Volume 259 Issue 6 15 September 2010 ISSN 0022-1236

JOURNAL OF
FUNCTIONAL
ANALYSIS

Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 259 (2010) 1380–1420

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

# An atomic decomposition of the Hajłasz Sobolev space $M_1^1$ on manifolds[☆]

N. Badr[a], G. Dafni[b,*]

[a] Universite de Lyon, CNRS, Université Lyon 1, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne cedex, France
[b] Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montréal, QC, Canada H3G1M8

Received 2 September 2009; accepted 29 May 2010

Communicated by I. Rodnianski

**Abstract**

Several possible notions of Hardy–Sobolev spaces on a Riemannian manifold with a doubling measure are considered. Under the assumption of a Poincaré inequality, the space $M_1^1$, defined by Hajłasz, is identified with a Hardy–Sobolev space defined in terms of atoms. Decomposition results are proved for both the homogeneous and the nonhomogeneous spaces.

*Keywords:* Hardy–Sobolev spaces; Atomic decomposition; Metric measure spaces; Hajłasz–Sobolev spaces

## 1. Introduction

The aim of this paper is to compare different definitions of Hardy–Sobolev spaces on manifolds. In particular, we consider characterizations of these spaces in terms of maximal functions, atomic decompositions, and gradients, some of which have been shown in the Euclidean setting, and apply them to the $L_1$ Sobolev space defined by Hajłasz.

---

[☆] Project funded in part by the Natural Sciences and Engineering Research Council, Canada, the Centre de recherches mathématiques and the Institut des sciences mathématiques, Montreal.

[*] Corresponding author. Fax: +1 (514) 848 2831.
*E-mail addresses:* badr@math.univ-lyon1.fr (N. Badr), gdafni@mathstat.concordia.ca (G. Dafni).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.05.020

In the Euclidean setting, specifically on a domain $\Omega \subset \mathbb{R}^n$, Miyachi [28] shows that for a locally integrable function $f$ to have partial derivatives $\partial^\alpha f$ (taken in the sense of distributions) belonging to the real Hardy space $H_p(\Omega)$, is equivalent to a certain maximal function of $f$ being in $L_p(\Omega)$. Earlier work by Gatto, Jiménez and Segovia [14] on Hardy–Sobolev spaces, defined via powers of the Laplacian, used a maximal function introduced by Calderón [6] in characterizing Sobolev spaces for $p > 1$ to extend his results to $p \leqslant 1$. Calderón's maximal function was subsequently studied by DeVore and Sharpley [12], who showed that it is pointwise equivalent to the following variant of the sharp function. For simplicity we only give the definition in the special case corresponding to one derivative in $L_1$, which is what this article is concerned with. We will call this function the *Sobolev sharp maximal function* (it is also called a "fractional sharp maximal function" in [21]):

**Definition 1.1.** For $f \in L_{1,\mathrm{loc}}$, define $Nf$ by

$$Nf(x) = \sup_{B:\, x \in B} \frac{1}{r(B)} \fint_B |f - f_B| \, d\mu,$$

where $B$ denotes a ball, $r(B)$ its radius and $f_B$ the average of $f$ over $B$.

Another definition of Hardy–Sobolev spaces on $\mathbb{R}^n$, using second differences, is given by Strichartz [30], who also obtains an atomic decomposition. Further characterizations of Hardy–Sobolev spaces on $\mathbb{R}^n$ by means of atoms are given in [8] and [25]. For related work see [20].

Several recent results provide a connection between Hardy–Sobolev spaces and the $p = 1$ case of Hajłasz's definition of $L_p$ Sobolev spaces on a metric measure space $(X, d, \mu)$:

**Definition 1.2** *(Hajłasz).* Let $1 \leqslant p \leqslant \infty$. The (homogeneous) Sobolev space $\dot{M}^1_p$ is the set of all functions $u \in L_{1,\mathrm{loc}}$ such that there exists a measurable function $g \geqslant 0$, $g \in L_p$, satisfying

$$\bigl|u(x) - u(y)\bigr| \leqslant d(x, y)\bigl(g(x) + g(y)\bigr), \quad \mu\text{-a.e.} \tag{1}$$

We equip $\dot{M}^1_p$ with the semi-norm

$$\|u\|_{\dot{M}^1_p} = \inf_{g \text{ satisfies } (1)} \|g\|_p.$$

In the Euclidean setting, Hajłasz [15] showed the equivalence of this definition with the usual one for $1 < p \leqslant \infty$. For $p \in (n/n+1, 1]$, Koskela and Saksman [22] proved that $\dot{M}^1_p(\mathbb{R}^n)$ coincides with the homogeneous Hardy–Sobolev space $\dot{H}^1_p(\mathbb{R}^n)$ defined by requiring all first-order partial derivatives of $f$ to lie in the real Hardy space $H_p$ (the same space defined by Miyachi [28]). In recent work [23], the Hajłasz Sobolev spaces $\dot{M}^s_p$, for $0 < s \leqslant 1$ and $\frac{n}{n+s} < p < \infty$, are characterized as homogeneous grand Triebel–Lizorkin spaces.

In the more general setting of a metric space with a doubling measure, Kinnunen and Tuominen [21] show that Hajłasz's condition is equivalent to Miyachi's maximal function characterization, extending to $p = 1$ a previous result of Hajłasz and Kinnunen [17] for $p > 1$:

**Theorem 1.3.** *(See [17,21].) For* $1 \leqslant p < \infty$,

$$\dot{M}^1_p = \{f \in L_{1,\mathrm{loc}}\colon\, Nf \in L_p\}$$

*with*

$$\|f\|_{\dot{M}^1_p} \sim \|Nf\|_p.$$

*Moreover, if* $f \in L_{1,\mathrm{loc}}$ *and* $Nf \in L_1$, *then* $f$ *satisfies*

$$\bigl|f(x) - f(y)\bigr| \leqslant Cd(x,y)\bigl(Nf(x) + Nf(y)\bigr) \tag{2}$$

*for* $\mu$*-a.e.* $x, y$.

We now restrict the discussion to a complete Riemannian manifold $M$ satisfying a doubling condition and a Poincaré inequality (see below for definitions). In this setting, Badr and Bernicot [5] defined a family of homogeneous atomic Hardy–Sobolev spaces $\dot{HS}^1_{t,\mathrm{ato}}$ and proved the following comparison between these spaces:

**Theorem 1.4.** *(See [5].) Let $M$ be a complete Riemannian manifold satisfying a doubling condition and a Poincaré inequality* $(P_q)$ *for some* $q > 1$. *Then* $\dot{HS}^1_{t,\mathrm{ato}} \subset \dot{HS}^1_{\infty,\mathrm{ato}}$ *for every* $t \geqslant q$ *and therefore* $\dot{HS}^1_{t_1,\mathrm{ato}} = \dot{HS}^1_{t_2,\mathrm{ato}}$ *for every* $q \leqslant t_1, t_2 \leqslant \infty$.

