ON THE VANISHING OF TWISTED L-FUNCTIONS OF ELLIPTIC CURVES

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Abstract. Let E be an elliptic curve over $\mathbb{Q}$ with L-function $L_E(s)$. We use the random matrix model of Katz and Sarnak to develop a heuristic for the frequency of vanishing of the twisted L-functions $L_E(1,\chi)$, as χ runs over the Dirichlet characters of order 3 (cubic twists). The heuristic suggests that the number of cubic twists of conductor less than X for which $L_E(1,\chi)$ vanishes is asymptotic to $b_E X^{1/2} \log^{e_E} X$ for some constants b_E, e_E depending only on E. We also compute explicitely the conjecture of Keating and Snaith about the moments of the special values $L_E(1,\chi)$ in the family of cubic twists. Finally, we present experimental data which is consistent with the conjectures for the moments and for the vanishing in the family of cubic twists of $L_E(s)$.

1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N_E, and let

$$L_E(s) = \prod_{p \nmid N_E} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \prod_{p \mid N_E} \left(1 - \frac{a_p}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1}$$

be the L-function of E. Then, from the work of Wiles, Taylor [30, 28] and Breuil, Conrad, Diamond, Taylor [1], $L_E(s)$ has analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda_E(s) = \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L_E(s) = \omega_E \Lambda_E(2-s)$$

where $-\omega_E = \pm 1$ is the eigenvalue of the Fricke involution. Let χ be a primitive character of conductor $\mathfrak{f}$ coprime to N_E. We can then form the twisted L-function

$$L_E(s,\chi) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}.$$

which also has analytic continuation to the whole complex plane, and satisfies the functional equation

$$\Lambda_E(s,\chi) = \left(\frac{\mathfrak{f}\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L_E(s,\chi) = \frac{\omega_E \chi(N_E) \tau(\chi)^2}{\mathfrak{f}} \Lambda_E(2-s,\overline{\chi}) \tag{2}$$

where $\tau(\chi)$ is the Gauss sum [25, Theorem 3.66].

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In the particular case where χ_d is a quadratic character of discriminant d, the functional equation is

$$\Lambda_E(s,\chi_d) = \left(\frac{|d|\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L_E(s,\chi_d) = \omega_E \chi_d(-N_E) \Lambda_E(2-s,\chi_d). \tag{3}$$

Then, for about half of the discriminants d, $\omega_E \chi_d(-N_E) = -1$ and $L_E(s,\chi_d)$ vanishes at $s=1$. For each quadratic character χ_d, let r_d be the order of vanishing of $L_E(s,\chi_d)$ at $s=1$. Goldfeld conjectured that [9]

$$\sum_{|d|\le X} r_d \sim \frac{1}{2} \sum_{|d|\le X} 1 \qquad \text{as } X \to \infty,$$

where both sums run over quadratic characters of discriminant $|d| \le X$. In particular, Goldfeld's conjecture implies that

$$N_{\ge 2}(X) = \#\left\{|d| \le X \ \text{ such that } \ r_d \ge 2\right\} = o(X).$$

There are lower bounds for $N_{\ge 2}(X)$, first obtained by Gouvêa and Mazur [11], and improved by Stewart and Top [27]. More precisely, $N_{\ge 2}(X) \gg X^{1/2}$ under the Parity Conjecture [11, 27]. See the review article [24] for a more complete account of these results, and for other similar results [23].

In the recent years, a new approach to the understanding of zeroes of L-functions in families emerged from the work of Katz and Sarnak on zeroes of L-functions and random matrix theory [16, 17]. For example, Goldfeld's conjecture is a particular case of their Density Conjecture, inspired by their work over function fields. Using similar ideas, Conrey, Keating, Rubinstein and Snaith [6] predicted a precise asymptotic for $N_{\ge 2}(X)$. Their work is described in more detail in Section 3.

In this paper, we study vanishing of the twisted L-functions $L_E(s,\chi)$ by Dirichlet characters of order 3 (cubic characters). In all the following, χ will be a cubic character of conductor $\mathfrak{f}$. Let E be an elliptic curve over $\mathbb{Q}$, and let

$$N(X) = \#\left\{\text{cubic characters } \chi \text{ of conductor } \mathfrak{f} \le X\right\} \tag{4}$$

$$\mathcal{F}_E = \left\{L_E(s,\chi) \ : \ \chi \text{ is a cubic character}\right\} \tag{5}$$

$$N_E(X) = \#\left\{L_E(s,\chi) \in \mathcal{F}_E \ : \ L_E(1,\chi) = 0 \text{ and } \mathfrak{f} \le X\right\}. \tag{6}$$

What can we say about the asymptotic behavior $N_E(X)$? The situation is different from the case of quadratic twists, as the functional equation (2) now relates $L_E(s,\chi)$ and $L_E(s,\overline{\chi})$ and does not force vanishing of $L_E(1,\chi)$ when the sign of the functional equation is not 1. There is then no reason to predict that the set of cubic characters for which $L_E(1,\chi)$ vanishes has positive density. We also note that in the case of cubic twists, the twisted L-function $L_E(s,\chi)$ is conjecturally related to the points that E acquires over cyclic cubic fields. More precisely, let K be cyclic cubic field, and let $\hat{G}$ be the character group of $\mathrm{Gal}(K/\mathbb{Q})$. Let $L(E/K,s)$ denote the L-function of E seen as an elliptic curve over the field K. Then,

$$L(E/K, s) = \prod_{\chi \in \hat{G}} L_E(s, \chi),$$

i.e. the vanishing of $L_E(s, \chi)$ is related (via the Birch and Swinnerton-Dyer conjecture) to the existence of rational points on $E(K)$.

Kuwata [20] and Fearnley and Kisilevsky [8] have shown that if there is one cubic twist χ such that $L_E(1, \chi)$ vanishes, then there are infinitely many. When E is a curve with rational 3-torsion with some additional conditions, Fearnley and Kisilevsky have shown that $N_E(X) \gg X^{1/2}$.

We give in this paper a heuristic, based on the connection between zeroes of L-functions in families and random matrix theory introduced by Katz and Sarnak, to predict the asymptotic behavior of $N_E(X)$. As in [6], we use the ideas of Keating and Snaith [18, 19] to predict the value distribution at the central critical point of the L-functions in our families. Similar heuristics can be developed for families of higher order twists, and this work is presently in progress [7].

We would like to emphasize that the cubic twists we discuss in this paper refer to the L-functions of elliptic curves over $\mathbb{Q}$ twisted by cubic Dirichlet characters. These are different from the L-functions arising from the family of (complex multiplication) elliptic curves $x^3 + y^3 = m$. Those curves are isomorphic to the elliptic curve $x^3+y^3 = 1$ by an isomorphism of order three, and are also called cubic twists. That family was studied by Zagier and Kramarz [31] who obtained some numerical data suggesting that a positive proportion of those curves have rank two or more. The numerical data for this family was extended recently by Watkins [29], suggesting that it is more likely that the proportion goes to zero. Watkins also shows that random matrix theory predicts that the number of curves in the family $x^3 + y^3 = m$ with even non-zero rank has density zero, following the ideas of [6] and the present paper.

