Average Frobenius Distributions of Elliptic Curves

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Abstract

Let E be an elliptic curve defined over $\mathbb{Q}$. For p a prime of good reduction, let $a_p(E)$ be the trace of the Frobenius morphism of the reduced curve E_p. The Lang-Trotter conjecture predicts the number of primes with a fixed value of $a_p(E)$. We prove in this paper average estimates which fit the estimate predicted by the conjecture, with an explicit constant confirming the conjectural constant of Lang and Trotter.

1 Introduction.

Let E be an elliptic curve defined over the rationals. For any prime p of good reduction, let E_p be the elliptic curve over $\mathbb{F}_p$ obtained by reducing E mod p. Let $a_p(E)$ be the trace of the Frobenius morphism of $E/\mathbb{F}_p$. Then, $\#E(\mathbb{F}_p) = p + 1 - a_p(E)$, and $|a_p(E)| \leq 2\sqrt{p}$. The case $a_p(E) = 0$ corresponds to supersingular reduction mod p.

For a fixed $r \in \mathbb{Z}$, what can be said about the number of primes p such that $a_p(E) = r$? If E has complex multiplication, Deuring showed that half of the primes are primes of supersingular reduction [3]. More precisely, let

$$\pi_E^r(x) = \#\left\{p \leq x : a_p\left(E\right) = r\right\}.$$

Then, if E has complex multiplication, $\pi_E^0(x) \sim \frac{1}{2}\ \pi(x)$ as $x \to \infty$. If E has complex multiplication and $r \neq 0$, the primes with a fixed trace of the Frobenius morphism are primes in quadratic progressions. For example, consider the elliptic curve $E : Y^2 = X^3 - X$ with complex multiplication by $\mathbb{Z}[i]$. It is easy to see that $a_p(E) = \pm 2$ if and only if $p = 1 + n^2$ for some integer n. If $q(n)$ is a quadratic progression, and

$$Q(x) = \#\left\{p \leq x : p = q(n) \text{ for some } n\right\},$$

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it was conjectured by Hardy and Littlewood [9] that $Q(x) \sim C \frac{\sqrt{x}}{\log x}$ as $x \to \infty$.

This conjecture is part of a more general conjecture of Lang and Trotter [11].

Conjecture 1.1 (Lang-Trotter Conjecture) *Let E be an elliptic curve defined over $\mathbb{Q}$, and let r be an integer. Except for the case where $r = 0$ and E has complex multiplication, there is a constant $C_{E,r}$ such that*

$$\pi_E^r(x) \sim C_{E,r} \frac{\sqrt{x}}{\log x} \qquad \text{as } x \to \infty. \tag{1}$$

Using their probabilistic model, Lang and Trotter gave an explicit description of the conjectural constant $C_{E,r}$ (see Section 2). The constant $C_{E,r}$ can be 0, and the asymptotic relation is then interpreted to mean that there is only a finite number of primes such that $a_p(E) = r$. It was shown by Elkies that this cannot happen when $r = 0$, i.e. that for any $E/\mathbb{Q}$, there are infinitely many primes of supersingular reduction [5]. However, if $r \neq 0$, there could be only finitely many primes p such that $a_p(E) = r$. For example, if $E/\mathbb{Q}$ has a rational torsion point of order t, then t divides $\#E(\mathbb{F}_p) = p + 1 - a_p(E)$ for all primes of good reduction, which imposes conditions on the values of $a_p(E)$.

We prove in this paper average estimates related to the Lang-Trotter conjecture. The average distribution fits the one predicted by the conjecture, and the conjectural constant $C_{E,r}$ of Lang and Trotter is confirmed by our results, as seen in Section 2. Average estimates for the case $r = 0$ were already obtained by Fouvry and Murty [6], and we obtain a generalization of their results for any $r \in \mathbb{Z}$. The techniques of Fouvry and Murty do not seem to extend to the general case $r \in \mathbb{Z}$. Our proof then differs significantly from theirs.

In the following, we fix $r \in \mathbb{Z}$ and we denote by $E(a,b)$ the elliptic curve $Y^2 = X^3 + aX + b$ with $a, b \in \mathbb{Z}$. Then

$$\pi_{E(a,b)}^r(x) = \# \left\{ p \leq x : a_p\left(E(a,b)\right) = r \right\}.$$

Following [11], we define

$$\pi_{1/2}(x) = \int_2^x \frac{dt}{2\sqrt{t}\log t} \sim \frac{\sqrt{x}}{\log x}.$$

Theorem 1.2 *Let r be an integer, $A, B \geq 1$. For every $c > 0$, we have*

$$\frac{1}{4AB} \sum_{|a| \leq A,\, |b| \leq B} \pi_{E(a,b)}^r(x) = C_r\, \pi_{1/2}(x) + O\left(\left(\frac{1}{A} + \frac{1}{B} \right) x^{3/2} + \frac{x^{5/2}}{AB} + \frac{\sqrt{x}}{\log^c x} \right) \tag{2}$$

where

$$C_r = \frac{2}{\pi} \prod_{l \mid r} \left(1 - \frac{1}{l^2} \right)^{-1} \prod_{l \nmid r} \frac{l(l^2 - l - 1)}{(l-1)(l^2-1)}. \tag{3}$$

The constants in the O-symbol depend only on c and r.

As the infinite products of (3) converge to positive numbers, the constant C_r is non-zero, even if some $C_{E,r}$ can be 0 as mentioned above.

From the last theorem, we immediately obtain that the Lang-Trotter conjecture is true "on average", i.e.

Corollary 1.3 *Let $\epsilon > 0$. If $A, B > x^{1+\epsilon}$, we have as $x \to \infty$,*

$$\frac{1}{4AB} \sum_{|a| \le A,\, |b| \le B} \pi^r_{E(a,b)}(x) \sim C_r \, \frac{\sqrt{x}}{\log x}. \qquad \square$$

In analogy with the classical terminology, we can say that the *average order* of $\pi^r_{E(a,b)}(x)$ is $C_r \frac{\sqrt{x}}{\log x}$. Using the same techniques, we can also prove that the *normal order* of $\pi^r_{E(a,b)}(x)$ is $C_r \frac{\sqrt{x}}{\log x}$. Then, $\pi^r_{E(a,b)}(x) \sim C_r \frac{\sqrt{x}}{\log x}$ for "almost all" $E(a,b)$ rather than on average (see Corollary 1.5). We are grateful to A. Granville for suggesting this application of our techniques.