In particular, under the assumption of the Poincaré inequality $(P_1)$, for every $t > 1$ we can take $1 < q \leqslant t$ for which $(P_q)$ holds, so all the atomic Hardy–Sobolev spaces $\dot{HS}^1_{t,\mathrm{ato}}$ coincide and can be denoted by $\dot{HS}^1_{\mathrm{ato}}$.

The main result of this paper is to identify this atomic Hardy–Sobolev space with Hajłasz's Sobolev space for $p = 1$:

**Theorem 1.5.** *Let $M$ be a complete Riemannian manifold satisfying a doubling condition and the Poincaré inequality* $(P_1)$. *Then*

$$\dot{M}^1_1 = \dot{HS}^1_{\mathrm{ato}}.$$

The definition of the atomic Hardy–Sobolev spaces, as well as the doubling condition, the Poincaré inequality, and other preliminaries, can be found in Section 2. The proof of Theorem 1.5, based on the characterization given by Theorem 1.3 and a Calderón–Zygmund decomposition, follows in Section 3. In Section 4, a nonhomogeneous version of Theorem 1.5 is obtained. Finally, in Section 5, we characterize our Hardy–Sobolev spaces in terms of derivatives. In particular, we show that the space of differentials $df$ of our Hardy–Sobolev functions coincides with the molecular Hardy space of differential one-forms defined by Auscher, McIntosh and Russ [4] (and by Lou and McIntosh [24] in the Euclidean setting).

## 2. Preliminaries

In all of this paper $M$ denotes a complete non-compact Riemannian manifold. We write $T_xM$ for the tangent space at the point $x \in M$, $\langle\cdot,\cdot\rangle_x$ for the Riemannian metric at $x$, and $\mu$ for the Riemannian measure (volume) on $M$. The Riemannian metric induces a distance function $\rho$ which makes $(M, \rho)$ into a metric space, and $B(x,r)$ will denote the ball of radius $r$ centered at $x$ in this space.

Let $T_x^*M$ be the cotangent space at $x$, $\Lambda T_x^*M$ the complex exterior algebra, and $d$ the exterior derivative acting on $C_0^\infty(\Lambda T^*M)$. We will work only with functions (0-forms) and hence for a smooth function $f$, $df$ will be a 1-form. In fact, in most of the paper we will deal instead with the gradient $\nabla f$, defined as the image of $df$ under the isomorphism between $T_x^*M$ and $T_xM$ (see [32], Section 4.10). Since this isomorphism preserves the inner product, we have

$$\langle df, df\rangle_x = \langle \nabla f, \nabla f\rangle_x. \tag{3}$$

Letting $L_p := L_p(M,\mu)$, $1 \leqslant p \leqslant \infty$, and denoting by $|\cdot|$ the length induced by the Riemannian metric on the tangent space (forgetting the subscript $x$ for simplicity), we can define $\|\nabla f\|_p := \||\nabla f|\|_{L_p(M,\mu)}$ and, in view of (3), $\|df\|_p = \|\nabla f\|_p$. If $d^*$ denotes the adjoint of $d$ on $L_2(\Lambda T_x^*M)$, then the Laplace–Beltrami operator $\Delta$ is defined by $dd^* + d^*d$. However since $d^*$ is null on 0-forms, this simplifies to $\Delta f = d^*df$ on functions and we have, for $f, g \in C_0^\infty(M)$, using (3),

$$\langle \Delta f, g\rangle_{L_2(M)} = \int_M \langle \Delta f, g\rangle_x \, d\mu = \int_M \langle df, dg\rangle_x \, d\mu = \langle \nabla f, \nabla g\rangle_{L_2(M)}.$$

We will use $\mathrm{Lip}(M)$ to denote the space of Lipschitz functions, i.e. functions $f$ satisfying, for some $C < \infty$, the global Lipschitz condition

$$\bigl|f(x) - f(y)\bigr| \leqslant C\rho(x,y) \quad \forall x, y \in M.$$

The smallest such constant $C$ will be denoted by $\|f\|_{\mathrm{Lip}}$. By $\mathrm{Lip}_0(M)$ we will denote the space of compactly supported Lipschitz functions. For such functions the gradient $\nabla f$ can be defined $\mu$-almost everywhere and is in $L_\infty(M)$, with $\|\nabla f\|_\infty \approx \|f\|_{\mathrm{Lip}}$ (see [7] for Rademacher's theorem on metric measure spaces and also the discussion of upper gradients in [18], Section 10.2).

### *2.1. The doubling property*

**Definition 2.1.** Let $M$ be a Riemannian manifold. One says that $M$ satisfies the (global) doubling property $(D)$ if there exists a constant $C > 0$, such that for all $x \in M$, $r > 0$ we have

$$\mu\bigl(B(x,2r)\bigr) \leqslant C\mu\bigl(B(x,r)\bigr). \tag{$D$}$$

Observe that if $M$ satisfies $(D)$ then

$$\mathrm{diam}(M) < \infty \quad \Leftrightarrow \quad \mu(M) < \infty$$

(see [1]). Therefore if $M$ is a complete non-compact Riemannian manifold satisfying $(D)$ then $\mu(M) = \infty$.

**Lemma 2.2.** *Let $M$ be a Riemannian manifold satisfying $(D)$ and let $s = \log_2 C_{(D)}$. Then for all $x, y \in M$ and $\theta \geqslant 1$,*

$$\mu\bigl(B(x,\theta R)\bigr) \leqslant C\theta^s \mu\bigl(B(x,R)\bigr). \tag{4}$$

**Theorem 2.3** *(Maximal theorem). (See [9].) Let $M$ be a Riemannian manifold satisfying $(D)$. Denote by $\mathcal{M}$ the non-centered Hardy–Littlewood maximal function over open balls of $M$, defined by*

$$\mathcal{M}f(x) := \sup_{\substack{B \text{ ball} \\ x \in B}} |f|_B,$$

*where $f_E := \fint_E f\,d\mu := \frac{1}{\mu(E)} \int_E f\,d\mu$. Then for every $1 < p \leqslant \infty$, $\mathcal{M}$ is $L_p$ bounded and moreover it is of weak type $(1,1)$. Consequently, for $r \in (0,\infty)$, the operator $\mathcal{M}_r$ defined by*

$$\mathcal{M}_r f(x) := \big[\mathcal{M}\big(|f|^r\big)(x)\big]^{1/r}$$

*is of weak type $(r,r)$ and $L_p$ bounded for all $r < p \leqslant \infty$.*

Recall that an operator $T$ is of weak type $(p,p)$ if there is $C > 0$ such that for any $\alpha > 0$, $\mu(\{x\colon |Tf(x)| > \alpha\}) \leqslant \frac{C}{\alpha^p} \|f\|_p^p$.