The structure of the paper is as follows. The second section presents a discretisation of the special values $L_E(1, \chi)$. The third section reviews the work of Keating and Snaith, which suggests that the value distribution of the L-functions at the critical point is related to the value distribution of characteristic polynomials of random matrices. This leads to a random matrix conjecture for the asymptotic behavior of $N_E(X)$. In the fourth section, we write a precise conjecture for the integral moments of $L_E(1, \chi)$ in our family, following from the work of Keating and Snaith. We compute explicitly the arithmetic constant for the family. The conjecture can then be tested numerically, providing support for the random matrix models of the L-functions $L_E(1, \chi)$ in the family of cubic twists. The fifth section contains asymptotics for $N(X)$ and related sums which are needed in the rest of the paper. Finally, the last section presents some experimental results.

2. Discretisation of the special values

Following Mazur, Tate and Teitelbaum [21], we define the algebraic part of $L_E(1,\chi)$ to be

$$L_E^{alg}(1,\chi) = \frac{2\,\mathfrak{f}\,L_E(1,\chi)}{\Omega\,\tau(\chi)} = \sum_{a \mod \mathfrak{f}} \overline{\chi}(a)\Lambda(a,\mathfrak{f}) \tag{7}$$

where $\Lambda(a,\mathfrak{f}) \in \mathbb{Z}$ and Ω is a non-zero rational multiple of the real period Ω_E. Then, $L_E^{alg}(1,\chi)$ is an algebraic integer in $\mathbb{Z}[\rho]$ where ρ is a third root of unity. In fact, we have

Theorem 2.1.

$$|L_E^{alg}(1,\chi)| = \begin{cases} n_\chi & \text{if } \omega_E = 1; \\ \sqrt{3}\, n_\chi & \text{if } \omega_E = -1; \end{cases}$$

for some non-negative integer n_χ.

Proof: As E is defined over $\mathbb{Q}$, we have that $\overline{L_E(1,\chi)} = L_E(1,\overline{\chi})$. Also, as χ is a cubic character, $\chi(-1) = 1$ and $\overline{\tau(\chi)} = \chi(-1)\tau(\overline{\chi}) = \tau(\overline{\chi})$. From (7), this gives $\overline{L_E^{alg}(1,\chi)} = L_E^{alg}(1,\overline{\chi})$. Now, using the functional equation

$$\begin{aligned} L_E^{alg}(1,\chi) &= \frac{2\,\mathfrak{f}\,L_E(1,\chi)}{\Omega\,\tau(\chi)} \\ &= \frac{2\,\omega_E\,\chi(N_E)\,\tau(\chi)}{\Omega} L_E(1,\overline{\chi}) \\ &= \omega_E\,\chi(N_E)\,\overline{L_E^{alg}(1,\chi)} \\ &= \zeta_\chi\,\overline{L_E^{alg}(1,\chi)} \qquad \text{with } \zeta_\chi = \omega_E\chi(N_E). \end{aligned}$$

Then, $L_E^{alg}(1,\chi)$ satisfies an equation

$$\lambda = \zeta_\chi \overline{\lambda} \tag{8}$$

for $\zeta_\chi \in \mathbb{C}^*$. It is easy to see that any two solutions λ_1, λ_2 of such an equation satisfy $\lambda_1 = \alpha\lambda_2$ with α real. Suppose that $\omega_E = 1$, which implies that ζ_χ is a third root of unity. If $\zeta_\chi = 1$, then $L_E^{alg}(1,\chi)$ is real, and as $L_E^{alg}(1,\chi) \in \mathbb{Z}[\rho]$, we must have $L_E^{alg}(1,\chi) \in \mathbb{Z}$. If ζ_χ is a primitive third root of unity, then $\lambda = \zeta_\chi^2$ satisfies (8) and we have $L_E^{alg}(1,\chi) = \alpha\zeta_\chi^2$ with α real. As $L_E^{alg}(1,\chi) \in \mathbb{Z}[\rho]$, we must have $\alpha \in \mathbb{Z}$. Suppose that $\omega_E = -1$. If $\zeta_\chi = -1$, then $\lambda = \sqrt{-3}$ satisfy (8) and we have $L_E^{alg}(1,\chi) = \alpha\sqrt{-3}$ with α real. As $L_E^{alg}(1,\chi) \in \mathbb{Z}[\rho]$, we must have $\alpha \in \mathbb{Z}$. If ζ_χ is a primitive sixth root of unity, then $\lambda = (\zeta_\chi - \overline{\zeta_\chi})\zeta_\chi^2$ satisfies (8) and we have $L_E^{alg}(1,\chi) = \alpha(\zeta_\chi - \overline{\zeta_\chi})\zeta_\chi^2$ with α real. As $L_E^{alg}(1,\chi) \in \mathbb{Z}[\rho]$, we must have $\alpha \in \mathbb{Z}$. □

As $L_E(1,\chi)$ vanishes if and only if the integer n_χ vanishes, this gives a discretisation on the special values $L_E(1,\chi)$. One should mention that the distribution of the integers n_χ is very interesting. For example, the experimental data suggests that there are infinitely many cubic characters χ for which $n_\chi = 1$ (see Table 5). This seems to be very difficult to prove. We also submit the following conjecture, obtained in part by observation of the experimental data, and in part by analogy with the genus theory of number fields.

Conjecture 2.2. *Suppose that E is isogenous to a curve with a rational 3-torsion point. For any positive integer n, let $\nu(n)$ be the number of distinct prime divisors of n. Let χ be a cubic character of conductor $\mathfrak{f}$, and let n_χ be the integer defined by Theorem 2.1. Then*

$$3^{\nu(\mathfrak{f})-1} \mid n_\chi.$$

In order to obtain a heuristic for the vanishing in the family $\mathcal{F}_E$, we have to make some assumptions on the distribution of the integers n_χ. From the above conjecture, it seems that we should distinguish between the cases where E has rational 3-torsion or not. This distinction is also suggested by the work of Fearnley and Kisilevsky discussed in the introduction, and fits the experimental data as we will see in Section 6.

3. Random Matrix Theory

Let $G(N)$ be one of the classical compact irreducible symmetric spaces. For each $A \in G(N)$, let $\lambda_1 = e^{i\theta_1}, \dots, \lambda_N = e^{i\theta_N}$ be the eigenvalues of A which are ordered by the eigenangles $\theta_1, \dots, \theta_N$ such that

$$0 \le \theta_1 \le \cdots \le \theta_N < 2\pi.$$

Let $\mathcal{F} = \{L_f(s)\}$ be a family of L-functions with symmetry type $G(N)$. It is conjectured by Katz and Sarnak that the statistics of the low-lying zeroes of $\mathcal{F}$ should fit those of the eigenangles of random matrices in $G(N)$ [16, 17].

Let $P_A(\lambda) = \det(A - \lambda I)$ be the characteristic polynomial of A, and let $\{L_f(1/2)\}_{f\in\mathcal{F}}$ be the central critical values of the L-functions in $\mathcal{F}$. Keating and Snaith [18, 19] suggest that the value distribution of the L-functions at the critical point is related to the value distribution of the characteristic polynomials $|P_A(1)|$ with respect to the Haar measure of $G(N)$.