Theorem 1.4 *Let $\epsilon > 0$. If $A, B > x^{1+\epsilon}$, then for every $c > 0$, we have*

$$\frac{1}{4AB} \sum_{|a| \le A,\, |b| \le B} \left| \pi^r_{E(a,b)}(x) - C_r \pi_{1/2}(x) \right|^2 = O\left(\frac{x}{\log^c x} + \left(\frac{1}{A} + \frac{1}{B} \right) x^3 + \frac{1}{AB} x^5 \right) \qquad (4)$$

where the constant in the O-symbol depends only on c and r.

The following Corollary is a standard application of the Turán normal order method.

Corollary 1.5 *Let $\epsilon > 0$ and fix $c > 0$. If $A,\ B > x^{2+\epsilon}$, then for all $d > 2c$ and for all elliptic curves $E(a,b)$ with $|a| \le A$ and $|b| \le B$ with at most $O(AB/\log^d x)$ exceptions, we have the inequality*

$$\left| \pi^r_{E(a,b)}(x) - C_r \ \pi_{1/2}(x) \right| \ll \frac{\sqrt{x}}{\log^c x}. \qquad \square$$

In Section 2, we compare the constant C_r with the constants $C_{E,r}$ predicted by Lang and Trotter. Sections 3 and 4 contain the proof of Theorem 1.2, and Section 5 contains the proof of Theorem 1.4.

2 The Lang-Trotter constant $C_{E,r}$.

To formulate their conjecture, Lang and Trotter considered a probabilistic model compatible with the Cebotarev Density Theorem and with the Sato-Tate conjecture. From the model, they obtained an explicit description of the constant $C_{E,r}$ in terms of Galois representations, as described below. We compare in this section the conjectural constants $C_{E,r}$ of Lang and Trotter with the constant C_r of Theorem 1.2.

Let $\rho_{E,m}$ be the Galois representation

$$\rho_{E,m} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{Aut}(E[m])$$

where $E[m]$ is the subgroup of m-torsion points of $E(\overline{\mathbb{Q}})$. Since $E[m] \simeq (\mathbb{Z}/m\mathbb{Z})^2$, after choosing a basis for $E[m]$, we can identify $\mathrm{Aut}(E[m])$ with $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$. Let $G(m)$ be the image of $\rho_{E,m}$ in $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$, and for any subgroup G of $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$, let G_r be the subset of elements of G of trace r modulo m.

Let E be an elliptic curve without complex multiplication. Serre proved in [13] that the image of the Galois representation on the full torsion subgroup of $E(\overline{\mathbb{Q}})$ is an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$. It follows that there exists an integer m_E such that $\rho_{E,l}$ is surjective for all primes l not dividing m_E, and such that the image in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of the Galois representation on the torsion subgroup of $E(\overline{\mathbb{Q}})$ is the full inverse image of $G(m_E)$. The Lang-Trotter constant $C_{E,r}$ is then defined as [11, p. 36]

$$\begin{aligned} C_{E,r} &= \frac{2}{\pi} \frac{m_E\,|G(m_E)_r|}{|G(m_E)|} \prod_{l \nmid m_E} \frac{l\,|G(l)_r|}{|G(l)|} \\ &= \frac{2}{\pi} \frac{m_E\,|G(m_E)_r|}{|G(m_E)|} \prod_{\substack{l \nmid m_E \\ l \mid r}} \left(1 - \frac{1}{l^2}\right)^{-1} \prod_{\substack{l \nmid m_E \\ l \nmid r}} \frac{l(l^2-l-1)}{(l-1)(l^2-1)}. \end{aligned} \tag{5}$$

The second equality follows from the easy estimates

$$\frac{l\,|\mathrm{GL}_2(\mathbb{F}_l)_r|}{|\mathrm{GL}_2(\mathbb{F}_l)|} = \begin{cases} \dfrac{l^3(l-1)}{l(l-1)^2(l+1)} & \text{when } r \equiv 0\ (l); \\[2ex] \dfrac{l^2(l^2-l-1)}{l(l-1)^2(l+1)} & \text{when } r \not\equiv 0\ (l). \end{cases}$$

Comparing (3) and (5), we see that the local factors are exactly the same for the primes $l \nmid m_E$. More precisely, is that true that

$$\frac{1}{4AB} \sum_{|a| \le A,\, |b| \le B} C_{E(a,b),r} \sim C_r \qquad \text{as } A, B \to \infty?$$

There are partial results for the case $r = 0$ due to Fouvry and Ullmo [7]. The recent estimates of Duke [4], who showed that for "most" elliptic curves $E/\mathbb{Q}$, $\rho_{E,l}(G) = \mathrm{GL}_2(\mathbb{F}_l)$ for all primes l, are also relevant to this problem. But this does not imply that $m_E = 1$ (and then $C_{E,r} = C_r$) for those curves. In fact, we never have $m_E = 1$ as shown in [13, Proposition 22].

3 An average of special value of L-series.

We will show in Section 4 how the average

$$\frac{1}{4AB} \sum_{|a| \leq A \, |b| \leq B} \pi^r_{E(a,b)}(x)$$

can be rewritten as an average of special values of Dirichlet L-series by counting the number of curves over the finite fields $\mathbb{F}_p$ with $a_p = r$. The proof of Theorem 1.2 is then obtained by studying this average of L-series, which we evaluate in this section.

Let $B(r) = \max(3, r, r^2/4)$. In particular, $p > B(r)$ ensures $|r| \leq 2\sqrt{p}$, a necessary condition for $a_p(E) = r$.

For $d \equiv 0, 1(4)$, d not a perfect square, and $n \neq 0$, let $\chi_d(n) = \left(\frac{d}{n}\right)$ be the Kronecker symbol (see for example [10, p. 304]). The Kronecker symbol is a real character modulo $|d|$, and for $n > 0$

$$\begin{aligned} d_1 \equiv d_2 \ (n) &\Rightarrow \left(\frac{d_1}{n}\right) = \left(\frac{d_2}{n}\right) \quad \text{for } n \text{ odd} \\ d_1 \equiv d_2 \ (4n) &\Rightarrow \left(\frac{d_1}{n}\right) = \left(\frac{d_2}{n}\right) \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{6}$$

We give the proof of Theorem 1.2 for r odd. The proof is similar when r is even therefore we omit it.

The main result of this section is

Theorem 3.1 *Let r be an odd integer, and let*

$$K_r = \sum_{\substack{f=1 \\ (2r,f)=1}}^{\infty} \sum_{n=1}^{\infty} \frac{c_f^r(n)}{nf\varphi(nf^2)} \quad \text{with} \quad c_f^r(n) = \sum_{\substack{a(4n)^* \\ (r^2-af^2,4n)=4}} \left(\frac{a}{n}\right), \tag{7}$$

where $\sum_{a(4n)^*}$ is the sum over a complete set of invertible residues mod $4n$.

