GENERALIZATIONS OF SOME ZERO-SUM THEOREMS

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1. Introduction

For a finite abelian group G, the Davenport Constant $D(G)$ is the smallest positive integer k such that any sequence of k elements in G has a non-empty subsequence whose sum is zero. For a finite abelian group G, with cardinality $|G| = n$, another combinatorial invariant $E(G)$ is defined to be the smallest positive integer k such that any sequence of k elements in G has a subsequence of length n whose sum is zero. These two constants were being studied independently before the following result of Gao [11]:

$$E(G) = D(G) + n - 1. \tag{1}$$

Generalizations of these constants with weights were considered in [5] and [6], for the particular group $\mathbb{Z}/n\mathbb{Z}$. Later in [4], the following generalizations of both $E(G)$ and $D(G)$ for an arbitrary finite abelian group G of order n have been introduced. One may look into [2] for an elaborate account of this theme.

For a finite abelian group G and a finite subset $A \subseteq \mathbb{Z}$, the Davenport Constant of G with weight A, denoted by $D_A(G)$, is defined to be the smallest positive integer k such that for any sequence $(x_1, \cdots, x_k)$ of k elements in G, there exists a non-empty subsequence $(x_{j_1}, \cdots, x_{j_r})$ and $a_1, \cdots, a_r \in A$ such that

$$\sum_{i=1}^{r} a_i \; x_{j_i} = 0.$$

To avoid trivial cases, one assumes that the weight set A does not contain 0 and it is non-empty. Further, if $|G| = n$, one can assume that $A \subset \{1, 2, \cdots, n-1\}$.

Similarly, for any such A and an abelian group G with $|G| = n$, the constant $E_A(G)$ is the smallest positive integer k such that for any sequence $(x_1, \cdots, x_k)$

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of k elements in G, there exists $x_{j_1}, \cdots, x_{j_n}$ such that

$$\sum_{i=1}^{n} a_i \; x_{j_i} = 0$$

with $a_i \in A$.

Taking $A = \{1\}$, we retrieve the classical constants $D(G)$ and $E(G)$. A result similar to the above result (1) of Gao is expected for the generalized constants with weights. In many special cases this relation has been established (see [5], [4], [13], [12], [4] [15], [3]).

One of the few general results known in this direction is the following one due to Adhikari and Chen [4]; one notes that it does not include the result (1) of Gao which corresponds to the case $|A| = 1$.

Theorem A. *Let G be a finite abelian group of order n and $A = \{a_1, \cdots, a_r\}$ be a finite subset of $\mathbb{Z}$ with $r \geq 2$. If $\gcd(a_2 - a_1, \ldots, a_r - a_1, n) = 1$, then*

$$E_A(G) = D_A(G) + n - 1.$$

When G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$, we denote $E_A(G)$ and $D_A(G)$ by $E_A(n)$ and $D_A(n)$ respectively. Exact values for $D_A(n)$ and $E_A(n)$ have been found in some cases (see [5], [13], [12], [6], [3]). For instance, it has been proved in [6] that for a prime p, when A is the set of quadratic residues modulo p, we have $D_A(p) = 3$ and $E_A(p) = p + 2$. In the present paper, we consider its natural generalization, that is, the problem of determining $E_A(n)$ and $D_A(n)$ where A is the set of squares in the group of units in the cyclic group $\mathbb{Z}/n\mathbb{Z}$ for a general integer n. In the rest of the paper, we will denote this set as $R_n = \{x^2 : x \in (\mathbb{Z}/n\mathbb{Z})^*\}$. When it is obvious from the context, we shall simply write R in place of R_n. We prove the following results.