Using this model, vanishing in the family of quadratic twists was studied in [6]. More precisely, let

$$\begin{aligned} \mathcal{F}_{E^+} &= \{L_E(s,\chi_d) \,:\, \chi_d \text{ quadratic with } \omega_E\chi_d(-N_E) = 1\} \\ N_{E^+}(X) &= \#\{L_E(s,\chi_d) \in \mathcal{F}_{E^+} \,:\, L_E(s,\chi_d) = 0 \text{ and } |d| \le X\}, \end{aligned}$$

i.e. $\mathcal{F}_{E^+}$ is the family of quadratic twists for which the sign of the functional equation is 1. Then, either $L_E(1,\chi_d) \ne 0$, or it vanishes with even order at least 2.

Conjecture 3.1 (Conrey, Keating, Rubinstein and Snaith [6]). *There are constants $b_E \neq 0$ and e_E such that*

$$N_{E^+}(X) \sim b_E X^{3/4} \log^{e_E} X$$

when $X \to \infty$.

In this section, we make a similar analysis for the family $\mathcal{F}_E$ of cubic twists. As the symmetry type of our family is the unitary group $U(N)$, we now review the work of Keating and Snaith for this symmetry group. All the results cited below are from [18]. Let

$$M_U(s,N) = \int_{U(N)} |P_A(1)|^s \, d\text{Haar}$$

be the moments for the distribution of $|P_A(1)|$ in $U(N)$ with respect to the Haar measure. Keating and Snaith prove that

$$M_U(s,N) \quad = \quad \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{\Gamma^2(j+s/2)}, \tag{9}$$

and then $M_U(s,N)$ is analytic for $\text{Re}(s) > -1$, and has meromorphic continuation to the whole complex plane. The probability density function is the Mellin transform

$$P_U(x,N) = \frac{1}{2\pi i} \int_{(c)} M_U(s,N) x^{-s-1} ds$$

for some $c > -1$. For x small, the value of $P_U(x,N)$ is determined by the first pole of $M_U(s,N)$ at $s=-1$, and this gives

$$P_U(x,N) \sim \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \frac{\Gamma(j)^2}{\Gamma^2(j-1/2)} = R(N) \qquad \text{as } x \to 0.$$

We have

$$R(N) \quad \sim \quad N^{1/4} G^2(1/2) \qquad \text{as } N \to \infty,$$

where G is the Barnes G-function defined by

$$G(1+z) = (2\pi)^{z/2} e^{-\left((1+\gamma)z^2+z\right)/2} \prod_{n=1}^{\infty} \left((1+z/n)^n e^{-z+z^2/2n} \right).$$

Let $M_E(s,X)$ be the moments

$$M_E(s,X) \quad = \quad \frac{1}{N(X)} \sum_{\mathfrak{f} \leq X} |L_E(1,\chi)|^s \tag{10}$$

where the sum runs over all cubic characters of conductor $\leq X$. As the family $\mathcal{F}_E$ of such L-function has symmetry type $U(N)$, we have

Conjecture 3.2 (Keating and Snaith Conjecture for cubic twists)**.**

$$M_E(s,X) \quad \sim \quad a_E(s/2) M_U(s,N)$$

where $N \sim 2\log X$ and $a_E(s/2)$ is an arithmetic factor depending only on the curve E.

In the conjecture, the relation between N and X is obtained by equating the mean density of eigenangles of matrices in the unitary group, and the mean density of non-trivial zeroes of the twisted L-functions $L_E(s,\chi)$ at a fixed height. More precisely, let

$$N(T,\chi) = \#\{s \in \mathbb{C} \,:\, 0 < \mathrm{Re}(s) < 2,\ 0 < \mathrm{Im}(s) < T \text{ and } L_E(s,\chi) = 0\}$$

be the number of zeroes of $L_E(s,\chi)$ in the critical strip up to height T. Then, using the Argument Principle, one proves that

$$N(T,\chi) = \frac{T}{\pi} \log\left(\frac{\sqrt{N_E}\,\mathfrak{f}\,T}{2\pi}\right) - \frac{T}{\pi} + O\left(\log T\right).$$

Equating the densities of zeroes at a fixed height T, one gets

$$\frac{N}{2\pi} \sim \frac{1}{\pi}\log\left(\frac{\sqrt{N_E}\,\mathfrak{f}\,T}{2\pi}\right) \quad \Rightarrow \quad N \sim 2\log\mathfrak{f}$$

as stated in Conjecture 3.2. The arithmetic factor $a_E(s/2)$ captures the arithmetic missing from the random matrix theory, and we can compute it for the family of cubic twists $\mathcal{F}_E$ in the next section. The conjectural moments can then be compared with the empirical ones (see Table 4), and our data is consistent with the Keating and Snaith Conjecture for the family $\mathcal{F}_E$.

From Conjecture 3.2, the probability density function for the distribution of the special values $|L_E(1,\chi)|$ for L-functions $L_E(s,\chi) \in \mathcal{F}_E$ is

$$\begin{aligned} P_E(x,X) \quad &= \quad \frac{1}{2\pi i}\int_{(c)} M_E(s,X)\, x^{-s-1}\, ds \\ &\sim \quad \frac{1}{2\pi i}\int_{(c)} a_E(s/2)\, M_U(s,N)\, x^{-s-1}\, ds \\ &\sim \quad a_E(-1/2)\, R(N) \quad \text{for small } x \qquad (11) \\ &\sim \quad a_E(-1/2) G^2(1/2) N^{1/4} \quad \text{for large } N. \qquad (12) \end{aligned}$$

Figure 6 compares the empirical distribution with the probability density function $P_U(x,N)$.

Let $k_E = 1$ when $\omega_E = 1$, and $k_E = \sqrt{3}$ when $\omega_E = -1$. From (7) and Theorem 2.1, we have

$$|L_E(1,\chi)| \quad = \quad \left|\frac{\Omega\,\tau(\chi)\,k_E\,n_\chi}{2\,\mathfrak{f}}\right| = \frac{|\Omega\,k_E|}{2}\frac{n_\chi}{\sqrt{\mathfrak{f}}} = n_\chi\,\frac{c_E}{\sqrt{\mathfrak{f}}} \qquad (13)$$

where c_E is a constant depending only on the curve E. We now use the properties of the integers n_χ to give the measure of the interval of vanishing for $|L_E(1,\chi)|$, i.e. we write

$$\mathrm{Prob}\,\{\, |L_E(1,\chi)| = 0 \,\} \quad = \quad \mathrm{Prob}\,\{\, |L_E(1,\chi)| < B(\mathfrak{f}) \,\}$$

for some function $B(\mathfrak{f})$ of the conductor of the character. In view of Theorem 2.1 and Conjecture 2.2, we set

$$B(\mathfrak{f}) \quad = \quad \begin{cases} \dfrac{c_E 3^{\nu(\mathfrak{f})-1}}{\sqrt{\mathfrak{f}}} & \text{if } E \text{ has rational 3-torsion;} \\ \\ \dfrac{c_E}{\sqrt{\mathfrak{f}}} & \text{otherwise} \end{cases}$$

which completely determines our probabilistic model. Using the probability density function $P_E(x,X) \sim a_E(-1/2)\,R(N)$ for small x, we have

$$\begin{aligned} \mathrm{Prob}\,\{\, |L_E(1,\chi)| = 0 \,\} \quad &= \quad \int_0^{B(f)} a_E(-1/2)\,R(N)\,dx \\ &= \quad a_E(-1/2)\ R(N)\ B(\mathfrak{f}). \end{aligned}$$