Furthermore let

$$\mathcal{S}_f(x) = \left\{ B(r) < p \leq x \;\middle|\; 4p \equiv r^2 \ (f^2), \text{ and } d = \frac{r^2 - 4p}{f^2} \equiv 0, 1 \ (4) \right\}.$$

Then for any $c > 0$

$$\sum_{f \leq 2\sqrt{x}} \frac{1}{f} \sum_{p \in \mathcal{S}_f(x)} L(1, \chi_d) \log p = K_r x + O\left(\frac{x}{\log^c x}\right). \tag{8}$$

Proof of Theorem 3.1: As r is odd, if $f^2 \mid r^2 - 4p$ then f is odd, and $d = (r^2 - 4p)/f^2 \equiv 1\ (4)$. Furthermore, since $(r, f) \mid p$, and $p > B(r)$, we have $(r, f) = 1$.

For a fixed parameter $U > 0$ to be chosen later, we have

$$\begin{aligned} L(1, \chi_d) = \sum_{n \geq 1} \left(\frac{d}{n}\right) \frac{1}{n} &= \sum_{n \leq U} \left(\frac{d}{n}\right) \frac{1}{n} + O\left(\frac{\sqrt{|d|} \log |d|}{U}\right) \\ &= \sum_{n \leq U} \left(\frac{d}{n}\right) \frac{1}{n} + O\left(\frac{\sqrt{p} \log p}{fU}\right) \end{aligned} \tag{9}$$

using the Polya-Vinogradov inequality [2, p. 135].

Using (9), we can rewrite the left hand side of (8) as

$$\sum_{\substack{f \leq 2\sqrt{x} \\ (2r,f)=1}} \frac{1}{f} \sum_{n \leq U} \frac{1}{n} \sum_{p \in \mathcal{S}_f(x)} \left(\frac{d}{n}\right) \log p + O\left(\frac{x^{3/2} \log x}{U}\right). \tag{10}$$

For a fixed parameter V with $1 \leq V \leq 2\sqrt{x}$ to be chosen later, the first part of (10) is

$$\sum_{\substack{f \leq V \\ (2r,f)=1}} \frac{1}{f} \sum_{n \leq U} \frac{1}{n} \sum_{p \in \mathcal{S}_f(x)} \left(\frac{d}{n}\right) \log p + \sum_{\substack{V < f \leq 2\sqrt{x} \\ (2r,f)=1}} \frac{1}{f} \sum_{n \leq U} \frac{1}{n} \sum_{p \in \mathcal{S}_f(x)} \left(\frac{d}{n}\right) \log p.$$

The summation for large values of f is easily evaluated as

$$\begin{aligned} \left| \sum_{\substack{V < f \ll 2\sqrt{x} \\ (2r,f)=1}} \frac{1}{f} \sum_{n \leq U} \frac{1}{n} \sum_{p \in \mathcal{S}_f(x)} \left(\frac{d}{n}\right) \log p \right| &\leq \log x \ \log U \sum_{V < f \leq 2\sqrt{x}} \frac{1}{f} \sum_{\substack{n \leq x \\ n \equiv 4^* r^2 (f^2)}} 1 \\ &\ll x \log x \log U \sum_{V < f \leq 2\sqrt{x}} \frac{1}{f^3} \ll \frac{x \log x \log U}{V^2} \end{aligned}$$

where 4^* is an integer such that $4^* \cdot 4 \equiv 1\ (f^2)$.

Therefore we can rewrite the left hand side of (8) as

$$\sum_{\substack{f \leq V \\ (2r,f)=1}} \frac{1}{f} \sum_{n \leq U} \frac{1}{n} \sum_{p \in \mathcal{S}_f(x)} \left(\frac{d}{n}\right) \log p + O\left(\frac{x^{3/2} \log x}{U} + \frac{x \log x \log U}{V^2}\right). \tag{11}$$

The sum over "small values" of f and n leads to the main term. It is evaluated by splitting the sum according to the residue of d mod $4n$. Since $d = (r^2 - 4p)/f^2$ is odd, and

$\left(\frac{d}{n}\right) = 0$ when $(d,n) > 1$, using(6) we get

$$\sum_{\substack{f \le V \\ (2r,f)=1}} \frac{1}{f} \sum_{n \le U} \frac{1}{n} \sum_{p \in \mathcal{S}_f(x)} \left(\frac{d}{n}\right) \log p = \sum_{\substack{n \le U, f \le V \\ (2r,f)=1}} \frac{1}{fn} \sum_{a(4n)^*} \left(\frac{a}{n}\right) \sum_{\substack{p \in \mathcal{S}_f(x) \\ d \equiv a(4n)}} \log p. \tag{12}$$

In the above sum, the two conditions $p \in \mathcal{S}_f(x)$ and $d = (r^2 - 4p)/f^2 \equiv a \ (4n)$ are equivalent to $B(r) < p \le x$ and $p \equiv (r^2 - af^2)/4 \ (nf^2)$. Furthermore, as $(2r, f) = 1$, $((r^2 - af^2)/4, nf^2) = 1 \iff (r^2 - af^2, 4n) = 4$.

We use the following standard notation

$$\begin{aligned} \psi_1(x; n, a) &= \sum_{\substack{p \le x \\ p \equiv a \ (n)}} \log p \\ E_1(x; n, a) &= \psi_1(x; n, a) - \frac{x}{\varphi(n)} \quad \text{for } (a, n) = 1. \end{aligned}$$

Lemma 3.2 (Theorem of Barban, Davenport and Halberstam)
With the notation above, for any $K > 0$ and $x/\log^K x \le Q \le x$, we have

$$\sum_{n \le Q} \sum_{a(n)^*} E_1^2(x; n, a) \ll Qx \log x. \qquad \square$$

This classical result can be found, for example, in Davenport [2, p. 169].