Theorem 1. *If $n = p_1 \cdots p_r$ is any square-free odd integer where $p_i \geq 5$ for all $i = 1, \ldots, r$, then*

$$\begin{aligned} \text{(i)} \quad D_R(n) &= 2r + 1, \\ \text{(ii)} \quad E_R(n) &= n + 2r. \end{aligned}$$

As it will be observed in Remark 1, when the prime 3 is involved, the constants $D_R(n)$ and $E_R(n)$ may be strictly greater than the values given in the above theorem. In this case we can prove the following

Theorem 2. *If $n = p_1 \cdots p_r$, $r \geq 2$, is any square-free odd integer where $p_1 = 3$ and $p_i \geq 5$ for all $i \geq 2$, we have the following bounds*

$$\begin{aligned} \text{(i)}\, 2r+1 \leq D_R(n) &\leq 6r-3, \\ \text{(ii)}\, n+2r \leq E_R(n) &\leq n+6r-4. \end{aligned}$$

However, in the case $n = 3p$, where $p \geq 7$ is prime, we can find the precise value of $D_R(n)$.

Theorem 3. *If $n = 3p$, where $p \geq 7$ is prime, then*

$$D_R(n) = 5.$$

When the prime 2 is involved we have the following results.

Theorem 4. *If $n = p_1 \cdots p_r$, $r \geq 2$, is any square-free integer where $p_1 = 2$ and $p_i \geq 5$ for all $i \geq 2$, we have the following bounds*

$$\begin{aligned} \text{(i)}\, 2r+1 \leq D_R(n) &\leq 4r-2, \\ \text{(ii)}\, n+2r \leq E_R(n) &\leq n+4r-3. \end{aligned}$$

Theorem 5. *If $n = p_1 \cdots p_r$, $r \geq 2$, is any square-free integer where $p_1 = 2$ and $p_2 = 3$, we have the following bounds*

$$\begin{aligned} \text{(i)}\, 2r+1 \leq D_R(n) &\leq 6r-6, \\ \text{(ii)}\, n+2r \leq E_R(n) &\leq n+6r-7. \end{aligned}$$

In the non-square-free case, we have the following result.

Theorem 6. *Let $n = p^r$, where $p > 3$ is a prime number and $r \in \mathbb{Z}^+$. Then,*

$$\begin{aligned} \text{(i)} \quad D_R(n) &= 2r+1, \\ \text{(ii)} \quad E_R(n) &= n+2r. \end{aligned}$$

Finally, we dedicate Section 3 to investigate other sets of weights. Among other remarks, we are able to prove the following result. We will use the usual notation $\lceil x \rceil$ to denote the closest integer to x bigger than x.

Theorem 7. *Let n, r be positive integers, $1 \leq r < n$ and consider the subset $A = \{1, \ldots, r\}$ of $\mathbb{Z}/n\mathbb{Z}$. Then,*

$$\begin{aligned} \text{(i)} \quad D_A(n) &= \left\lceil \frac{n}{r} \right\rceil, \\ \text{(ii)} \quad E_A(n) &= n-1+D_A(n). \end{aligned}$$

This theorem also generalizes a result in [6], where the case $n = p$, prime, had been proved.

2. Proofs of Theorems

We shall need the following version of *Cauchy-Davenport Theorem* ([7], [9], can also see [14] for instance).

Theorem B (Cauchy-Davenport Theorem). *If p is a prime and $A_1, A_2, \cdots, A_h$ are non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$, then*

$$|A_1 + A_2 + \cdots + A_h| \geq min\left(p, \sum_{i=1}^{h} |A_i| - (h-1)\right).$$

We shall also need the following generalization of the above result (see [8] and [14]).

Theorem C (Chowla). *Let n be a natural number, and let A and B be two nonempty subsets of $\mathbb{Z}$, such that $0 \in B$ and $A + B \neq \mathbb{Z}/n\mathbb{Z}$. If $(x, n) = 1$ for all $\in B \setminus \{0\}$, then $|A + B| \geq |A| + |B| - 1$.*

Lemma 8. *For an odd prime $p \geq 7$, if a sequence $(x_1, \ldots, x_k)$ with $x_i \in \mathbb{Z}/p\mathbb{Z}$, contains at least three non-zero elements, then*

$$\sum_{i=1}^{k} a_i \; x_i = 0,$$

with $a_i \in R_p$.