We first consider the case where E does not have rational 3-torsion. Summing the probabilities, this gives

$$N_E(X) \quad = \quad c_E\, a_E(-1/2)\, R(N) \sum_{\mathfrak{f} \le X} \frac{1}{\sqrt{\mathfrak{f}}}.$$

As

$$N(X) = \sum_{\mathfrak{f} \le X} 1 \sim c_3 X \qquad \text{as } X \to \infty$$

for some constant c_3 (see Corollary 5.3), we obtain using partial summation

$$\begin{aligned} N_E(X) \quad &\sim \quad 2\ c_3\ c_E\ a_E(-1/2)\ R(N)\ X^{1/2} \\ &\sim \quad 2^{5/4}\ G^2(1/2)\ c_3\ c_E\ a_E(-1/2)\ X^{1/2}\ \log^{1/4} X \\ &\sim \quad b_E\ X^{1/2}\ \log^{1/4} X \qquad \text{as } X \to \infty. \end{aligned}$$

Similarly, if E has rational 3-torsion,

$$N_E(X) \quad = \quad c_E\, a_E(-1/2)\, R(N) \sum_{\mathfrak{f} \le X} \frac{3^{\nu(\mathfrak{f})-1}}{\sqrt{\mathfrak{f}}}$$

As

$$\sum_{\mathfrak{f} \le X} 3^{\nu(\mathfrak{f})} \sim {c_3}' X \log^2 X \qquad \text{as } X \to \infty$$

for some constant ${c_3}'$ (see Theorem 5.5), we obtain using partial summation

$$\begin{aligned} N_E(X) \quad &= \quad \frac{2}{3}\ {c_3}'\ c_E\ a_E(-1/2)\ R(N)\ \sqrt{X}\ \log^2 X \\ &\sim \quad \frac{2^{5/4}}{3}\ G^2(1/2)\ {c_3}'\ c_E\ a_E(-1/2)\ \sqrt{X}\ \log^{9/4} X \\ &\sim \quad b_E X^{1/2} \log^{9/4} X \qquad \text{as } X \to \infty. \end{aligned}$$

Hence the nature of the logarithmic factor seems to depend subtly on the arithmetic of the curve E. On the other hand, the heuristic model points to a growth rate satisfying

$$\log N_E(X) \quad \sim \quad \frac{1}{2} \log X.$$

This is supported by the empirical data in Section 6, and is consistent with the lower bounds for curves with rational 3–torsion proved in [8]. In fact, the empirical data seems to indicate a more refined conclusion of the type conjectured in [6]

$$N_E(X) \sim b_E X^{1/2} \log^{e_E} X$$

for some constants b_E and e_E depending on E (see Figures 2 and 3).

4. Moments

As mentioned in the last section, the work of Keating and Snaith led to some remarkable conjectures for the moments of special values in families of L-functions. Their conjectures agree with the known results for the first few integral moments of the Riemann zeta-function (see [12, 13]), and with the known results the first few integral moments of twists by quadratic Dirichlet characters (see [10, 15, 26]). They also agree with the number theoretic heuristics of [3, 4]. In order to verify that our empirical data also provide support for the Keating and Snaith conjectures, we need to compute the arithmetical factor $a_E(s/2)$ of Conjecture 3.2.

Let k be a positive integer. We now consider the $2k$th moments

$$M_E(2k, X) \quad = \quad \frac{1}{N(X)} \sum_{\mathfrak{f} \leq X} |L_E(1, \chi)|^{2k}$$

where the sum runs over cubic characters of conductor less than X. In this special case, the Keating and Snaith conjectures can be stated as

Conjecture 4.1 (Keating and Snaith Conjecture for cubic twists)**.** *Let k be a positive integer. Then,*

$$M_E(2k, X) \sim a_E(k)\, g_k \ (2 \log X)^{k^2}$$

where

$$g_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

and $a_E(k)$ is some arithmetical factor related to the curve E.

The arithmetical factor $a_E(k)$ cannot be obtained from the random matrix model which contains no arithmetic, but can be computed using an number-theoretic heuristic as explained in [5]. We consider

$$L(s) = \frac{1}{N(X)} \sum_{\mathfrak{f} \leq X} |L_E(s, \chi)|^{2k}$$

in some half plane $\mathrm{Re}(s) > c$. Following [5], one keeps only the diagonal terms, and neglects all error terms to write $L(s)$ as $\zeta(s)^{k^2} f(s)$ for some function $f(s)$ analytic at $s=1$. Then, specialising at $s=1$, $\zeta(s)^{k^2}$ corresponds to $(\log X)^{k^2}$ and $f(s)$ to $a_E(k)$. One can then evaluate $a_E(k)$ at any $k \in \mathbb{C}$, and in particular at $k = -1/2$ as in Section 3.

We write

$$\begin{aligned} L(s) &= \frac{1}{N(X)} \sum_{\mathfrak{f} \le X} |L_E(s,\chi)|^{2k} \\ &= \frac{1}{N(X)} \sum_{\mathfrak{f} \le X} L_E(s,\chi)^k \, L_E(s,\overline{\chi})^k \\ &= \frac{1}{N(X)} \sum_{\mathfrak{f} \le X} \sum_{n_1,\dots,n_{2k}} \frac{a_{n_1} \dots a_{n_{2k}}}{(n_1 \dots n_{2k})^s} \, \chi(n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}) \\ &= \sum_{n_1,\dots,n_{2k}} \frac{a_{n_1} \dots a_{n_{2k}}}{(n_1 \dots n_{2k})^s} \, \frac{1}{N(X)} \sum_{\mathfrak{f} \le X} \chi(n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}). \end{aligned}$$

If $n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}$ is a rational cube, the innner sum is

$$\frac{1}{N(X)} \sum_{\substack{\mathfrak{f} \le X \\ (n_1 \dots n_{2k}, \mathfrak{f}) = 1}} 1 \sim c_3(d)$$

as $X \to \infty$, where for $d = n_1 \dots n_{2k}$ and $c_3(d)$ as defined in Corollary 5.4.