We rewrite (12) as

$$\sum_{\substack{n \le U, f \le V \\ (2r,f)=1}} \frac{1}{fn} \sum_{a(4n)^*} \left(\frac{a}{n}\right) \psi_1\left(x; nf^2, \frac{r^2 - af^2}{4}\right) + O\left(U \log V\right),$$

where the term $O(U \log V)$ comes from the primes less than $B(r)$, and the O-constant depends on r only. Using the notation defined above, we rewrite the last equation as

$$\begin{aligned} & x \sum_{\substack{n \le U, f \le V \\ (2r,f)=1}} \frac{c_f^r(n)}{fn\varphi(nf^2)} + O(U \log V) \\ & \quad + \sum_{\substack{n \le U, f \le V \\ (2r,f)=1}} \frac{1}{fn} \sum_{\substack{a(4n)^* \\ (r^2 - af^2, 4n) = 4}} \left(\frac{a}{n}\right) E_1\left(x; nf^2, \frac{r^2 - af^2}{4}\right). \end{aligned} \tag{13}$$

The second sum of (13) is dominated by the error term. Indeed, using the Cauchy-Schartz inequality, we bound it by

$$\sum_{\substack{f\le V\\(2r,f)=1}} \frac{1}{f}\left(\sum_{n\le U}\frac{\varphi(4n)}{n^2}\right)^{1/2}\left(\sum_{n\le U}\sum_{\substack{a\,(4n)^*\\(r^2-af^2,4n)=4}} E_1^2\left(x;nf^2,\frac{r^2-af^2}{4}\right)\right)^{1/2}$$

$$\le \quad (\log U)^{1/2}\sum_{\substack{f\le V\\(2r,f)=1}} \frac{1}{f}\left(\sum_{n\le U}\sum_{\substack{a\,(4n)^*\\(r^2-af^2,4n)=4}} E_1^2\left(x;nf^2,\frac{r^2-af^2}{4}\right)\right)^{1/2}$$

$$\le \quad (\log U)^{1/2}\sum_{\substack{f\le V\\(2r,f)=1}} \frac{1}{f}\left(\sum_{n\le U}\sum_{b(nf^2)^*} E_1^2(x;nf^2,b)\right)^{1/2},$$

as $a_1 \not\equiv a_2 \ (4n)$ ensures $b_1 = \dfrac{r^2-a_1f^2}{4} \not\equiv b_2 = \dfrac{r^2-a_2f^2}{4} \ (nf^2)$.

Fix any $c>0$. Then the last sum is bounded by

$$\le \log V\,(\log U)^{1/2}\left(\sum_{n\le UV^2}\sum_{a(n)^*} E_1^2(x;n,a)\right)^{1/2},$$

which is

$$\le \log V\,(\log U)^{1/2}\ \frac{x}{\log^{c+2} x} \tag{14}$$

when

$$UV^2 \le \frac{x}{\log^{B(c)} x} \tag{15}$$

from Lemma 3.2, with $B(c)=2c+6$.

Finally, using (11), (13) and (14), we obtain

$$\sum_{f\le 2\sqrt{x}} \frac{1}{f}\sum_{p\in\mathcal{S}_f(x)} L(1,\chi_d)\ \log p = x\sum_{\substack{n\le U, f\le V\\(2r,f)=1}} \frac{c_f^r(n)}{fn\varphi(nf^2)}$$

$$+O\left(U\log V + \frac{x\log V\log^{1/2} U}{\log^{c+2} x} + \frac{x^{3/2}\log x}{U} + \frac{x\log x\log U}{V^2}\right) \tag{16}$$

for any U,V satisfying (15).

In order to find the asymptotic behavior of the main term, we have to estimate the growth of

$$c_f^r(n) = \sum_{\substack{a\,(4n)^* \\ (r^2-af^2,4n)=4}} \left(\frac{a}{n}\right).$$

Let $\kappa(n)$ be the multiplicative arithmetic function generated by the identity

$$\kappa(l^\alpha) = \begin{cases} l & \alpha \text{ odd} \\ 1 & \alpha \text{ even} \end{cases}$$

for any prime l and any positive integer α. Then for a positive integer n, $\kappa(n)$ is the smallest integer dividing n such that $n/\kappa(n)$ is a square.

Lemma 3.3 *The following hold:*

1. *if n is odd, then* $c_f^r(n) = \displaystyle\sum_{\substack{a\,(n)^* \\ (r^2-af^2,n)=1}} \left(\frac{a}{n}\right)$;
2. *$c_f^r(n)$ is a multiplicative function of n;*
3. *for any prime l, $c_f^r(l^\alpha) = c_{(f,l)}^r(l^\alpha)$;*
4. *if $\alpha \geq 1$, then $c_1^r(2^\alpha) = (-2)^\alpha/2$;*
5. *if l is an odd prime, then* $\frac{c_1^r(l^\alpha)}{l^{\alpha-1}} = \begin{cases} l-1-\left(\frac{r^2}{l}\right) & \text{if } \alpha \text{ is even} \\ -\left(\frac{r^2}{l}\right) & \text{if } \alpha \text{ is odd;} \end{cases}$
6. *if l is an odd prime ($l \nmid r$), then* $\frac{c_l^r(l^\alpha)}{l^{\alpha-1}} = \begin{cases} 0 & \text{if } \alpha \text{ is odd} \\ l-1 & \text{if } \alpha \text{ is even;} \end{cases}$
7. *for all n, $\left|c_f^r(n)\right| \leq \frac{n}{\kappa(n)}$.*

Proof: 1. By definition,

$$c_f^r(n) = \sum_{\substack{a\,(4n)^*, a\equiv 1(4) \\ (r^2-af^2,4n)=4}} \left(\frac{a}{n}\right) + \sum_{\substack{a\,(4n)^*, a\equiv 3(4) \\ (r^2-af^2,4n)=4}} \left(\frac{a}{n}\right).$$

The second sum is empty since $r^2+f^2 \equiv 2 \ (4)$. If $a \equiv 1 \ (4)$, then $(r^2-af^2,4n)=4$ if and only if $(r^2-af^2,n)=1$. This gives

$$c_f^r(n) = \sum_{\substack{a\,(4n)^*, a\equiv 1(4) \\ (r^2-af^2,n)=1}} \left(\frac{a}{n}\right).$$

As n is odd, there is a bijection between the invertible residues modulo $4n$ which are congruent to 1 modulo 4 and the residues modulo n. We then use the property (6) of the Kronecker symbols to deduce the claim;

2. Clearly, $c_f^r(1) = 1$. Let $n = n_1 n_2$ with $(n_1, n_2) = 1$. We can suppose without loss of generality that $2 \not| n_1$. By 1., we have

$$c_f^r(n_1)\, c_f^r(n_2) = \sum_{\substack{a_1 (n_1)^* \\ (r^2 - a_1 f^2, n_1) = 1}} \sum_{\substack{a_2 (4n_2)^* \\ (r^2 - a_2 f^2, 4n_2) = 4}} \left(\frac{a_1}{n_1}\right)\left(\frac{a_2}{n_2}\right). \tag{17}$$

Now for any a_1 and a_2 in the above sums, let a be the unique integer such that $1 \leq a \leq 4n$, $(a, 4n) = 1$, $a = a_1 + k_1 n_1 = a_2 + k_2 4n_2$ for some integers k_1 and k_2.