Proof. Without loss of generality, let x_1, x_2, x_3 be units.

By Cauchy-Davenport Theorem (stated as Theorem B above),

$$|x_1 R_p + x_2 R_p + x_3 R_p| \geq min\left(p, \frac{3(p-1)}{2} - 2\right) = p.$$

Therefore, one can write

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = -(x_4 + x_5 + \cdots + x_k),$$

where $\alpha_i \in R_p$. This proves the lemma.

Lemma 9. *If a sequence $(x_1, \ldots, x_k)$ with $x_i \in \mathbb{Z}/5\mathbb{Z}$, contains at least four non-zero elements, then*

$$\sum_{i=1}^{k} a_i \; x_{j_i} = 0$$

with $a_i \in R_5$.

Proof. The proof is similar to that of Lemma 8.

Theorem 1 will be an easy corollary of Propositions 10 and 11 below. As can be seen, bulk of the work goes towards the proof of Proposition 11.

Proposition 10. *If $n = p_1 \cdots p_r$, $r \geq 1$, is any square-free odd integer where $p_i \geq 7$ for all $i = 1, \ldots, r$, then, given any sequence $(x_1, \ldots, x_{m+2r})$ of $m+2r$ elements in $\mathbb{Z}/n\mathbb{Z}$ for an integer $m \geq 3r$, there exists a subsequence $(x_{i_1}, \ldots, x_{i_m})$ such that*

$$\sum_{j=1}^{m} a_j \ x_{i_j} = 0,$$

with $a_j \in R_n$.

Proposition 11. *If $n = p_1 \cdots p_r$, $r \geq 1$, is any square-free odd integer where $p_1 = 5$ and $p_i \geq 7$ for all $i \geq 2$, then, given any sequence $(x_1, \cdots, x_{m+2r})$ of $m+2r$ elements in $\mathbb{Z}/n\mathbb{Z}$ for an integer $m \geq 3r+1$, there exists a subsequence $(x_{i_1}, \cdots, x_{i_m})$ such that*

$$\sum_{j=1}^{m} a_j \ x_{i_j} = 0,$$

with $a_j \in R_n$.

We observe that in the above propositions, the results would be true if we have more than $m + 2r$ elements, by not considering the extra elements.

Proof of Proposition 10.

We proceed by induction on r. When $r = 1$, we have $n = p$, a prime.

By Lemma 8 given any sequence $(x_1, \cdots, x_{m+2})$ with $x_i \in \mathbb{Z}/p\mathbb{Z}$, where $m \geq 3$, if it contains at least three units say x_1, x_2, x_3, then

$$\sum_{i=1}^{m} a_i \ x_i = 0$$

with $a_i \in R_p$.

Otherwise, at most two elements of the sequence are units which implies that at least m elements say $x_{j_1}, \cdots, x_{j_m}$ are divisible by p and hence

$$\sum_{i=1}^{m} a_i \ x_{j_i} = 0$$

for any choice of $a_i \in R_p$ for each i. This establishes the case with $r = 1$.

Now, suppose that $r \geq 2$ and the result is true for any square-free odd integer with number of prime factors not exceeding $r-1$ provided all its prime factors are ≥ 7. Suppose we are given a sequence $(x_1, \cdots, x_{m+2r})$ of $m+2r$ elements of $\mathbb{Z}/n\mathbb{Z}$.

Suppose that, for each prime $p|n$, the sequence contains three elements coprime to p. Then without loss of generality, let $S = (x_1, \cdots, x_t)$ be a subsequence of $t \leq 3r \leq m$ elements such that S has three units corresponding to each prime.

Then, by Lemma 8, for each prime p_i, we have

$$\sum_{j=1}^{m} a_j^{(i)} \, x_j \equiv 0 \pmod{p_i},$$

with $a_j^{(i)} \in R_{p_i}$.