### 2.2. *Poincaré inequality*

**Definition 2.4** *(Poincaré inequality on $M$).* We say that a complete Riemannian manifold $M$ admits *a Poincaré inequality* $(P_q)$ for some $q \in [1,\infty)$ if there exists a constant $C > 0$ such that, for every function $f \in \mathrm{Lip}_0(M)$ and every ball $B$ of $M$ of radius $r > 0$, we have

$$\left(\fint_B |f - f_B|^q\,d\mu\right)^{1/q} \leqslant Cr\left(\fint_B |\nabla f|^q\,d\mu\right)^{1/q}. \tag{$P_q$}$$

We also recall the following result

**Theorem 2.5.** *(See [16], Theorem 8.7.) Let $u \in \dot{M}_1^1$ and $g \in L_1$ such that $(u,g)$ satisfies (1). Take $\frac{s}{s+1} \leqslant q < 1$ and $\lambda > 1$. Then $(u,g)$ satisfies the following Sobolev–Poincaré inequality: there is a constant $C > 0$ depending on $(D)$ and $\lambda$, independent of $(u,g)$ such that for all balls $B$ of radius $r > 0$,*

$$\left(\fint_B |u - u_B|^{q^*}\,d\mu\right)^{1/q^*} \leqslant Cr\left(\fint_{\lambda B} g^q\,d\mu\right)^{1/q}, \tag{5}$$

*where $q^* = \frac{sq}{s-q}$.*

Applying this together with Theorem 1.3, for $u \in \dot{M}_1^1$ we have

$$\left(\fint_B |u - u_B|^{q^*}\,d\mu\right)^{1/q^*} \leqslant Cr\left(\fint_{\lambda B} (Nu)^q\,d\mu\right)^{1/q} \tag{6}$$

for all balls $B$.

### 2.3. Comparison between $Nf$ and $|\nabla f|$

The following Proposition shows that the maximal function $Nf$ controls the gradient of $f$ in the pointwise almost everywhere sense. In the Euclidean setting this result was demonstrated by Calderón (see [6], Theorem 4) for his maximal function $N(f,x)$ (denoted by $f^\star$ in Section 4.2 below), which was shown to be pointwise equivalent to our $Nf$ by DeVore and Sharpley (see also the stronger inequality (5.5) in [28], which bounds the maximal function of the partial derivatives).

Recall that if $u \in C_0^\infty(M)$, given any smooth vector field $\boldsymbol{\Phi}$ with compact support, we can write, based on (3) and the definition of $d^*$,

$$\int\limits_M \langle \nabla u, \boldsymbol{\Phi}\rangle_x \, d\mu := \int\limits_M \langle du, \omega_{\boldsymbol{\Phi}}\rangle_x \, d\mu = \int\limits_M u\big(d^*\omega_{\boldsymbol{\Phi}}\big)\, d\mu,$$

where $\omega_{\boldsymbol{\Phi}}$ is the 1-form corresponding to $\Phi$ under the isomorphism between the tangent space $T_xM$ and the co-tangent space $T_x^*M$ (see [32], Section 4.10). Denoting $d^*\omega_{\boldsymbol{\Phi}}$ by $\operatorname{div}\boldsymbol{\Phi}$, we can define, for $u \in L_{1,\mathrm{loc}}$, the gradient $\nabla u$ in the sense of distributions by

$$\langle \nabla u, \boldsymbol{\Phi}\rangle := -\int\limits_M u(\operatorname{div}\boldsymbol{\Phi})\, d\mu \tag{7}$$

for all smooth vector fields $\boldsymbol{\Phi}$ with compact support (see [27]). When $M$ is orientable, $\operatorname{div}\boldsymbol{\Phi}$ is given by $*d*\omega_{\boldsymbol{\Phi}}$ with $*$ the Hodge star operator (see [32]), and in the Euclidean case this corresponds to the usual notion of divergence of a vector field.

**Proposition 2.6.** *Assume that $M$ satisfies* $(D)$*, and suppose $u \in L_{1,\mathrm{loc}}$ with $Nu \in L_1$. Then $\nabla u$, initially defined by* (7)*, is given by an $L_1$ vector field and satisfies*

$$|\nabla u| \leqslant CNu, \quad \mu\text{-a.e.}$$

**Proof.** Fix $r > 0$. We begin with a covering of $M$ by balls $B_i = B(x_i, r)$, $i = 1, 2 \ldots$, such that

1. $M \subset \bigcup_i B_i$,
2. $\sum_i \mathbb{1}_{6B_i} \leqslant K$.

Note that the constant $K$ can be taken independent of $r$. Then we take $\{\varphi_i\}_i$ a partition of unity related to the covering $\{B_i\}_i$ such that $0 \leqslant \varphi_i \leqslant 1$, $\varphi_i = 0$ on $(6B_i)^c$, $\varphi_i \geqslant c$ on $3B_i$ and $\sum_i \varphi_i = 1$. The $\varphi_i$'s are $C/r$ Lipschitz. For details concerning this covering we refer to [13,21,19,10]. Now let (see [13], p. 1908 and [21], Section 3.1)

$$u_r(x) = \sum_j \varphi_j(x) u_{3B_j}. \tag{8}$$

The sum is locally finite and defines a Lipschitz function so we can take its gradient and we have, for $\mu$-almost every $x$,

$$\begin{aligned}
\left|\nabla u_r(x)\right| &= \left|\sum_j \nabla\varphi_j(x)u_{3B_j}\right| \\
&= \left|\sum_{\{j:\, x\in 6B_j\}} \nabla\varphi_j(x)\left(u_{3B_j} - u_{B(x,9r)}\right)\right| \\
&\leqslant CK\frac{1}{r}\ \fint_{B(x,9r)} |u - u_{B(x,9r)}|\,d\mu \\
&\leqslant CKNu(x). \qquad (9)
\end{aligned}$$

We used the fact that $\sum\nabla\phi_j = 0$ and that for $x \in 6B_j$, $3B_j \subset B(x,9r)$.

To see that $u_r \to u$, $\mu$-a.e. and moreover in $L_1$ when $r \to 0$ (see also [13], p. 1908), write, for $x$ a Lebesgue point of $\mu$,

$$\left|u_r(x) - u(x)\right| \leqslant \sum_j \left|\varphi_j(x)\right|\left|u(x) - u_{3B_j}\right| \leqslant \sum_{\{j:\, x\in 6B_j\}} \left|u(x) - u_{3B_j}\right| \leqslant CKr\mathcal{M}_q(Nu)(x)$$

where $\frac{s}{s+1} \leqslant q < 1$. The last inequality follows from estimates of $|u(x) - u_{B(x,9r)}|$ and $|u_{3B_j} - u_{B(x,9r)}|$, $x \in 6B_j$, which are the same as estimates (12)–(14) in the proof of Lemma 1 in [21], using the doubling property and (6).