It is easy to see that $(r^2 - a_1 f^2, n_1) = 1$ and $(r^2 - a_2 f^2, 4n_2) = 4$ if and only if $(r^2 - af^2, 4n) = 4$. Therefore we can write the right hand side of (17) as

$$\sum_{\substack{a_1 (n_1)^*,\ a_2 (4n_2)^* \\ (r^2 - af^2, 4n) = 4}} \left(\frac{a}{n_1}\right)\left(\frac{a}{n_2}\right) = \sum_{\substack{a_1 (n_1)^*,\ a_2 (4n_2)^* \\ (r^2 - af^2, 4n) = 4}} \left(\frac{a}{n}\right).$$

The statement now follows by the Chinese Remainder Theorem;

3. If $(f, l) = 1$ then the two sets $\{a \mid a(\mathrm{mod} 4l^\alpha),\ (a, 4l^\alpha) = 1\}$ and $\{af^2 \mid a(\mathrm{mod} 4l^\alpha),\ (a, 4l^\alpha) = 1\}$ are equal. So, if l is odd,

$$c_f^r(l^\alpha) = \sum_{\substack{a (l^\alpha)^* \\ (r^2 - af^2, l) = 1}} \left(\frac{a}{l}\right)^\alpha = \sum_{\substack{b (l^\alpha)^* \\ (r^2 - b, l) = 1}} \left(\frac{b}{l}\right)^\alpha \left(\frac{f^*}{l}\right)^{2\alpha} = c_1^r(l^\alpha) \tag{18}$$

where $f^* f \equiv 1\ (l)$. The proof is similar for $l = 2$. If $(f, l) = l$, then l is odd and since $(r, f) = 1$, we have that $(r^2 - af^2, l) = 1$ for all a invertible residues modulo l^α. Therefore,

$$c_f^r(l^\alpha) = \sum_{a (l^\alpha)^*} \left(\frac{a}{l}\right)^\alpha = c_l^r(l^\alpha); \tag{19}$$

4. Since $\left(\frac{a_1}{2}\right) = \left(\frac{a_2}{2}\right)$ when $a_1 \equiv a_2\ (8)$, we have

$$c_1^r(2^\alpha) = 2^{\alpha - 1} \sum_{\substack{a (8)^* \\ (r^2 - a, 8) = 4}} \left(\frac{a}{2}\right)^\alpha.$$

Now $r^2 \equiv 1\ (8)$, so $(r^2 - a, 8) = 4$ if and only if $a \equiv 5\ (8)$. Therefore

$$c_1^r(2^\alpha) = 2^{\alpha - 1} \left(\frac{5}{2}\right)^\alpha = (-2)^\alpha / 2;$$

5. Using (18), we write

$$c_1^r(l^\alpha) = l^{\alpha-1} \sum_{\substack{a(l)^* \\ (r^2-a,l)=1}} \left(\frac{a}{l}\right)^\alpha = l^{\alpha-1} \sum_{a(l)^*} \left(\frac{a}{l}\right)^\alpha - l^{\alpha-1} \left(\frac{r^2}{l}\right)$$

and the claim is deduced from the orthogonality relations of the Legendre symbols;

6. is a consequence of (19), by the orthogonality relations of the Legendre symbols;

7. is a consequence of 2., 4., 5. and 6. $\square$

Lemma 3.4 *Let* $c = \prod_{l \text{ prime}} \left(1 + \frac{1}{l(\sqrt{l}-1)}\right)$. *Then*

$$\sum_{n>U} \frac{1}{\kappa(n)\varphi(n)} \sim \frac{c}{\sqrt{U}}.$$

In particular, $\sum_{n=1}^{\infty} \frac{1}{\kappa(n)\varphi(n)}$ *converges.*

Proof: Let

$$C(t) = \sum_{n \le t} \frac{n^{3/2}}{\kappa(n)\varphi(n)}.$$

Using the partial summation formula [12, Exercise 1.1], we immediately deduce that

$$\sum_{n>U} \frac{1}{\kappa(n)\varphi(n)} = \frac{3}{2} \int_U^\infty \frac{C(t)}{t^{5/2}} + \lim_{N\to\infty} \frac{C(N)}{N^{3/2}} - \frac{C(U)}{U^{3/2}}. \tag{20}$$

We claim that the following asymptotic formula holds:

$$C(t) \sim \frac{c}{2} t. \tag{21}$$

The lemma then follows easily by substituting (21) in (20).

To prove (21), we consider the Dirichlet series

$$K(s) = \sum_{n=1}^{\infty} \frac{n^{3/2}}{\kappa(n)\varphi(n)} n^{-s},$$

which clearly converges for $\Re(s) > 5/2$. Since both κ and φ are multiplicative functions, a straightforward computation gives the Euler product expansion

$$K(s) = \prod_{l \text{ prime}} \left(1 + \frac{l(l^{s-3/2}+1)}{(l-1)(l^{2s-1}-1)}\right).$$

This shows that $K(s)$ converges for $\Re(s) > 1$.

By computing the product

$$K(s) \cdot \frac{1}{\zeta(2s-1)} = \prod_{l \text{ prime}} \left(1 + \frac{1 + l^{s-1/2}}{l^{2s-1}(l-1)}\right)$$

which converges for $\Re(s) > 1/2$, we deduce that $K(s)$ admits a meromorphic continuation in the half plane $\Re(s) > 1/2$, with only a simple pole at $s = 1$ and residue

$$\frac{1}{2} \prod_{l \text{ prime}} \left(1 + \frac{1}{l(\sqrt{l}-1)}\right).$$

Since $K(s)$ is regular on the vertical line $\Re(s) = 1$ ($s \neq 1$), we apply the Wiener-Ikehara Tauberian Theorem (see for example [12] Theorem 1.1) to $K(s)$ to deduce (21). This proves the lemma. $\square$