For integers $n_1, \dots, n_{2k}$, let $c(n_1, \dots, n_{2k}) = c_3(d)$ for $d = n_1 \dots n_{2k}$, and let $\psi(n_1, \dots, n_{2k}) = 1$ when $n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}$ is a rational cube, and $\psi(n_1, \dots, n_{2k}) = 0$ otherwise. Considering only the contribution from the terms where $n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}$ is a rational cube, we obtain

$$\begin{aligned} L(s) &\sim \sum_{n_1,\dots,n_{2k}} \frac{a_{n_1} \dots a_{n_{2k}}}{(n_1 \dots n_{2k})^s} \, c(n_1, \dots, n_{2k}) \, \psi(n_1, \dots, n_{2k}) \\ &= \sum_{n_1,\dots,n_{2k}} f(n_1, \dots, n_{2k}), \end{aligned}$$

where $f(n_1, \dots, n_{2k})$ is a multiplicative function of the $2k$ variables. Then, $L(s)$ has the Euler product

$$\begin{aligned} L(s) &= \prod_p \sum_{\substack{e_1,\dots,e_{2k} \\ e_1+\cdots+e_k \equiv e_{k+1}+\cdots+e_{2k} \mod 3}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{(p^{e_1+\cdots+e_{2k}})^s} c(p^{e_1}, \dots, p^{e_{2k}}) \\ &= \prod_{p \equiv 2 \mod 3} \sum_{\substack{e_1,\dots,e_{2k} \\ e_1+\cdots+e_k \equiv e_{k+1}+\cdots+e_{2k} \mod 3}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{(p^{e_1+\cdots+e_{2k}})^s} \\ & \quad \prod_{p \equiv 1 \mod 3} 1 + \frac{p}{p+2} \sideset{}{^*}\sum_{\substack{e_1,\dots,e_{2k} \\ e_1+\cdots+e_k \equiv e_{k+1}+\cdots+e_{2k} \mod 3}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{(p^{e_1+\dots e_{2k}})^s} \\ & \quad \prod_{p=3} 1 + \frac{9}{11} \sideset{}{^*}\sum_{\substack{e_1,\dots,e_{2k} \\ e_1+\cdots+e_k \equiv e_{k+1}+\cdots+e_{2k} \mod 3}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{(p^{e_1+\dots e_{2k}})^s} \qquad (14) \\ &= \prod_p E(p,s) \qquad (15) \end{aligned}$$

where $*$ indicates that the term $e_1 = \cdots = e_{2k} = 0$ is missing from the sum.

Lemma 4.2. *Let $E(p,s)$ be the Euler factor defined by Equation (15). For any $\epsilon > 0$,*

$$E(p,s) = 1 + k^2 \frac{a_p^2}{p^{2s}} + O_k\left(p^{-3s+\epsilon}\right).$$

Proof: Suppose $p \equiv 2 \mod 3$. Then,

$$E(p,s) = \sum_{\substack{e_1,\dots,e_{2k} \\ e_1+\cdots+e_k \equiv e_{k+1}+\cdots+e_{2k} \mod 3}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{(p^{e_1+\cdots+e_{2k}})^s}.$$

Using $n = \sum_{i=1}^{2k} e_i$, $n_1 = \sum_{i=1}^{k} e_i$ and $n_2 = \sum_{i=k+1}^{2k} e_i$, and collecting the terms with the same n, we write the above sum as

$$\sum_{n=0}^{\infty} \frac{c_n}{p^{ns}}.$$

Clearly, $c_0 = 1$ and $c_1 = 0$. For $n = 2$, the only choice with $n_1 \equiv n_2 \mod 3$ is $n_1 = n_2 = 1$. There are k^2 tuples $(e_1, \dots, e_{2k})$ with $n_1 = n_2 = 1$ and for each such tuple,

$$\frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{(p^{e_1+\cdots+e_{2k}})^s} = \frac{a_p^2}{p^{2s}},$$

and $c_2 = k^2 a_p^2$. In general, there are $O(n^k)$ tuples with $\sum_{i=1}^{2k} e_i = n$, and for each such tuple $a_{p^{e_1}} \dots a_{p^{e_{2k}}}$ is at most $O\left(p^{2kn\epsilon}\right)$ for any $\epsilon > 0$. This gives

$$\begin{aligned} E(p,s) &= 1 + k^2 \frac{a_p^2}{p^{2s}} + O_k\left(\sum_{n=3}^{\infty} \left(p^{-s+\epsilon}\right)^n\right) \\ &= 1 + k^2 \frac{a_p^2}{p^{2s}} + O_k\left(p^{-3s+\epsilon}\right) \end{aligned}$$

for any $\epsilon > 0$. The proof for $p \equiv 0, 1 \mod 3$ is similar. □

From Lemma 4.2, $L(s)$ has a pole of order k^2 at $s = 1$ as does the Rankin-Selberg convolution

$$L(E \otimes E, s) = \sum_{n=1}^{\infty} \left(\frac{a_n}{\sqrt{n}}\right)^2 \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{a_n^2}{n^{s+1}}$$

(see [14, Section 13.8] for more details). Then,

$$L(s) = \zeta(s)^{k^2} \prod_p \left(1 - \frac{1}{p^s}\right)^{k^2} E(p,s)$$

where

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{k^2} E(p,s)$$

is analytic at $s = 1$. We then set

$$a_E(k) = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} E(p,1). \tag{16}$$

We now write the Euler factors $E(p,1)$ in a more suitable form using the multiplicativity of the a_p's.

Lemma 4.3. *Let ρ be a primitive third root of 1, and let*

$$F(p) = \sum_{\substack{e_1,\dots,e_{2k} \\ e_1+\dots+e_k \equiv e_{k+1}+\dots+e_{2k} \mod 3}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{p^{e_1+\dots+e_{2k}}}.$$

Then, as a formal series, $F(p)$ is

$$\begin{cases} \frac{1}{3}\left(1 - \frac{a_p}{p} + \frac{1}{p}\right)^{-2k} + \frac{2}{3}\left(1 - \frac{\rho a_p}{p} + \frac{\rho^2}{p}\right)^{-k}\left(1 - \frac{\rho^{-1} a_p}{p} + \frac{\rho^{-2}}{p}\right)^{-k} & \textit{for } p \nmid N_E; \\ \frac{1}{3}\left(1 - \frac{a_p}{p}\right)^{-2k} + \frac{2}{3}\left(1 - \frac{\rho a_p}{p}\right)^{-k}\left(1 - \frac{\rho^{-1} a_p}{p}\right)^{-k} & \textit{for } p \mid N_E. \end{cases}$$

Proof: Using $n_1 = \sum_{i=1}^{k} e_i$, $n_2 = \sum_{i=k+1}^{2k} e_i$, and the characteristic function

$$\frac{1}{3}\left(1 + \rho^{n_1-n_2} + \rho^{-n_1+n_2}\right) = \begin{cases} 1 & \text{if } n_1 \equiv n_2 \mod 3; \\ 0 & \text{otherwise,} \end{cases}$$

we have the formal equalities

$$\begin{aligned}\sum_{\substack{e_1,\dots,e_{2k}\\ n_1\equiv n_2 \mod 3}} \frac{a_{p^{e_1}}\dots a_{p^{e_{2k}}}}{p^{e_1+\dots+e_{2k}}} &= \frac{1}{3}\sum_{e_1,\dots,e_{2k}} \frac{a_{p^{e_1}}\dots a_{p^{e_{2k}}}}{p^{e_1+\dots+e_{2k}}} + \frac{2}{3}\sum_{e_1,\dots,e_{2k}} \frac{a_{p^{e_1}}\dots a_{p^{e_{2k}}}}{p^{e_1+\dots+e_{2k}}}\rho^{n_1-n_2}\\ &= \frac{1}{3}\left(\sum_{e=1}^{\infty}\frac{a_{p^e}}{p^e}\right)^{2k} + \frac{2}{3}\left(\sum_{e=1}^{\infty}\frac{a_{p^e}\,\rho^e}{p^e}\right)^{k}\left(\sum_{e=1}^{\infty}\frac{a_{p^e}\,\rho^{-e}}{p^e}\right)^{k}. \qquad (17)\end{aligned}$$