Now let $\boldsymbol{\Phi}$ be a smooth vector field with compact support. Using the convergence in $L_1$, the fact that $\operatorname{div}\boldsymbol{\Phi} \in C_0^\infty(M)$, and the estimate on $|\nabla u_r|$ above, we have

$$\begin{aligned}
\left|\int_M \langle\nabla u, \boldsymbol{\Phi}\rangle_x\, d\mu\right| &= \left|\int_M u(\operatorname{div}\boldsymbol{\Phi})\,d\mu\right| \\
&= \left|\lim_{r\to 0}\int_M u_r(\operatorname{div}\boldsymbol{\Phi})\,d\mu\right| \\
&\leqslant \limsup_{r\to 0}\int_M |\nabla u_r||\boldsymbol{\Phi}|\,d\mu \\
&\leqslant CK\int |Nu||\boldsymbol{\Phi}|\,d\mu.
\end{aligned}$$

Taking the supremum of the left-hand side over all such $\boldsymbol{\Phi}$ with $|\boldsymbol{\Phi}| \leqslant 1$, we get that the total variation of $u$ is bounded (see [27], (1.4), p. 104), i.e.

$$|Du|(M) \leqslant C\|Nu\|_{L_1(M)} < \infty,$$

hence $u$ is a function of bounded variation on $M$, and $|Du|$ defines a finite measure on $M$. We can write the distributional gradient as

$$\langle\nabla u, \boldsymbol{\Phi}\rangle = \int_M \langle X_u, \boldsymbol{\Phi}\rangle_x\, d|Du|$$

for some vector field $X_u$ with $|X_u| = 1$ a.e. (see again [27], p. 104 where this is expressed in terms of the corresponding 1-form $\sigma_u$). Moreover, from the above estimates and the fact that $Nu \in L_1$, we further deduce that the measure $|Du|$ is absolutely continuous with respect to the Riemannian measure $\mu$, so there is an $L_1$ function $g$ such that we can write $\nabla u = gX_u$, and $|\nabla u| \leqslant CNu$, $\mu$-a.e. □

**Corollary 2.7.** *Assume that $M$ satisfies $(D)$. Then*

$$\dot{M}_1^1 \subset \dot{W}_1^1.$$

**Proof.** The result follows from Proposition 2.6 and Theorem 1.3. □

### 2.4. *Hardy spaces*

We begin by introducing the maximal function characterization of the real Hardy space $H_1$.

**Definition 2.8.** Let $f \in L_{1,\mathrm{loc}}(M)$. We define its grand maximal function, denoted by $f^+$, as follows:

$$f^+(x) := \sup_{\varphi \in \mathcal{T}_1(x)} \left| \int f\varphi \, d\mu \right| \tag{10}$$

where $\mathcal{T}_1(x)$ is the set of all test functions $\psi \in \mathrm{Lip}_0(M)$ such that for some ball $B := B(x,r)$ containing the support of $\psi$,

$$\|\psi\|_\infty \leqslant \frac{1}{\mu(B)}, \qquad \|\nabla\psi\|_\infty \leqslant \frac{1}{r\mu(B)}. \tag{11}$$

Set $H_{1,\max}(M) = \{f \in L_{1,\mathrm{loc}}(M)\colon\ f^+ \in L_1(M)\}$.

While this definition assumes $f$ to be only locally integrable, by taking an appropriate sequence $\varphi_\epsilon \in \mathcal{T}_1(x)$, the Lebesgue differentiation theorem implies that

$$\left|f(x)\right| = \lim_{\epsilon \to 0} \left| \int f\varphi_\epsilon \right| \leqslant f^+(x) \quad \text{for } \mu\text{-a.e. } x, \tag{12}$$

so $H_{1,\max}(M) \subset L_1(M)$.

Another characterization is given in terms of atoms (see [10]).

**Definition 2.9.** Fix $1 < t \leqslant \infty$, $\frac{1}{t} + \frac{1}{t'} = 1$. We say that a function $a$ is an $H_1$-atom if

1. $a$ is supported in a ball $B$,
2. $\|a\|_t \leqslant \mu(B)^{-\frac{1}{t'}}$, and
3. $\int a \, d\mu = 0$.

We say $f$ lies in the atomic Hardy space $H_{1,\mathrm{ato}}$ if $f$ can be represented, in $L_1(M)$, by

$$f = \sum \lambda_j a_j \tag{13}$$

for sequences of $H_1$-atoms $\{a_j\}$ and scalars $\{\lambda_j\} \in \ell^1$. Note that this representation is not unique and we define

$$\|f\|_{H_{1,\mathrm{ato}}} := \inf \sum |\lambda_j|,$$

where the infimum is taken over all atomic decompositions (13).

A priori this definition depends on the choice of $t$. However, we claim

**Proposition 2.10.** *Let $M$ be a complete Riemannian manifold satisfying* $(D)$. *Then*

$$H_{1,\mathrm{ato}}(M) = H_{1,\max}(M)$$

*with equivalent norms*

$$\|f\|_{H_{1,\mathrm{ato}}} \approx \left\|f^+\right\|_1$$

(*where the constants of proportionality depend on the choice of $t$*).

In the case of a space of homogeneous type $(X, d, \mu)$, this was shown in [26] (Theorem 4.13) for a normal space of order $\alpha$ and in [31] (Theorem C) under the assumption of the existence of a family of Lipschitz kernels (see also the remarks following Theorem (4.5) in [10]). For the manifold $M$ this will follow as a corollary of the atomic decomposition for the Hardy–Sobolev space below. We first prove the inclusion

$$H_{1,\mathrm{ato}}(M) \subset H_{1,\max}(M). \tag{14}$$

**Proof.** We show that if $f \in H_{1,\mathrm{ato}}$ then $f^+ \in L_1$. Let $t > 1$ and $a$ be an atom supported in a ball $B_0 = B(x_0, r_0)$. We want to prove that $a^+ \in L_1$. First take $x \in 2B_0$. We have $a^+(x) = \sup_{\varphi \in \mathcal{T}_1(x)} |\int_B a\varphi\, d\mu| \leqslant C\mathcal{M}(a)(x)$. Then by the $L_t$-boundedness of the Hardy–Littlewood maximal function for $t > 1$ (Theorem 2.3) and the size condition on $a$,

$$\begin{aligned}\int\limits_{2B_0} \left|a^+(x)\right| d\mu &\leqslant \mu(B_0)^{1/t'} \Biggl(\int\limits_{2B_0} \left|a^+\right|^t d\mu\Biggr)^{1/t} \leqslant C\mu(B_0)^{1/t'} \|\mathcal{M}a\|_t \\ &\leqslant C_t \mu(B_0)^{1/t'} \|a\|_t \leqslant C_t.\end{aligned} \tag{15}$$

Note that the constant depends on $t$ due to the dependence of the constant in the boundedness of the Hardy–Littlewood maximal function, which blows up as $t \to 1^+$.