Now, the result follows by the Chinese Remainder Theorem.

If, on the other hand, the sequence does not contain three elements coprime to every prime p_i, there is a prime p_l such that the sequence does not contain more than two elements coprime to it.

We remove those elements and consider a subsequence of $m+2(r-1)$ elements all whose elements are 0 in $\mathbb{Z}/p_l\mathbb{Z}$.

By the induction hypothesis, there will be a subsequence $(x_{i_1}, \cdots, x_{i_m})$ such that

$$\sum_{j=1}^{m} a_j^{(i)} \, x_{i_j} \equiv 0 \pmod{p_i},$$

with $a_j^{(i)} \in R_{p_i}$, for all $i \neq l$.

However,

$$\sum_{j=1}^{m} a_j^{(l)} \, x_{i_j} \equiv 0 \pmod{p_l},$$

where $a_j^{(l)} = 1$, for all j.

Once again, we are through by the Chinese Remainder Theorem.

Proof of Proposition 11.

Case 0 (When $n = 5$).

In this case, $r = 1$ and we are given a sequence $(x_1, \cdots, x_{m+2})$ where $m \geq 4$.

If there are at least four non-zero elements of $\mathbb{Z}/5\mathbb{Z}$ in the given sequence, the result is true by Lemma 9.

If there are not more than two non-zero elements, then the sequence had at least m multiples of 5 and the result follows for these elements and any choice of $a_i \in R_5$.

If there are exactly three non-zero elements of $\mathbb{Z}/5\mathbb{Z}$ in the given sequence, let x_1, x_2, x_3, be those three elements without loss of generality. Since $D_R(p) = 3$, for any prime p, where R is the set of quadratic residues modulo p (see Theorem 3 of [6]), we have $\sum_{i\in I} a_i x_i = 0$, $a_i \in R_5$ for some subset I of $\{1, 2, 3\}$ with $|I| \geq 2$.

Taking $(x_4, \cdots, x_t)$ with $t = m + (3 - |I|)$, we have

$$\sum_{i\in I} a_i x_i + \sum_{i=4}^{t} a_i x_i = 0,$$

where $a_4 = \cdots = a_t = 1$, thus giving us an m-sum with $a_i \in R_5$.

So let us now suppose that $n > 5$, that is, we have $r \geq 2$.

Let $n = 5n_1n_2$ where n_2 is the product of all primes $p|n$, $p \neq 5$ such that the sequence does not contain more than two elements coprime to p. We then remove a sequence of length $t \leq 2\omega(n_2) \leq 2r - 2$ so that each of the remaining elements are divisible by n_2.

Hence, we just have to prove the theorem for the new $N = 5n_1 = p_1 \cdots p_{r_1}$, and, in this case, we have a sequence $(x_1, \ldots, x_{m+2r_1})$ of at least $m + 2r_1$ elements containing at least three elements coprime to p for any prime $p|n_1$.

Case I (The sequence contains four units modulo 5).

Without loss of generality, let $S = (x_1, \cdots, x_t)$ be a subsequence of $t \leq 3r_1 + 1 \leq m$ elements such that S has three units corresponding to each prime p_i for $i = 2, \cdots r_1$, and four elements coprime to 5.

Then, by Lemma 8 and Lemma 9, we have

$$\sum_{j=1}^{m} a_j^{(i)}\ x_j \equiv 0 \,(\mathrm{mod}\ p_i)\,,$$

for each prime $p_i|N$, with $a_j^{(i)} \in R_{p_i}$ and the result follows by the Chinese Remainder Theorem.

Case II (The sequence contains at most two units modulo 5).

We remove the elements coprime with 5, and apply Proposition 10 to the remaining subsequence to obtain another one $x_{j_1}, \cdots, x_{j_m}$ with

$$\sum_{i=1}^{m} a_i \, x_{j_i} \equiv 0 \, (\mathrm{mod}\; n_1) \, ,$$

with $a_i \in R_{n_1}$. The result now follows since every element in this subsequence is a multiple of 5.