Using the multiplicativity of the Fourier coefficients a_n, we get for any $\alpha \in C^*$

$$\sum_{e=1}^{\infty}\frac{a_{p^e}\,\alpha^e}{p^e} = \begin{cases}\left(1-\dfrac{\alpha\,a_p}{p}+\dfrac{\alpha^2}{p}\right)^{-1} & \text{if } p\nmid N_E;\\[2ex] \left(1-\dfrac{\alpha\,a_p}{p}\right)^{-1} & \text{if } p\mid N_E.\end{cases}$$

Replacing in (17), this proves the lemma. □

Using the above lemma in (14), we can write the Euler factors as

$$E(p,1) = \begin{cases}\dfrac{p}{p+2}F(p)+\dfrac{2}{p+2} & \text{for } p\equiv 1 \mod 3;\\[2ex] F(p) & \text{for } p\equiv 2 \mod 3;\\[2ex] \dfrac{9}{11}F(p)+\dfrac{2}{11} & \text{for } p=3.\end{cases}$$

This expression is now valid for all $k\in\mathbb{C}$, and not only integers. This value of $a_E(k)$ is used to compute the conjectural moments of Table 4.

5. Number of cubic conductors

We give in this section asymptotics for

$$\begin{aligned}N(X) &= \#\{\text{cubic characters of conductor } \mathfrak{f}\le X\}\\ N_d(X) &= \#\{\text{cubic characters of conductor } \mathfrak{f}\le X \text{ with } (\mathfrak{f},d)=1\}\\ S(X) &= \sum_{\mathfrak{f}\le X} 3^{\nu(\mathfrak{f})}\end{aligned}$$

which are needed in the rest of the paper. The estimate for $N(X)$ can also be found in [2].

Lemma 5.1. *Let χ be a cubic character of conductor $\mathfrak{f}$. Then, $\mathfrak{f}=(9)^{\alpha}p_1\dots p_t$ where $p_1,\dots,p_t$ are distinct primes congruent to 1 modulo 3, and $\alpha = 0$ or 1. Furthermore, for each such conductor, there are $2^{(t+\alpha)} = 2^{\nu(\mathfrak{f})}$ distinct cubic characters with conductor $\mathfrak{f}$.*

Proof: A cubic Dirichlet character of conductor $\mathfrak{f}$ can be written uniquely as a product of cubic Dirichlet characters of prime power conductor. Since the prime power conductors of cubic characters are either 9 or a prime p

congruent to 1 modulo 3, the first statement of the lemma follows. Furthemore, writing $\mathfrak{f} = (9)^\alpha p_1 \dots p_t$, we see that there are $2^{\alpha+t} = 2^{\nu(\mathfrak{f})}$ cubic characters with conductor $\mathfrak{f}$ since there are two characters of order 3 for each such prime power conductor. □

Let $a(n)$ be the number of cubic characters of conductor n. Then, it follows from the above lemma that

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \mod 3} \left(1 + \frac{2}{p^s}\right),$$

and the above series converges for $\mathrm{Re}(s) > 1$. We then have to analyse the analytic behavior of $L(s)$ at $s = 1$. We find out that

Proposition 5.2.

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

has a simple pole at $s = 1$ *with residue*

$$c_3 = \frac{11\sqrt{3}}{18\pi} \prod_{p \equiv 1 \mod 3} \left(1 - \frac{2}{p(p+1)}\right).$$

Proof:

$$L(s) = \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \mod 3} \left(1 + \frac{2}{p^s}\right) = g(s) \prod_{p \equiv 1 \mod 3} \left(1 - \frac{1}{p^s}\right)^{-2}$$

where

$$\begin{aligned} g(s) &= \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \mod 3} \left(1 - \frac{1}{p^s}\right)^2 \left(1 + \frac{2}{p^s}\right) \\ &= \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \mod 3} \left(1 - \frac{3}{p^{2s}} + \frac{2}{p^{3s}}\right) \end{aligned}$$

is analytic at $s = 1$.

Let K be the field obtained by adding a third root of 1. Then, $K = \mathbb{Q}(\sqrt{-3})$ and the Dedekind zeta function

$$\zeta_K(s) = \left(1 - \frac{1}{3^s}\right)^{-1} \prod_{p \equiv 1 \mod 3} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \equiv 2 \mod 3} \left(1 - \frac{1}{p^{2s}}\right)^{-1}.$$

has a simple pole at $s = 1$ with residue

$$\rho = \frac{2^{r+s} \pi^s \mathrm{reg}(K) h_K}{\omega_K |\Delta_K|^{1/2}} = \frac{\pi}{3\sqrt{3}}.$$

Using this fact, we get

$$\begin{aligned} L(s) &= g(s) \prod_{p\equiv 1 \mod 3} \left(1-\frac{1}{p^s}\right)^{-2} \\ &= g(s)\left(1-\frac{1}{3^s}\right) \prod_{p\equiv 2 \mod 3} \left(1-\frac{1}{p^{2s}}\right) \zeta_K(s) \\ &= h(s)\zeta_K(s) \end{aligned}$$

where

$$h(s) = \left(1+\frac{2}{9^s}\right)\left(1-\frac{1}{3^s}\right) \prod_{p\equiv 1 \mod 3} \left(1-\frac{3}{p^{2s}}+\frac{2}{p^{3s}}\right) \prod_{p\equiv 2 \mod 3} \left(1-\frac{1}{p^{2s}}\right)$$

is analytic at $s=1$. One computes

$$\begin{aligned} h(1) &= \frac{11}{9}\,\frac{2}{3} \prod_{p\equiv 1,2 \mod 3} \left(1-\frac{1}{p^2}\right) \prod_{p\equiv 1 \mod 3} \frac{(1-3p^{-2}+2p^{-3})}{(1-p^{-2})} \\ &= \frac{11}{12\zeta(2)} \prod_{p\equiv 1 \mod 3} \left(1-\frac{2}{p(p+1)}\right) \end{aligned}$$

Then, $L(s)$ has a simple pole at $s=1$ with residue

$$c_3 = \frac{\pi}{3\sqrt{3}}\, h(1) = \frac{11\sqrt{3}}{18\pi} \prod_{p\equiv 1 \mod 3} \left(1-\frac{2}{p(p+1)}\right) = 0.3170564\ldots$$

□

Corollary 5.3. $N(X) \sim c_3 X \qquad$ *as* $X \to \infty$.

Proof: Using Proposition 5.2 and the Tauberian Theorem (see for example [22]), we have

$$N(X) = \sum_{n\le X} a(n) \sim c_3 X.$$

□

Remark: The constant $c_{\mathbb{Q}}(C_3)$ on [2, p. 104] is half of our constant as there are two characters per cyclic cubic field.

