,d_2)\\ (H,f)=1}} |H|_q^{1+\frac{3}{2}-3\sigma}$$

$$\ll q^{\sigma g-\frac{g}{2}+\varepsilon g+(g-A)(2-\sigma)} \ll q^{\frac{3g}{2}-A(2-\sigma)+\varepsilon g}$$

as long as $\sigma \geq 7/6$.

Then the error from $S_{2,\mathrm{dual}}$ will be bounded by

$$E_{\mathrm{dual}} \ll q^{\left(\frac{23}{12}-\frac{\sigma}{2}+\varepsilon\right)g-\left(\frac{13}{12}-\frac{\sigma}{2}\right)A} + q^{\frac{3g}{2}-A(2-\sigma)+\varepsilon g}.$$

This finishes the proof of Lemma 5.3.

5.5. **The proof of Theorem 1.2.** Combining Lemmas 5.1, 5.2 and 5.3, it follows that

$$\sum_{\substack{d_1+d_2=g+1\\ d_1+2d_2\equiv 1 \,(\mathrm{mod}\, 3)}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1}\\ F_2\in\mathcal{H}_{q,d_2}\\ (F_1,F_2)=1}} L_q\left(\frac{1}{2},\chi_{F_1F_2^2}\right) = C_{\mathrm{K},1}gq^{g+1} + C_{\mathrm{K},2}q^{g+1} + D_{\mathrm{K},1}gq^{g+1-\frac{A}{6}} + D_{\mathrm{K},2}q^{g+1-\frac{A}{6}}$$

$$+ O\left(q^{\frac{g}{3}+\varepsilon g} + q^{g-\frac{5A}{6}+\varepsilon g} + q^{\frac{A+g}{2}+\varepsilon g} + q^{(1+\varepsilon)g-\frac{A}{6}} + q^{\left(\frac{23}{12}-\frac{\sigma}{2}+\varepsilon\right)g-\left(\frac{13}{12}-\frac{\sigma}{2}\right)A} + q^{\frac{3g}{2}-A(2-\sigma)+\varepsilon g}\right),$$

where $7/6 \leq \sigma < 4/3$. Picking $\sigma = \frac{13-2\sqrt{7}}{6}$ and $A = 3\left[\frac{g(\sqrt{7}-1)}{6}\right]$ (so that $A \equiv 0 \,(\mathrm{mod}\, 3)$) gives a total upper bound of size $q^{g\frac{1+\sqrt{7}}{4}+\varepsilon g}$ and finishes the proof of Theorem 1.2.

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