Now if $x \in M \setminus 2B_0$, there exists $k \in \mathbb{N}^*$ such that $x \in C_k(B_0) := 2^{k+1}B_0 \setminus 2^k B_0$. Let $\varphi \in \mathcal{T}_1(x)$ and take a ball $B = B(x, r)$ such that $\varphi$ is supported in and satisfies (11) with respect to $B$. Using the moment condition for $a$ and the Lipschitz bound on $\varphi$, we get

$$\begin{aligned}\left|\int_B a\varphi\,d\mu\right| &= \left|\int_{B\cap B_0} a(y)\big(\varphi(y)-\varphi(x_0)\big)\,d\mu(y)\right| \\ &\leqslant C\int_{B\cap B_0} \big|a(y)\big|\frac{d(y,x_0)}{r\mu(B)}\,d\mu(y) \\ &\leqslant C\frac{r_0}{r\mu(B)}\|a\|_1.\end{aligned}$$

Note that for the integral not to vanish we must have $B\cap B_0\neq\emptyset$. We claim that this implies

$$r>2^{k-1}r_0 \quad\text{and}\quad 2^{k+1}B_0\subset 8B. \tag{16}$$

To see this, let $y\in B\cap B_0$. Then $r>d(y,x)\geqslant d(x,x_0)-d(y,x_0)\geqslant 2^k r_0-r_0\geqslant 2^{k-1}r_0$. Thus if $d(z,x_0)<2^{k+1}r_0$ then $d(z,x)\leqslant d(z,x_0)+d(x,x_0)<2^{k+1}r_0+2^{k+1}r_0<8r$ and we deduce that $2^{k+1}B_0\subset 8B$. We then have

$$\mu\big(2^{k+1}B_0\big)\leqslant C8^s\mu(B)$$

by (4). Using this estimate and the fact that $\|a\|_1\leqslant 1$, we have

$$\begin{aligned}\int_{x\notin 2B_0} \big|a^+\big|(x)\,d\mu &= \sum_{k\geqslant 1}\int_{C_k(B_0)} \big|a^+\big|(x)\,d\mu \\ &\leqslant C\|a\|_1\sum_{k\geqslant 1}\frac{8^s 2^{1-k}}{\mu(2^{k+1}B_0)}\mu\big(C_k(B_0)\big) \\ &\leqslant C8^s\sum_{k\geqslant 1}2^{1-k} \\ &\leqslant C.\end{aligned}$$

Thus $a^+\in L_1$ with $\|a^+\|_1\leqslant C_t$.

Now for $f\in H_{1,\mathrm{ato}}$, take an atomic decomposition of $f$ as in (13). By the convergence of the series in $L_1$, we have, for each $x$ and each $\varphi\in\mathcal{T}_1(x)$,

$$\left|\int f\varphi\,d\mu\right|\leqslant\sum|\lambda_j|\left|\int a_j\varphi\,d\mu\right|\leqslant\sum|\lambda_j|a_j^+(x)$$

so $f^+$ is pointwise dominated by $\sum|\lambda_j|a_j^+$, giving

$$\big\|f^+\big\|_1\leqslant\sum_j|\lambda_j|\big\|a_j^+\big\|_1\leqslant C_t\sum_j|\lambda_j|.$$

Taking the infimum over all the atomic decompositions of $f$ yields $\|f^+\|_1\leqslant C_t\|f\|_{H_1}$. □

The proof of the converse, namely that if $f^+ \in L_1$ then $f \in H_{1,\mathrm{ato}}$, relies on an atomic decomposition and will follow from the proof of Proposition 3.4 below.

### 2.5. *Atomic Hardy–Sobolev spaces*

In [5], the authors defined atomic Hardy–Sobolev spaces. Let us recall their definition of homogeneous Hardy–Sobolev atoms. These are similar to $H_1$ atoms but instead of the usual $L_t$ size condition they are bounded in the Sobolev space $\dot{W}_t^1$.

**Definition 2.11.** (See [5].) For $1 < t \leqslant \infty$, $\frac{1}{t} + \frac{1}{t'} = 1$, we say that a function $a$ is a homogeneous Hardy–Sobolev $(1, t)$-atom if

1. $a$ is supported in a ball $B$,
2. $\|a\|_{\dot{W}_t^1} := \|\nabla a\|_t \leqslant \mu(B)^{-\frac{1}{t'}}$, and
3. $\int a \, d\mu = 0$.

They then define, for every $1 < t \leqslant \infty$, the homogeneous Hardy–Sobolev space $\dot{HS}_{t,\mathrm{ato}}^1$ as follows: $f \in \dot{HS}_{t,\mathrm{ato}}^1$ if there exists a sequence of homogeneous Hardy–Sobolev $(1, t)$-atoms $\{a_j\}_j$ such that

$$f = \sum_j \lambda_j a_j \tag{17}$$

with $\sum_j |\lambda_j| < \infty$. This space is equipped with the semi-norm

$$\|f\|_{\dot{HS}_{t,\mathrm{ato}}^1} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all possible decompositions (17).

**Remarks 2.12.**

1. Since condition 2 implies that the homogeneous Sobolev $\dot{W}_1^1$ semi-norm of the atoms is bounded by a constant, the sum in (17) converges in $\dot{W}_1^1$ and therefore we can consider $\dot{HS}_{t,\mathrm{ato}}^1$ as its subspace.
2. Since we are working with homogeneous spaces, we can modify functions by constants so the cancellation conditions are, in a sense, irrelevant. As we will see below, and when comparing to other definitions in the literature (see, for example, [25]), condition 3 can be replaced by one of the following:

   3′. $\|a\|_1 \leqslant r(B)$, or
   3″. $\|a\|_t \leqslant r(B)\mu(B)^{-\frac{1}{t'}}$,

   where $r(B)$ is the radius of the ball $B$. Clearly condition 3″ implies 3′, and conditions 2 and 3 imply 3′ (respectively 3″) if we assume the Poincaré inequality $(P_1)$ (respectively $(P_t)$). It is most common to consider the case $t = 2$ under the assumption $(P_2)$.

3. As mentioned in the introduction, from Theorem 1.4 we have that under $(P_1)$ all the spaces $\dot{HS}^1_{t,\mathrm{ato}}$ can be identified as one space $\dot{HS}^1_{\mathrm{ato}}$. As we will see, in this case the atomic decomposition can be taken with condition $3'$ instead of 3.

## 3. Atomic decomposition of $\dot{M}^1_1$ and comparison with $\dot{HS}^1_{t,\mathrm{ato}}$

We begin by proving that under the Poincaré inequality $(P_1)$, $\dot{HS}^1_{\mathrm{ato}} \subset \dot{M}^1_1$. While under this assumption the space $\dot{HS}^1_{\mathrm{ato}}$ is equivalent to any one of the spaces $\dot{HS}^1_{t,\mathrm{ato}}$ defined above, if we want to consider the norms we need to fix some $t > 1$.