Corollary 5.4. *Let d be a positive integer. Then,*

$$N_d(X) \sim c_3(d)\, N(X) \qquad \text{as } X \to \infty$$

where

$$c_3(d) = \begin{cases} \displaystyle\prod_{\substack{p\equiv 1 \mod 3 \\ p\mid d}} \frac{p}{p+2} & \text{for } 3\nmid d; \\ \displaystyle\frac{9}{11} \prod_{\substack{p\equiv 1 \mod 3 \\ p\mid d}} \frac{p}{p+2} & \text{for } 3\mid d. \end{cases}$$

Proof: Suppose that $3 \nmid d$. Let $b(n)$ be the number of cubic characters of conductor n when $(n,d)=1$, and $b(n)=0$ otherwise. We consider the L-function

$$\begin{aligned} L_2(s) &= \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \\ &= \left(1+\frac{2}{9^s}\right) \prod_{\substack{p \equiv 1 \mod 3 \\ p \mid d}} \left(1+\frac{2}{p^s}\right)^{-1} \prod_{p\equiv 1 \mod 3} \left(1+\frac{2}{p^s}\right) \\ &= f(s)L(s) \end{aligned}$$

where

$$f(s) = \prod_{\substack{p \equiv 1 \mod 3 \\ p \mid d}} \left(1+\frac{2}{p^s}\right)^{-1} = \prod_{\substack{p \equiv 1 \mod 3 \\ p \mid d}} \frac{p^s}{p^s+2}$$

is analytic at $s=1$. Then, using Proposition 5.2 and the Tauberian Theorem, this gives

$$\sum_{n \leq X} b(n) \sim f(1) c_3 X,$$

and the result follows. The proof for $3 \mid d$ is similar. □

Corollary 5.5.

$$S(X) = \sum_{\mathfrak{f} \leq X} 3^{\nu(\mathfrak{f})} \sim c_3{}' X \log^2 X \qquad \text{as } X \to \infty$$

for some constant $c_3{}'$.

Proof: Using Lemma 5.1, we write

$$\sum_{\mathfrak{f} \leq X} 3^{\nu(\mathfrak{f})} = \sum_{n \leq X} a(n)$$

where

$$a(n) = \begin{cases} 6^{\nu(n)} & \text{if } n \text{ is the conductor of cubic character;} \\ 0 & \text{otherwise.} \end{cases}$$

Now, working exactly as above, consider the L-function

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \left(1+\frac{6}{9^s}\right) \prod_{p \equiv 1 \mod 3} \left(1+\frac{6}{p^s}\right) = \zeta_K(s)^3 g(s)$$

where $g(s)$ is analytic at $s=1$. Then, $L(s)$ has a pole of order 3 with residue c_3' (say) at $s=1$, and it follows from the Tauberian Theorem that

$$S(X) = \sum_{n \leq X} a(n) \sim c_3' \, X \, \log^2 X.$$

□

6. Numerical Data

In order to effectively compute twisted L-functions, we use the series representation

$$L_E(1,\chi) = \sum_{n=1}^{\infty} \frac{a_n}{n} \exp\left(-\frac{2\pi n}{\mathfrak{f}\sqrt{N_E}}\right) \left(\chi(n) + \omega_E \chi(N_E) \frac{\tau(\chi)^2}{\mathfrak{f}} \overline{\chi}(n)\right)$$

derived from the functional equation (2). This series is rapidly convergent for small values of $\mathfrak{f}\sqrt{N_E}$ and has an easily computable (though conservative) bound on the truncation error after k terms, namely

$$\frac{4}{1-q} q^k \quad \text{where } q = \exp\left(-\frac{2\pi}{\mathfrak{f}\sqrt{N_E}}\right).$$

A small sample of eight elliptic curves was selected and computer runs of varying lengths were performed to establish a database of cubic twists. The curves were chosen to represent a variety of torsion and rank. Curves of small conductor are chosen in order to maintain precision in the calculations; in the case of E11A and E14A, up to 16,000,000 terms were summed for the highest conductor twists. The computations were greatly assisted by the fact that n_χ is an integer. At least four decimal place accuracy was maintained in these integers throughout the calculations. The empirical results are shown the next figures.

Curve	Torsion	Rank	Maximal conductor	Number of characters	Number of vanishing
E11A	5	0	2,023,513	320,795	1152
E14A	6	0	2,108,767	260,001	4347
E15A	8	0	399,979	51,890	807
E32A	4	0	300,217	47,577	117
E36A	6	0	283,051	36,718	346
E37A	1	1	279,211	41,991	559
E37B	3	0	364,723	54,830	1899
E389A	1	2	99,991	15,851	408

Figure 1. The eight elliptic curves selected for this study with the sample sizes used. The number of characters is the number of characters χ with conductor $\mathfrak{f}$ smaller than the maximal conductor and such that $(\mathfrak{f}, N_E) = 1$. For each conductor $\mathfrak{f}$, there are 2 conjugate cubic characters $\chi, \overline{\chi}$ with $L_E(1,\overline{\chi}) = \overline{L_E(1,\chi)}$, and only of them is counted. The number of vanishing is the number of such characters with $L_E(1,\chi) = 0$.

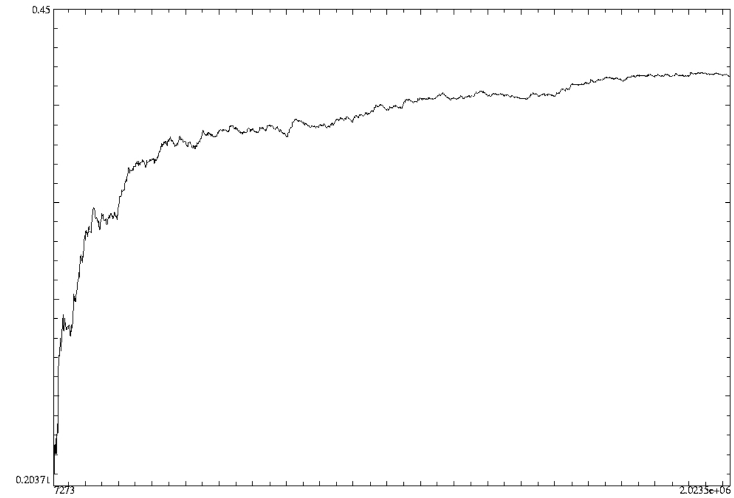

FIGURE 2. Ratio of the empirical $N_E(X)$ with $\sqrt{X} \log^{1/4} X$ for the curve $E11A$ and $1 \leq X \leq 2,023,513$.

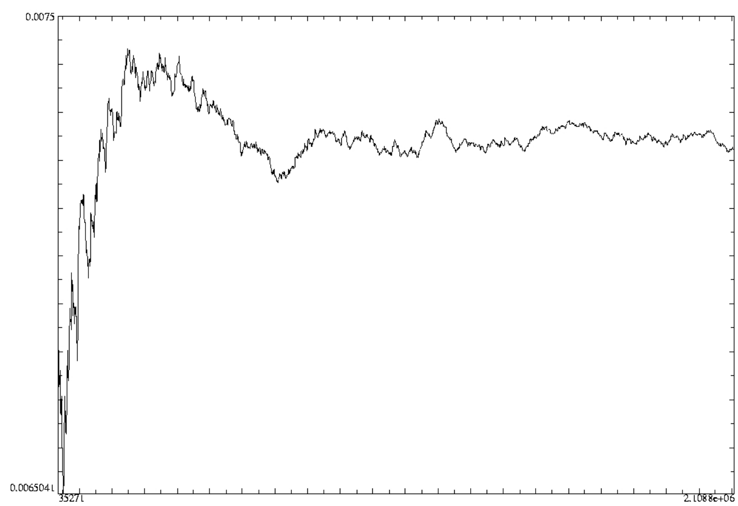

FIGURE 3. Ratio of the empirical $N_E(X)$ with $\sqrt{X} \log^{9/4} X$ for the curve $E14A$ and $1 \leq X \leq 2,108,767$.