Case III (The sequence contains exactly three units modulo 5).

Let x_1, x_2, x_3 be those elements. Once again, since $D_R(p) = 3$, we have $\sum_{i \in I} a_i x_i \equiv 0 \pmod 5$, $a_i \in R_5$, for some subset I of $\{1, 2, 3\}$ with $|I| \geq 2$. If $|I| = 3$, we have a subsequence of length less than $3r_1$ and hence, not exceeding m, which will contain x_1, x_2, x_3 and three elements coprime to each of the remaining primes. We complete to a subsequence of length m, say $x_1, \cdots, x_m$.

Now, $\sum_{i=1}^{m} a_i \, x_i \equiv 0 \pmod 5$, where a_1, a_2, a_3 are as above and $a_4, \cdots, a_m \in R_5$ are chosen arbitrarily.

Applying Lemma 8 we get

$$\sum_{i=1}^{m} a_i^{(j)} \, x_i \equiv 0 \pmod{p_j},$$

with $a_i^{(j)} \in R_{p_j}$, for any prime $p_j | n_1$.

Now, the result follows by the Chinese Remainder Theorem.

If however, $|I| = 2$, let us suppose $1 \notin I$. Now, we remove x_1. Let $\hat{n}$ be the product of those primes $p | n_1$ such that, after removing x_1, there are only two elements coprime to p remaining. We remove all those coprime elements for these primes; in particular, we are removing less than $2\omega(\hat{n}) + 1$ elements in the whole process. If after this, there remains at most one unit modulo 5 we remove it. So, in total, we are removing at most $2\omega(\hat{n}) + 2$ elements, and now the result follows by Proposition 10. If after this, there remains two units modulo 5 we argue as in the previous case ($|I| = 3$), but for this new sequence and integer $n/\hat{n}$, which is enough since every remaining element is multiple of $\hat{n}$.

Deduction of Theorem 1 from Propositions 10 and 11.

Since, trivially, $n \geq 3r + 1$ we can apply Propositions 10 and 11 with $m = n$ to get

$$E_R(n) \leq n + 2r.$$

Moreover, it is easy to see that the sequence $x_1, y_1, \ldots, x_r, y_r$ given by $x_i = \frac{n}{p_i}u_i$, $y_i = -\frac{n}{p_i}v_i$, where u_i is a square modulo p_i and v_i is a non-square modulo p_i does not have any subsequence which sum up to zero with coefficients from R_n. Hence,

$$D_R(n) \geq 2r + 1.$$

Now, trivially

$$D_R(n) + n - 1 \leq E_R(n)$$

and so

$$n + 2r \leq D_R(n) + n - 1 \leq E_R(n) \leq n + 2r$$

which gives the result.

Remark 1. If $(n, 15) > 1$, we do not expect Theorem 1 to be true in general. In particular the sequence obtained by repeating 1 five times does not contain any subsequence whose sum is zero with coefficients squares of units modulo 15. We just have to note that such a subsequence, to be multiple of 3, would have exactly three elements. On the other hand, we can assume the squares modulo 5 to be ± 1. Then, the sum of any three elements would be $-3 \leq \sum a_i \leq 3$, and the only way to be a multiple of 5 is that it is 0, which needs an even number of ± 1.

Proof of Theorem 2.

By Erdős-Ginzburg-Ziv theorem [10] (can also see [1] or [14], for instance), given any five integers, there is a subsequence of three elements which sums up to $0 \pmod 3$.

Therefore given a sequence $(x_1, \cdots, x_{n+6r-4})$ of $n+6r-4$ elements of $\mathbb{Z}/n\mathbb{Z}$, we can pick up $t = p_2 \cdots p_r + 2(r-1)$ disjoint subsequences $I_1, I_2, \cdots, I_t$ one after another each of length 3 such that

$$\sum_{i \in I_j} x_i = 0 \pmod 3,$$

for $i = 1, 2, \cdots, t$. Now, considering the sequence $(y_1, \cdots, y_t)$ where $y_j = \sum_{i \in I_j} x_i$, by Theorem 1 there exists a subsequence $(y_{i_1}, \cdots, y_{i_l})$ with $l = p_2 \cdots p_r$ such that

$$\sum_{j=1}^{l} a_j y_{i_j} = 0 \pmod l,$$

with $a_j \in R_l$.