**Proposition 3.1.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$ and $(P_1)$. Let $1 < t \leqslant \infty$ and $a$ be a homogeneous Hardy–Sobolev $(1,t)$-atom. Then $a \in \dot{M}^1_1$ with $\|a\|_{\dot{M}^1_1} \leqslant C_t$, the constant $C$ depending only on $t$, the doubling constant and the constant appearing in $(P_1)$, and independent of $a$.*

*Consequently $\dot{HS}^1_{t,\mathrm{ato}} \subset \dot{M}^1_1$ with*

$$\|f\|_{\dot{M}^1_1} \leqslant C_t \|f\|_{\dot{HS}^1_{t,\mathrm{ato}}}.$$

**Proof.** Let $a$ be an $(1,t)$-atom supported in a ball $B_0 = B(x_0, r_0)$. We want to prove that $Na \in L_1$. For $x \in 2B_0$ we have, using $(P_1)$,

$$\begin{aligned} Na(x) &= \sup_{B:\, x\in B} \frac{1}{r(B)} \fint_B |a - a_B|\, d\mu \leqslant C \sup_{B:\, x\in B} \fint_B |\nabla a|\, d\mu \\ &= C\mathcal{M}\big(|\nabla a|\big)(x). \end{aligned}$$

Then, exactly as in (15), by the $L_t$ boundedness of $\mathcal{M}$ for $t > 1$ (with a constant depending on $t$), and properties 1 and 2 of $(1,t)$-Hardy–Sobolev atoms,

$$\begin{aligned} \int_{2B_0} \big|Na(x)\big|\, d\mu &\leqslant C\mu(B_0)^{1/t'} \bigg( \int_{2B_0} \big(\mathcal{M}\big(|\nabla a|\big)\big)^t\, d\mu \bigg)^{1/t} \\ &\leqslant C_t \mu(B_0)^{1/t'} \|\nabla a\|_t \leqslant C_t. \end{aligned}$$

Now if $x \notin 2B_0$, then there exists $k \in \mathbb{N}^*$ such that $x \in C_k(B_0) := 2^{k+1}B_0 \setminus 2^k B_0$. Let $B = B(y, r(B))$ be a ball containing $x$. Then

$$\begin{aligned} \frac{1}{r(B)} \fint_B |a - a_B|\, d\mu &= \frac{1}{r(B)} \frac{1}{\mu(B)} \bigg( \int_{B\cap B_0} |a - a_B|\, d\mu + \int_{B\cap B_0^c} |a_B|\, d\mu \bigg) \\ &\leqslant \frac{3}{r(B)} \frac{1}{\mu(B)} \int_{B\cap B_0} |a|\, d\mu. \end{aligned} \tag{18}$$

From (16) we have that $B \cap B_0 \neq \emptyset$ implies $r(B) > 2^{k-1} r_0$ and $\mu(2^{k+1}B_0) \leqslant C8^s \mu(B)$. This, together with the doubling and Poincaré assumptions $(D)$ and $(P_1)$, the cancellation condition 3 for $a$ and the size condition 2 for $\nabla a$, yield

$$Na(x) \leqslant \frac{3}{2^{k-1} r_0} \frac{8^s}{\mu(2^{k+1}B_0)} \int_{B_0} |a|\,d\mu \leqslant \frac{3}{2^{k-1}} \frac{8^s}{\mu(2^{k+1}B_0)} \int_{B_0} |\nabla a|\,d\mu \leqslant 3 \frac{2^{-k+1} 8^s}{\mu(2^{k+1}B_0)}.$$

Note that at this point we could have used condition $3'$ (see Remarks 2.12) instead of conditions $2, 3, (D)$ and $(P_1)$.

Therefore

$$\begin{aligned}\int_{x \notin 2B_0} |Na|(x)\,d\mu &= \sum_{k \geqslant 1} \int_{C_k(B_0)} |Na|(x)\,d\mu \\ &\leqslant C8^s \sum_{k \geqslant 1} 2^{-k+1} \int_{C_k(B_0)} \frac{1}{\mu(2^{k+1}B_0)}\,d\mu(x) \\ &\leqslant C8^s \sum_{k \geqslant 1} 2^{-k+1} = C_s.\end{aligned}$$

Thus $Na \in L_1$ with $\|Na\|_1 \leqslant C_{s,t}$.

Now if $f \in \dot{HS}^1_{t,\mathrm{ato}}$, take an atomic decomposition of $f$: $f = \sum_j \lambda_j a_j$ with $a_j$ $(1,t)$-atoms and $\sum_j |\lambda_j| < \infty$. Then the sum $\sum_j \lambda_j N a_j$ converges absolutely in $L_1$ so by Theorem 1.3 the sequence of functions $f_k = \sum_{j=1}^k \lambda_j a_j$ has a limit, $g$, in the Banach space $\dot{M}^1_1$. By Proposition 2.6, this implies convergence in $\dot{W}^1_1$. Since (as pointed out in Remarks 2.12) the convergence of the decomposition $f = \sum_j \lambda_j a_j$ also takes place in $\dot{W}^1_1$, we get that $f = g$ in $\dot{W}^1_1$. This allows us to consider $f$ as a (locally integrable) element of $\dot{M}^1_1$, take $Nf$ and estimate

$$\|Nf\|_1 \leqslant \sum_j |\lambda_j| \|Na_j\|_1 \leqslant C_t \sum_j |\lambda_j|.$$

Taking the infimum over all the atomic decompositions of $f$ yields $\|Nf\|_1 \leqslant C_t \|f\|_{\dot{HS}^1_{t,\mathrm{ato}}}$. □

**Remark 3.2.** As pointed out in the proof, Proposition 3.1 remains valid if we take, for the definition of a $(1,t)$-atom, instead of condition 3 of Definition 2.11, condition $3'$ or $3''$ of Remarks 2.12.

Now for the converse, that is, to prove that $\dot{M}^1_1 \subset \dot{HS}^1_{t,\mathrm{ato}}$, we establish an atomic decomposition for functions $f \in \dot{M}^1_1$. To attain this goal, we need a Calderón–Zygmund decomposition for such functions. We refer to [2] for the original proof of the Calderón–Zygmund decomposition for Sobolev spaces on Riemannian manifolds.