Curve		$s = 1/2$	$s = 1$	$s = 3/2$	$s = 2$	$s = 3$	$s = 4$
E11A	Empirical	1.420	2.878	7.349	22.02	274.3	4617.
	Conjectural	1.436	2.962	7.621	22.34	227.7	2288.
	Ratio	0.990	0.972	0.964	0.985	1.205	2.017
E14A	Empirical	1.268	2.196	4.696	11.76	104.6	1302.
	Conjectural	1.282	2.243	4.796	11.66	83.18	599.8
	Ratio	0.990	0.979	0.979	1.008	1.257	2.171
E15A	Empirical	1.384	2.609	5.995	15.86	149.4	1874.
	Conjectural	1.400	2.677	6.175	15.87	117.9	816.9
	Ratio	0.989	0.974	0.971	1.000	1.266	2.294
E32A	Empirical	1.221	1.928	3.641	7.863	49.23	407.2
	Conjectural	1.225	1.946	3.629	7.468	35.42	154.8
	Ratio	0.996	0.991	1.003	1.052	1.389	2.630
E36A	Empirical	1.184	1.792	3.202	6.491	35.34	253.6
	Conjectural	1.193	1.814	3.188	6.101	24.29	86.90
	Ratio	0.992	0.988	1.004	1.063	1.454	2.919
E37A	Empirical	1.468	3.196	8.935	29.50	441.3	8592.
	Conjectural	1.483	3.280	9.197	29.40	341.3	3547.
	Ratio	0.990	0.974	0.972	1.003	1.292	2.421
E37B	Empirical	1.119	1.656	2.946	6.060	36.15	311.3
	Conjectural	1.127	1.646	2.829	5.395	22.69	93.66
	Ratio	0.993	1.006	1.041	1.123	1.593	3.323
E389A	Empirical	1.594	3.960	13.08	52.36	1210.	38636.
	Conjectural	1.614	4.088	13.68	53.95	1015.	17901.
	Ratio	0.988	0.969	0.956	0.971	1.192	2.158

FIGURE 4. Moments of cubic twists for the eight selected elliptic curves. The empirical moments are the moments (10) for various values of s and up to X given in Table 1. The conjectural moments are computed following Conjecture 3.2 with the arithmetic factor $a_E(s)$ of Section 4. For small values of s, our data supports Conjecture 3.2. The divergence between the conjectural and empirical data for higher moments can be explained by the asymptotic nature of the moments. We use only the leading order asymptotic for the conjectural moments, but there are several other terms which will contribute strongly when the sample size is relatively small. For integral moments, there is a new heuristic which gives all the main terms for the asymptotic behavior of the moments [5], and using this new heuristic, one would get a better empirical fit with a smaller sampling size.

Curve	E11A	E14A	E15A	E32A	E36A	E37A	E37B	E389A
Factor	10	6	8	4	2	4	12	12
$n_\chi = 0$	1152	4347	807	117	346	559	1899	408
$n_\chi = 1$	1662	344	287	695	118	1096	150	962
$n_\chi = 2$	1117	440	229	509	108	645	136	493
$n_\chi = 3$	2676	1379	414	209	683	1264	1419	761
$n_\chi = 4$	1328	336	660	201	54	799	171	521
$n_\chi = 5$	1069	288	219	515	97	657	147	427
$n_\chi = 6$	1711	2707	470	194	535	715	785	374
$n_\chi = 7$	1827	390	327	789	161	879	188	542
$n_\chi = 8$	1125	442	504	768	107	527	105	330
$n_\chi = 9$	2578	2365	378	209	959	836	2853	414
$n_\chi = 10$	631	293	174	329	58	376	88	223
$n_\chi = 11$	1336	299	227	534	95	533	118	325
$n_\chi = 12$	2183	2188	872	66	260	702	993	301
$n_\chi = 13$	1607	365	274	666	122	624	149	327
$n_\chi = 14$	1015	489	229	559	119	429	98	225
$n_\chi = 15$	1625	1044	288	164	466	618	810	252
$n_\chi = 16$	1182	330	790	516	71	388	103	216
$n_\chi = 17$	1262	273	217	519	104	439	99	199
$n_\chi = 18$	1624	5605	353	138	766	385	1489	187
$n_\chi = 19$	1433	331	256	605	106	459	151	221
$n_\chi = 20$	770	272	388	131	44	349	60	162
$n_\chi = 21$	2562	1439	409	219	679	686	1173	250
$n_\chi = 22$	786	255	139	327	65	237	75	116
$n_\chi = 23$	1193	276	203	467	80	400	84	167
$n_\chi = 24$	1634	2304	721	211	487	405	606	138
$n_\chi = 25$	952	241	203	478	73	323	116	136
$n_\chi = 26$	852	385	187	477	90	306	67	112
$n_\chi = 27$	2169	2571	319	182	919	450	3111	176
$n_\chi = 28$	1199	315	526	183	43	346	107	138
$n_\chi = 29$	1119	236	156	408	91	330	85	128
$n_\chi = 30$	920	1719	218	80	256	254	461	87
Maximal value	10139	9872	4250	1867	2322	1968	1935	1443

FIGURE 5. Frequency distribution for n_χ. Each line of the table is the number of incidences of $n_\chi = 0, 1, \ldots, 30$ for all characters with conductor $1 \leq \mathfrak{f} \leq X$ for the sample sizes given in Table 1. The maximal value is the largest n_χ in this sample. The factor of the first line is the multiple of the period Ω_E used to make the values n_χ integral without common factor.

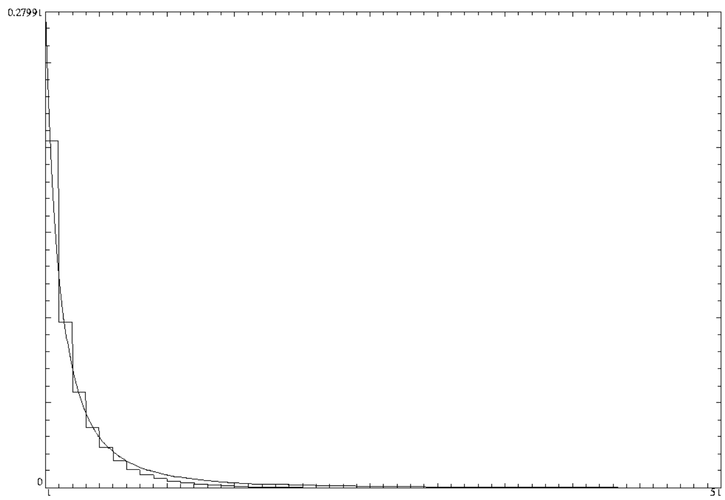

FIGURE 6. Histogram of the empirical values $|L_E(1,\chi)|$ for the curve $E14$ and the sample size of Table 1 supersimposed with the probability distribution function $P_U(x,N)$ with $N=12$. The probability distribution is computed using the approximations of [18].

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