Theorem 3.1 now follows easily from Lemmas 3.3 and 3.4, as

$$x \sum_{\substack{n \leq U, f \leq V \\ (2r,f)=1}} \frac{c_f^r(n)}{fn\varphi(nf^2)} = x \sum_{\substack{f \leq V \\ (2r,f)=1}} \sum_{n=1}^{\infty} \frac{c_f^r(n)}{fn\varphi(nf^2)} + O\left(x \sum_{f \leq V} \frac{1}{f} \sum_{n > U} \frac{1}{\kappa(n)\varphi(nf^2)}\right)$$

from Lemma 3.3(7). But $\varphi(nf^2) \geq \varphi(n)\varphi(f^2)$, which gives

$$O\left(x \sum_{f \leq V} \frac{1}{f} \sum_{n > U} \frac{1}{\kappa(n)\varphi(nf^2)}\right) = O\left(\frac{x}{U^{1/2}}\right)$$

from Lemma 3.4. Finally,

$$\begin{aligned}
x \sum_{\substack{n \leq U, f \leq V \\ (2r,f)=1}} \frac{c_f^r(n)}{fn\varphi(nf^2)} &= x \sum_{\substack{f=1 \\ (2r,f)=1}}^{\infty} \sum_{n=1}^{\infty} \frac{c_f^r(n)}{fn\varphi(nf^2)} \\
&\quad + O\left(x \sum_{f > V} \frac{1}{f\varphi(f^2)} \sum_{n=1}^{\infty} \frac{1}{\kappa(n)\varphi(n)}\right) + O\left(\frac{x}{U^{1/2}}\right) \\
&= K_r x + O\left(\frac{x}{V^2}\right) + O\left(\frac{x}{U^{1/2}}\right) \qquad (22)
\end{aligned}$$

This completes the proof of the theorem. Indeed, from (16) and (22)

$$\begin{aligned}
\sum_{f \leq 2\sqrt{x}} \frac{1}{f} \sideset{}{^*}\sum_{p \leq x} L(1, \chi_d) \log p &= K_r x + O\left(\frac{x}{V^2} + \frac{x}{U^{1/2}}\right) \\
&\quad + O\left(U \log V + \frac{x \log V (\log U)^{1/2}}{\log^{c+2} x} + \frac{x^{3/2} \log x}{U} + \frac{x \log x \log U}{V^2}\right)
\end{aligned}$$

for any U, V satisfying (15). Choosing

$$\begin{array}{rcl} U & = & \sqrt{x}\log^{c+1} x \\ V & = & (\log x)^{1/2(c+2)}, \end{array}$$

we deduce the result. $\square$

4 Proof of Theorem 1.2.

For $r \leq 2\sqrt{p}$, the number of $\mathbb{F}_p$-isomorphism classes of elliptic curves over $\mathbb{F}_p$ with $p+1-r$ points is the total number of ideal classes of the ring $\mathbb{Z}\left[\left(D+\sqrt{D}\right)/2\right]$ where $D = r^2 - 4p$ is a negative integer which is congruent to 0 or 1 modulo 4. This total number of ideal classes is the Kronecker class number

$$H(r^2-4p) = 2 \sum_{\substack{f^2 | r^2-4p \\ d \equiv 0,1(4)}} \frac{h(d)}{w(d)}, \tag{23}$$

where the sum ranges over positive integers f such that f^2 divides r^2-4p, and $d = (r^2-4p)/f^2$ is congruent to 0 or 1 modulo 4. As usual, $h(d)$ and $w(d)$ denote the class number and the number of units respectively, of the order of discriminant d.

Suppose that $p \neq 2, 3$. Then, any elliptic curve over $\mathbb{F}_p$ has a model

$$E : Y^2 = X^3 + aX + b$$

with $a, b \in \mathbb{F}_p$. The elliptic curves $E'(a', b')$ over $\mathbb{F}_p$ which are $\mathbb{F}_p$-isomorphic to E are given by all the choices

$$a' = \mu^4 a \text{ and } b' = \mu^6 b$$

with $\mu \in \mathbb{F}_p^*$. The number of such E' is

$$\begin{array}{ll} (p-1)/6 & \text{when } a = 0 \text{ and } p \equiv 1(3); \\ (p-1)/4 & \text{when } b = 0 \text{ and } p \equiv 1(4); \\ (p-1)/2 & \text{otherwise.} \end{array}$$

Then, the number of curves $E(a, b)$ with $a, b \in \mathbb{Z}$, $0 \leq a, b < p$ and $a_p(E(a, b)) = r$ is

$$H(r^2-4p)\left(\frac{p-1}{2}\right) + O(p) = \frac{p\,H(r^2-4p)}{2} + O(p) \tag{24}$$

This result can be found in Birch [1].

We then write

$$\frac{1}{4AB}\sum_{|a|\le A,\,|b|\le B}\pi^r_{E(a,b)}(x) \quad = \quad \frac{1}{4AB}\sum_{p\le x}\#\left\{|a|\le A, |b|\le B : a_p\left(E(a,b)\right)=r\right\}$$

as

$$\frac{1}{4AB}\sum_{B(r)<p\le x}\left(\frac{2A}{p}+O(1)\right)\left(\frac{2B}{p}+O(1)\right)\left(\frac{p\,H(r^2-4p)}{2}+O(p)\right).$$

This last equation can be rewritten as

$$\frac{1}{2}\sum_{B(r)<p\le x}\frac{H(r^2-4p)}{p}+O\left(\sum_{B(r)<p\le x}H(r^2-4p)\left(\frac{1}{A}+\frac{1}{B}+\frac{p}{AB}\right)\right)+O\left(\log\log x\right). \quad (25)$$

Using

$$H(r^2-4p)\le\sum_{\substack{f^2|r^2-4p\\ d\equiv 0,1(4)}}h(d)\ll\sqrt{p}\log p\sum_{f^2|r^2-4p}\frac{1}{f},$$

and the Brun-Titchmarsh Theorem (see for example [8]), we have the estimates

$$\begin{aligned}\sum_{p\le x}H(r^2-4p) &\ll \sqrt{x}\log x\sum_{p\le x}\sum_{f^2|r^2-4p}\frac{1}{f}\\ &\ll \sqrt{x}\log x\sum_{f\le 2\sqrt{x}}\frac{1}{f\varphi(f)}\frac{x}{\log x}\ll x^{3/2} \qquad (26)\\ \sum_{p\le x}pH(r^2-4p) &\ll x^{5/2}. \qquad (27)\end{aligned}$$

Finally, replacing these estimates in (25), we have

$$\begin{aligned}\frac{1}{4AB}\sum_{|a|\le A,\,|b|\le B}\pi^{(r)}_{E(a,b)}(x) &= \frac{1}{2}\sum_{B(r)<p\le x}\frac{H(r^2-4p)}{p}\\ &+ O\left(\left(\frac{1}{A}+\frac{1}{B}\right)x^{3/2}+\frac{x^{5/2}}{AB}+\log\log x\right) \qquad (28)\end{aligned}$$