**Proposition 3.3** *(Calderón–Zygmund decomposition). Let $M$ be a complete Riemannian manifold satisfying $(D)$. Let $f \in \dot{M}^1_1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Then one can find a collection of balls $\{B_i\}_i$, functions $b_i \in W^1_1$ and a Lipschitz function $g$ such that the following properties hold:*

$$f = g + \sum_i b_i,$$

$$\left|\nabla g(x)\right| \leqslant C\alpha \quad \textit{for } \mu\textit{-a.e. } x \in M, \tag{19}$$

$$\operatorname{supp} b_i \subset B_i, \qquad \|b_i\|_1 \leqslant C\alpha\mu(B_i)r_i, \qquad \|\nabla b_i\|_q \leqslant C\alpha\mu(B_i)^{1/q},$$

$$\sum_i \mu(B_i) \leqslant \frac{C}{\alpha}\int Nf\,d\mu, \tag{20}$$

*and*

$$\sum_i \chi_{B_i} \leqslant K. \tag{21}$$

*The constants $C$ and $K$ only depend on the constant in $(D)$.*

**Proof.** Let $f \in \dot{M}_1^1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Consider the open set

$$\Omega = \{x\colon \mathcal{M}_q(Nf)(x) > \alpha\}.$$

If $\Omega = \emptyset$, then set

$$g = f, \quad b_i = 0 \quad \text{for all } i$$

so that (19) is satisfied according to the Lebesgue differentiation theorem. Otherwise

$$\begin{aligned}\mu(\Omega) &\leqslant \frac{C}{\alpha}\int_M \mathcal{M}_q(Nf)\,d\mu \\ &\leqslant \frac{C}{\alpha}\int_M \left(\mathcal{M}(Nf)^q\right)^{1/q}d\mu \\ &\leqslant \frac{C}{\alpha}\int_M Nf\,d\mu < \infty.\end{aligned} \tag{22}$$

We used the fact the $\mathcal{M}$ is $L_{1/q}$ bounded since $1/q > 1$ and Theorem 1.3. In particular $\Omega \neq M$ as $\mu(M) = +\infty$.

Let $F$ be the complement of $\Omega$. Since $\Omega$ is an open set distinct from $M$, let $\{\underline{B_i}\}_i$ be a Whitney decomposition of $\Omega$ (see [10]). That is, the $\underline{B_i}$ are pairwise disjoint, and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \bigcup_i B_i$ with $B_i = C_1\underline{B_i}$, and the balls $B_i$ have the bounded overlap property;
2. $r_i = r(B_i) = \frac{1}{2}d(x_i, F)$ and $x_i$ is the center of $B_i$;
3. each ball $\overline{B_i} = C_2B_i$ intersects $F$ ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i\colon\ x \in B_i\}$. By the bounded overlap property of the balls $B_i$, we have that $\sharp I_x \leqslant K$, and moreover, fixing $k \in I_x$, $\frac{1}{r_i} \leqslant r_k \leqslant 3r_i$ and $B_i \subset 7B_k$ for all $i \in I_x$.

Condition (21) is nothing but the bounded overlap property of the $B_i$'s and (20) follows from (21) and (22). Note also that using the doubling property, we have

$$\int\limits_{B_i} |Nf|^q \, d\mu \leqslant C\mu(B_i) \fint\limits_{\overline{B_i}} |Nf|^q \, d\mu \leqslant \mu(B_i)\mathcal{M}_q^q(Nf)(y) \leqslant C\alpha^q \mu(B_i) \tag{23}$$

for some $y \in \overline{B_i} \cap F$, whose existence is guaranteed by property 3 of the Whitney decomposition.

Let us now define the functions $b_i$. For this, we construct a partition of unity $\{\chi_i\}_i$ of $\Omega$ subordinate to the covering $\{B_i\}_i$. Each $\chi_i$ is a Lipschitz function supported in $B_i$ with $0 \leqslant \chi_i \leqslant 1$ and $\|\nabla \chi_i\|_\infty \leqslant \frac{C}{r_i}$ (see for example [13], p. 1908).

We set $b_i = (f - c_i)\chi_i$ where $c_i := \frac{1}{\chi_i(B_i)} \int_{B_i} f\chi_i \, d\mu$ and $\chi_i(B_i)$ means $\int_{B_i} \chi_i \, d\mu$, which is comparable to $\mu(B_i)$. Note that by the properties of the $\chi_i$ we have the trivial estimate

$$\|b_i\|_1 \leqslant \int\limits_{B_i} |f - c_i| \, d\mu \leqslant \int\limits_{B_i} |f| \, d\mu + \frac{\mu(B_i)}{\chi_i(B_i)} \int\limits_{B_i} |f| \, d\mu \leqslant C \int \mathbb{1}_{B_i} |f| \, d\mu, \tag{24}$$

but we need a better estimate, as follows:

$$\begin{aligned}
\|b_i\|_1 &\leqslant \frac{1}{\chi_i(B_i)} \int\limits_{B_i} \left| \int\limits_{B_i} \big(f(x) - f(y)\big)\chi_i(y) \, d\mu(y) \right| d\mu(x) \\
&\leqslant \frac{1}{\chi_i(B_i)} \int\limits_{B_i} \int\limits_{B_i} \big|f(x) - f(y)\big| \, d\mu(y) \, d\mu(x) \\
&\leqslant 2\frac{\mu(B_i)}{\chi_i(B_i)} \int\limits_{B_i} \big|f(x) - f_{B_i}\big| \, d\mu(x) \\
&\leqslant Cr_i \Bigg( \fint\limits_{\overline{B_i}} |Nf|^q \, d\mu \Bigg)^{1/q} \mu(B_i) \\
&\leqslant Cr_i \mathcal{M}_q(Nf)(y)\mu(B_i) \\
&\leqslant Cr_i \alpha \mu(B_i),
\end{aligned} \tag{25}$$

as in (23). Here we have used the Sobolev–Poincaré inequality (6) with $\lambda = 4$ and the fact that $q^* > 1$.

Together with the estimate on $\|b_i\|_1$, we use the fact that $|\nabla f|$ is in $L_1$ (see Proposition 2.6) to bound $\|\nabla b_i\|_1$ and conclude that $b_i \in W_1^1$:

$$\|\nabla b_i\|_1 \leqslant \int\limits_{B_i} |f - c_i||\nabla \chi_i| \, d\mu + \int\limits_{B_i} |\nabla f| \, d\mu$$

$$
\begin{aligned}
&\leqslant C\frac{1}{r_i} r_i \mu(B_i)\Bigl(\fint_{4B_i} |Nf|^q \, d\mu\Bigr)^{1/q} + \int_{B_i} |\nabla f| \, d\mu \\
&\leqslant C\alpha\mu(B_i) + \int_{B_i} |\nabla f| \, d\mu < \infty. \qquad (26)
\end{aligned}
$$

Similarly, we can estimate $b_i$ in the Sobolev space $\dot{W}^1_q$; note again that by Proposition 2.6, $|\nabla f|$ is in $L_1$ and can be bounded pointwise $\mu$-a.e. by $Nf$:

$$
\begin{aligned}
\|\nabla b_i\|_q &\leqslant \bigl\| |(f-c_i)\nabla \chi_i| \bigr\|_q + \bigl\| |\nabla f| \chi_i \bigr\|_q \\
&\leqslant \frac{\mu(B_i)^{\frac{1}{q}-1}}{\chi_i(B_i)} \int_{B_i}\int_{B_i} \bigl| f(x) - f(y) \bigr| \chi_i(y) \bigl| \nabla \chi_i(x) \bigr| \, d\mu(y) \, d\mu(x) + \Bigl( \int_{B_i} |\nabla f|^q \, d\mu \Bigr)^{1/q} \\
&\leqslant C \Bigl( \fint_{\overline{B_i}} |Nf|^q \, d\mu \Bigr)^{1/q} \mu(B_i)^{1/q} + \Bigl( \int_{\overline{B_i}} |Nf|^q \, d\mu \Bigr)^{1/q} \\
&\leqslant C\alpha\mu(B_i)^{1/q} \qquad (27)
\end{aligned}
$$

by (23).