Now, observing that $y_j = \sum_{i \in I_j} x_i$, where $|I_j| = 3$ for each j, by the Chinese Remainder Theorem we get the result since $n = 3l$. From here we deduce the upper bound for $E_R(n)$ and, hence, the upper bound for $D_R(n)$ follows from the inequality $n-1+D_R(n) \leq E_R(n)$. For the lower bounds we just have to consider the analogous counterexample as in Theorem 1.

Proof of Theorem 3.

It is interesting to observe that in the case when $n = 3p$, for $p \geq 7$ prime, we again reach the identity of Theorem 1, $D_R(n) = 2r + 1 = 5$. Indeed, given a sequence $\{x_1, \ldots, x_5\}$, (in all the arguments we will assume that none of these elements is zero modulo $3p$), with at most two units modulo p, or at most two units modulo 3, then removing those elements, the result is true since $D_R(q) = 3$ for any prime q. Now suppose the sequence has at least three units modulo p and three units modulo 3. The interesting case is when the sequence has precisely three units modulo p. So suppose $p|(x_4, x_5)$, and hence, are coprime with 3. If $x_4 \equiv -x_5 \pmod 3$ then $x_4+x_5 = 0 \pmod{3p}$. Otherwise, since there are at least three units modulo 3, we can assume that $(x_3, 3p) = 1$. Then, for some $\{b_4, b_5\} \subset \{0, 1\}$ we have $x_1+x_2+x_3 \equiv -(b_4x_4+b_5x_5) \pmod 3$. We fix those b_i. On the other hand, there exist squares $a_i \in (\mathbb{Z}/p\mathbb{Z})^*$ for $i = 1, 2, 3$, such that $\sum_{i=1}^{3} a_ix_i \equiv -(b_4x_4 + b_5x_5) \equiv 0 \pmod p$. We just have to apply the Chinese remainder theorem to get the result.

When the sequence has five units modulo p the result is trivial by Lemma 8, since by Erdős-Ginzburg-Ziv theorem, the sum of three of them will be a multiple of 3.

If the sequence has exactly four units modulo p then suppose $p|x_5$ and $3 \nmid x_4x_5$. Then, as before, we will choose $\{b_4, b_5\} \subset \{0, 1\}$ so that $\sum_{i=1}^{3} a_ix_i \equiv -(b_4x_4 + b_5x_5) \equiv 0 \pmod{3p}$.

In this way we get $D_R(3p) \leq 5$, and we get the identity by using the same counterexample as in Theorem 1.

It is important to note that Theorem A does not apply because the only square modulo 3 is 1, so $a^2 - b^2$ will always be a multiple of 3.

Proofs of Theorems 4 and 5.

The proof of Theorem 4 relies on the trivial observation that given any three integers, there is a subsequence of two elements which sums up to 0 (mod 2). Similarly, for the proof of Theorem 5, one has to observe that by Erdős-Ginzburg-Ziv theorem, given any eleven integers, there is a subsequence of six elements which sums up to 0 (mod 6). Then, one has to follow the arguments as in the proof of Theorem 2.

Proof of Theorem 6.