We now use Theorem 3.1 to evaluate the main term of (28). Using (23), we write

$$\frac{1}{2}\sum_{B(r)<p\le x}\frac{H(r^2-4p)}{p}=\sum_{f\le 2\sqrt{x}}\sum_{p\in\mathcal{S}_f(x)}\frac{1}{p}\frac{h(d)}{w(d)}, \quad (29)$$

where

$$\mathcal{S}_f(x) = \{B(r) < p \le x \mid 4p - r^2 \equiv 0 \ (f^2), \text{ and } d = \frac{r^2 - 4p}{f^2} \equiv 0, 1 \ (4)\}.$$

As $d = (r^2 - 4p)/f^2$ is a negative integer, the class number formula reads as

$$h(d) = \frac{w(d)|d|^{1/2}}{2\pi} L(1, \chi_d), \tag{30}$$

where χ_d is the Kronecker symbol defined in Section 3. Therefore, replacing (30) in (29), we have

$$\begin{aligned} \frac{1}{2} \sum_{B(r)<p\le x} \frac{H(r^2 - 4p)}{p} &= \frac{1}{2\pi} \sum_{f \le 2\sqrt{x}} \frac{1}{f} \sum_{p \in \mathcal{S}_f(x)} \frac{\sqrt{4p - r^2}}{p} L(1, \chi_d) \\ &= \frac{1}{\pi} \sum_{f \le 2\sqrt{x}} \frac{1}{f} \sum_{p \in \mathcal{S}_f(x)} \frac{L(1, \chi_d)}{\sqrt{p}} + O\left(\log^2 x\right). \end{aligned}$$

With partial summation, we write

$$\begin{aligned} \sum_{f \le 2\sqrt{x}} \frac{1}{f} \sum_{p \in \mathcal{S}_f(x)} \frac{L(1, \chi_d)}{\sqrt{p}} &= \frac{1}{\sqrt{x}\log x} \sum_{f \le 2\sqrt{x}} \frac{1}{f} \sum_{p \in \mathcal{S}_f(x)} L(1, \chi_d) \log p \\ &- \int_2^x \sum_{f \le 2\sqrt{t}} \left(\frac{1}{f} \sum_{p \in \mathcal{S}_f(t)} L(1, \chi_d) \log p \right) \frac{d}{dt} \left(\frac{1}{\sqrt{t}\log t} \right) dt, \end{aligned} \tag{31}$$

since $\mathcal{S}_f(t) = \emptyset$ for $f > 2\sqrt{t}$. Using Theorem 3.1, (31) can be rewritten as

$$\begin{aligned} K_r \int_2^x \frac{dt}{2\sqrt{t}\log t} &+ K_r \left(\frac{\sqrt{x}}{\log x} + \int_2^x \frac{dt}{\sqrt{t}\log^2 t} \right) \\ &+ O\left(\frac{\sqrt{x}}{\log^{c+1} x} \right) + O\left(\int_2^x \frac{dt}{\sqrt{t}\log^{c+1} t} \right) \end{aligned}$$

which gives

$$\frac{1}{2} \sum_{B(r)<p\le x} \frac{H(r^2 - 4p)}{p} = \frac{2K_r}{\pi} \pi_{1/2}(x) + O\left(\frac{\sqrt{x}}{\log^c x} \right). \tag{32}$$

Replacing (32) in (28), the only thing left to show is that $C_r = \dfrac{2}{\pi} K_r$ has the correct Euler product expansion.

Lemma 4.1 *Suppose that r is an odd integer. Let*

$$K_r = \sum_{\substack{f=1 \\ (2r,f)=1}}^{\infty} \sum_{n=1}^{\infty} \frac{c_f^r(n)}{nf\varphi\left(nf^2\right)} \quad \text{with} \quad c_f^r(n) = \sum_{\substack{a(4n)^* \\ (r^2-af^2,4n)=4}} \left(\frac{a}{n}\right),$$

where $\sum_{a(n)^*}$ is the sum over a complete set of invertible residues mod n. Then

$$K_r = \prod_{l|r} \left(1 - \frac{1}{l^2}\right)^{-1} \prod_{l \nmid r} \frac{l(l^2-l-1)}{(l-1)(l^2-1)}.$$

Proof of Lemma 4.1: Since $c_f^r(n)$ is a multiplicative function of n (2. of Lemma 3.3), we have that

$$K_r = \sum_{\substack{f=1 \\ (f,2r)=1}}^{\infty} \frac{1}{f\varphi(f^2)} \prod_{l} \left(\sum_{\alpha \geq 0} \frac{c_f^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \cdot \frac{\varphi((f^2,l^\alpha))}{(f^2,l^\alpha)} \right). \tag{33}$$

Using 3. of Lemma 3.3, we rewrite the above product as

$$\begin{aligned} \prod_{l} \left(\sum_{\alpha \geq 0} \frac{c_f^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \cdot \frac{\varphi((f^2,l^\alpha))}{(f^2,l^\alpha)} \right) &= \prod_{l \nmid f} \left(\sum_{\alpha \geq 0} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \right) \prod_{l|f} \left(\sum_{\alpha \geq 0} \frac{c_f^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \frac{\varphi((f^2,l^\alpha))}{(f^2,l^\alpha)} \right) \\ &= \prod_{l} \left(\sum_{\alpha \geq 0} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \right) \prod_{l|f} \left(\frac{\sum_{\alpha \geq 0} \frac{c_f^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \frac{\varphi((f^2,l^\alpha))}{(f^2,l^\alpha)}}{\sum_{\alpha \geq 0} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)}} \right). \end{aligned}$$

Replacing inside (33), and using the multiplicativity of the functions in the outer sum of (33), we obtain

$$\begin{aligned} K_r &= \prod_{l} \left(\sum_{\alpha \geq 0} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \right) \prod_{l \nmid 2r} \left(1 + \sum_{\beta \geq 1} \frac{1}{l^\beta \varphi(l^{2\beta})} \cdot \frac{\sum_{\alpha \geq 0} \frac{c_l^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \cdot \frac{\varphi((l^{2\beta},l^\alpha))}{(l^{2\beta},l^\alpha)}}{\sum_{\alpha \geq 0} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)}} \right) \\ &= \prod_{l|2r} \left(\sum_{\alpha \geq 0} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \right) \prod_{l \nmid 2r} \left(1 + \sum_{\alpha \geq 1} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} + \frac{1}{l^3-1} \left(\frac{l}{l-1} + \sum_{\alpha \geq 1} \frac{c_l^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} \right) \right) \end{aligned}$$

With Lemma 3.3, we compute

$$\sum_{\alpha \geq 1} \frac{c_1^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} = \begin{cases} \dfrac{2}{3} & \text{if } l = 2 \\[2ex] \dfrac{1}{l^2-1} & \text{if } l | r \\[2ex] \dfrac{-2}{(l-1)(l^2-1)} & \text{if } l \nmid 2r \end{cases}$$

and for $l \not| 2r$

$$\sum_{\alpha \geq 1} \frac{c_l^r(l^\alpha)}{l^\alpha \varphi(l^\alpha)} = \frac{1}{l^2 - 1}.$$

Replacing in the expression for K_r above, this gives

$$\begin{aligned} K_r &= \frac{2}{3} \prod_{l|r} \left(1 - \frac{1}{l^2}\right)^{-1} \prod_{l \not| 2r} \left(1 - \frac{2}{(l-1)(l^2-1)} + \frac{1}{(l-1)(l^2-1)}\right) \\ &= \prod_{l|r} \left(1 - \frac{1}{l^2}\right)^{-1} \prod_{l \not| r} \frac{l(l^2 - l - 1)}{(l-1)(l^2-1)}. \end{aligned}$$

This completes the proof of Theorem 1.2 for r odd. □

5 Proof of Theorem 1.4.

We prove in this section Theorem 1.4. As in the proof of Theorem 1.2, the main ingredient is Theorem 3.1.

Let

$$\mu = \frac{1}{4AB} \sum_{|a| \leq A,\, |b| \leq B} \pi^r_{E(a,b)}(x).$$

Fix any $c > 0$. Then, for $A, B > x^{1+\epsilon}$, Theorem 1.2 gives

$$\mu = C_r \pi_{1/2}(x) + O\left(\frac{\sqrt{x}}{\log^c x}\right). \tag{34}$$

Thus, by the triangle inequality, the left hand side of (4) is

$$\ll \frac{1}{4AB} \sum_{|a| \leq A,\, |b| \leq B} \left|\pi^r_{E(a,b)}(x) - \mu\right|^2 + O\left(\frac{x}{\log^{2c} x}\right). \tag{35}$$

Now in general, if $\mu = \frac{1}{N} \sum_{n=1}^N \lambda_n$, then $\frac{1}{N} \sum_{n=1}^N (\lambda_n - \mu)^2 = \frac{1}{N} \sum_{n=1}^N \lambda_n^2 - \mu^2$. Therefore, the left hand side of (4) is

$$\ll \frac{1}{4AB} \sum_{|a| \leq A,\, |b| \leq B} \left(\pi^r_{E(a,b)}(x)\right)^2 - \mu^2 + O\left(\frac{x}{\log^{2c} x}\right). \tag{36}$$

We then write

$$\left(\pi^r_{E(a,b)}(x)\right)^2 = \pi^r_{E(a,b)}(x) + \#\left\{p, q \leq x \mid p \neq q,\ a_p(E(a,b)) = a_q(E(a,b)) = r\right\}$$

where the pairs p, q and q, p are both counted.

Proceeding as in the proof of Theorem 1.2 and using the Chinese Remainder theorem, we obtain from the last line that

$$\frac{1}{4AB} \sum_{|a| \leq A,\, |b| \leq B} \left(\pi^r_{E(a,b)}(x) \right)^2 = \mu + \frac{1}{4} \sum_{B(r) < p, q \leq x,\, p \neq q} \frac{H(r^2-4p)}{p} \frac{H(r^2-4q)}{q} + E(x, A, B)$$

where

$$E(x, A, B) \ll \sum_{p,q \leq x} \frac{H(r^2-4p) + H(r^2-4q)}{pq} + \sum_{p,q \leq x} H(r^2-4p)\ H(r^2-4q) \ \left(\frac{1}{A} + \frac{1}{B} + \frac{pq}{AB} \right).$$

Therefore, using the estimates (26) and (27), we obtain

$$E(x, A, B) \ll \frac{\sqrt{x} \log\log x}{\log x} + \left(\frac{1}{A} + \frac{1}{B} \right) x^3 + \frac{x^5}{AB}. \tag{37}$$

Then

$$\frac{1}{4AB} \sum_{|a| \leq A,\, |b| \leq B} \left(\pi^r_{E(a,b)}(x) \right)^2 = \mu + \left(\frac{1}{2} \sum_{B(r) < p \leq x} \frac{H(r^2-4p)}{p} \right)^2 - \frac{1}{4} \sum_{p \leq x} \frac{H(r^2-4p)^2}{p^2} + E(x, A, B),$$

and using (32) and (37), this gives

$$\frac{1}{4AB} \sum_{|a| \leq A,\, |b| \leq B} \left(\pi^r_{E(a,b)}(x) \right)^2 = \left(C_r \pi_{1/2}(x) \right)^2 + O\left(\frac{x}{\log^c x} + \left(\frac{1}{A} + \frac{1}{B} \right) x^3 + \frac{x^5}{AB} \right).$$

Replacing in (36), and using (34), this completes the proof of Theorem 1.4 □

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References

[1] B. J. Birch, How the number of points of an elliptic curve over a fixed prime field varies, *J. London Math. Soc.* **43** (1968), 57–60.

[2] H. Davenport, *Multiplicative Number Theory*, Springer-Verlag, 1980.

[3] M. DEURING, Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, *Abh. Math. Sem. Hansischen Univ.* **14** (1941), 197-272.

[4] W. DUKE, Rational elliptic curves with no exceptional primes, *C.R. Acad. Sci. Paris* **325** (1997), 813–818.

[5] N. D. ELKIES, The existence of infinitely many supersingular primes for every elliptic curve over $\mathbb{Q}$, *Invent. Math.* **89** (1987), 561-567.

[6] E. FOUVRY and M. R. MURTY, On the distribution of supersingular primes, *Can. J. Math.* **48** (1996), 81-104.

[7] E. FOUVRY and E. ULLMO, personal communication.

[8] H. HALBERSTAM and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[9] G. H. HARDY and J. E. LITTLEWOOD, Some problems of partitio numenorum III, *Acta. Math.* **44** (1923), 1-70.

[10] L.-K. HUA, *Introduction to number theory*, Springer-Verlag, 1982.

[11] S. LANG and H. Trotter, *Frobenius distributions in GL_2-extensions*, Springer-Verlag Lecture Notes in Mathematics **504**, 1976.

[12] M. R. MURTY and V. M. MURTY, *Non-vanishing of L-functions and applications*, Birkhauser, 1997.

[13] J.-P. SERRE, Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, *Inventiones Math.*, **15** (1972), 259-331.

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