Observe that, by Theorem A, we just have to prove $D_R(n) = 2r + 1$ since $\{1, 4\} \subset R$. Now, let $S = \{x_1 \dots, x_{2r+1}\} \subset \mathbb{Z}$. To prove the upper bound $D_R(n) \leq 2r + 1$ we note that three of the integers in S will be divisible by the same power of p so, without loss of generality, we can suppose that $\{y_1, y_2, y_3\} \subset (Z/p^r\mathbb{Z})^*$ where $y_i = x_i/p^\alpha$ for some $0 \leq \alpha \leq r - 1$. Then, by Theorem C we see that

$$|Ry_1 + Ry_2 \cup 0 + Ry_3 \cup 0| \geq \min\{n, 3|R|\} = n,$$

since $|R| = \frac{n}{2}(1 - \frac{1}{p})$, and $\frac{3}{2}n(1 - \frac{1}{p}) > n$ for any $p > 3$, and the result follows. Observe that $Ry \cup 0$ satisfies the conditions of Theorem C for any $y \in (Z/p^r\mathbb{Z})^*$. On the other hand, consider the set S, with $2r$ elements, given by $x_i = ap^{i/2}$, for even $0 \leq i < 2r$ and $x_i = -bx_{i-1}$, for odd $1 \leq i < 2r$ where $a \in R_p$ and $b \notin R_p$. We are going to prove, by induction on r, that this set does not contain any subset $S_0 \subset \{0, \dots 2r - 1\}$ such that $\sum_{i \in S_0} a_i x_i = 0$ for any $a_i \in R_n$. The case $r = 1$ is trivial. Now suppose that there exist such a set. Then, either $\{0, 1\} \subset S_0$, or $S_0 \cap \{0, 1\} = \emptyset$. In the first case we note that, then, we must have $a_0 x_0 + a_1 x_1 \equiv 0 \pmod{p}$ for some $\{a_0, a_1\} \subset R_p$ which is impossible since b is not a square. In the later, we just have to divide by p to note that $\sum_{i \in S_0} a_i(x_i/p) \equiv 0 \pmod{p^{r-1}}$, which is impossible by induction. This concludes the proof of the Theorem.

3. Other weights

In this section we include some zero sum results concerning different sets of weights. We start with the remark that Theorems 1, 2 and 3 remain true if we replace the set R_n by the set $S_n = \{a \in (\mathbb{Z}/n\mathbb{Z})^*, \left(\frac{a}{n}\right) = 1\}$, where $\left(\frac{a}{n}\right)$ is the Jacobi symbol. Indeed, R_n is a subset of S_n, which gives the upper bound. For the lower bound we just have to use the similar counterexample as in those theorems, using the sequence $x_1, y_1, \dots, x_r, y_r$ given by $x_i = \frac{n}{p_i}u_i$, $y_i = -\frac{n}{p_i}v_i$, where $\left(\frac{u_i}{p_i}\right) = -\left(\frac{v_i}{p_i}\right)$. On the other hand, it is interesting to observe that, $|S_n| = \varphi(n)/2$ whereas, in general, R_n gets much smaller when n is composite.

We now proceed to prove Theorem 7, where one considers a completely different set of weights.

Proof of Theorem 7.

For the proof of the first part we use the argument in [6]. Given a sequence $S = (s_1, \ldots, s_{\lceil \frac{n}{r} \rceil})$ we consider the sequence

$$S' = (s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots, s_{\lceil \frac{n}{r} \rceil}, \ldots, s_{\lceil \frac{n}{r} \rceil}),$$

where each element is repeated r times. Then $|S'| \geq n$, and noting that $D_1(n) \leq n$ we obtain

$$D_A(n) \leq \left\lceil \frac{n}{r} \right\rceil.$$

On the other hand, let us consider the sequence of $\lceil \frac{n}{r} \rceil - 1$ elements all equal to 1. Then, for any nonempty subsequence, $(s_{j_1}, \ldots, s_{j_l})$ and $a_i \in A$, $i = 1, \ldots, l$ we have

$$0 < \sum_{i=1}^{l} a_i s_{j_i} < rl \leq n - 1,$$

which gives us the lower bound,

$$D_A(n) \geq \left\lceil \frac{n}{r} \right\rceil,$$

and, hence, part one follows.

Noting that $\{1, 2\} \subset A$, the second part of the theorem is a consequence of Theorem A.

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