Set now $g = f - \sum_i b_i$. Since the sum is locally finite on $\Omega$, $g$ is defined almost everywhere on $M$ and $g = f$ on $F$. Observe that $g$ is a locally integrable function on $M$. Indeed, let $\varphi \in L_\infty$ with compact support. Since $d(x, F) \geqslant r_i$ for $x \in \operatorname{supp} b_i$, we obtain

$$
\int \sum_i |b_i||\varphi| \, d\mu \leqslant \Bigl( \int \sum_i \frac{|b_i|}{r_i} \, d\mu \Bigr) \sup_{x \in M} \bigl( d(x, F) \bigl| \varphi(x) \bigr| \bigr).
$$

Hence by (25) and the bounded overlap property,

$$
\int \sum_i |b_i||\varphi| \, d\mu \leqslant C\alpha \sum_i \mu(B_i) \sup_{x \in M} \bigl( d(x, F) \bigl| \varphi(x) \bigr| \bigr) \leqslant CK\alpha\mu(\Omega) \sup_{x \in M} \bigl( d(x, F) \bigl| \varphi(x) \bigr| \bigr).
$$

Since $f \in L_{1,\mathrm{loc}}$, we conclude that $g \in L_{1,\mathrm{loc}}$.

It remains to prove (19). Indeed, using the fact that on $\Omega$ we have $\sum \chi_i = 1$ and $\sum \nabla \chi_i = 0$, we get

$$
\begin{aligned}
\nabla g &= \nabla f - \sum_i \nabla b_i \\
&= \nabla f - \Bigl( \sum_i \chi_i \Bigr) \nabla f - \sum_i (f - c_i) \nabla \chi_i \\
&= \mathbb{1}_F \nabla f - \sum_i (f - c_i) \nabla \chi_i. \qquad (28)
\end{aligned}
$$

From Proposition 2.6, the definition of $F$ and the Lebesgue differentiation theorem, we have that $\mathbb{1}_F|\nabla f| \leqslant \mathbb{1}_F Nf \leqslant \alpha$, $\mu$-a.e. We claim that a similar estimate holds for

$$h = \sum_i (f - c_i)\nabla\chi_i,$$

i.e. $|h(x)| \leqslant C\alpha$ for all $x \in M$. For this, note first that by the properties of the balls $B_i$ and the partition of unity, $h$ vanishes on $F$ and the sum defining $h$ is locally finite on $\Omega$. Then fix $x \in \Omega$ and let $B_k$ be some Whitney ball containing $x$. Again using the fact that $\sum_i \nabla\chi_i(x) = 0$, we can replace $f(x)$ by any constant in the sum above, so we can write

$$h(x) = \sum_{i\in I_x}\biggl(\fint_{7B_k} f\,d\mu - c_i\biggr)\nabla\chi_i(x).$$

For all $i, k \in I_x$, by the construction of the Whitney collection, the balls $B_i$ and $B_k$ have equivalent radii and $B_i \subset 7B_k$. Thus

$$\begin{aligned}\biggl|c_i - \fint_{7B_k} f\,d\mu\biggr| &\leqslant \frac{1}{\chi_i(B_i)}\int_{B_i}\biggl|f - \fint_{7B_k} f\,d\mu\biggr|\chi_i\,d\mu \\ &\lesssim \fint_{7B_k}|f - f_{7B_k}|\,d\mu \\ &\lesssim r_k\biggl(\fint_{7\lambda B_k}|Nf|^q\,d\mu\biggr)^{1/q} \\ &\lesssim \alpha r_k. \end{aligned} \tag{29}$$

We used $(D)$, (6) , $\chi_i(B_i) \simeq \mu(B_i)$ and (23) for $7B_k$. Hence

$$\bigl|h(x)\bigr| \lesssim \sum_{i\in I_x}\alpha r_k(r_i)^{-1} \leqslant CK\alpha. \qquad \Box \tag{30}$$

**Proposition 3.4.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$. Let $f \in \dot{M}_1^1$. Then for all $\frac{s}{s+1} < q < 1$, $q^* = \frac{sq}{s-q}$, there is a sequence of homogeneous $(1, q^*)$ Hardy–Sobolev atoms $\{a_j\}_j$, and a sequence of scalars $\{\lambda_j\}_j$, such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } \dot{W}_1^1, \quad \text{and} \quad \sum|\lambda_j| \leqslant C_q\|f\|_{\dot{M}_1^1}.$$

*Consequently, $\dot{M}_1^1 \subset \dot{HS}^1_{q^*,\mathrm{ato}}$ with $\|f\|_{\dot{HS}^1_{q^*,\mathrm{ato}}} \leqslant C_q\|f\|_{\dot{M}_1^1}$.*

**Remark 3.5.** Note that for the inclusion $\dot{M}_1^1 \subset \dot{HS}^1_{q^*,\mathrm{ato}}$, we do not need to assume any additional hypothesis, such as a Poincaré inequality, on the doubling manifold.

**Proof of Proposition 3.4.** Let $f \in \dot{M}_1^1$. We follow the general scheme of the atomic decomposition for Hardy spaces, found in [29], Section III.2.3. For every $j \in \mathbb{Z}^*$, we take the Calderón–Zygmund decomposition, Proposition 3.3, for $f$ with $\alpha = 2^j$. Then

$$f = g^j + \sum_i b_i^j$$

with $b_i^j$, $g^j$ satisfying the properties of Proposition 3.3.

We want to write

$$f = \sum_{-\infty}^{\infty} \left(g^{j+1} - g^j\right) \tag{31}$$

in $\dot{W}_1^1$. First let us see that $g^j \to f$ in as $j \to \infty$. Indeed, since the sum is locally finite we can write

$$\left\|\nabla\left(g^j - f\right)\right\|_1 = \left\|\nabla\left(\sum_i b_i^j\right)\right\|_1 \leqslant \sum_i \left\|\nabla b_i^j\right\|_1.$$

By (26),

$$\begin{aligned}\sum_i \left\|\nabla b_i^j\right\|_1 &\leqslant CK2^j \mu(\Omega_j) + K \int\limits_{\Omega_j} |\nabla f| \, d\mu \\ &= I_j + II_j.\end{aligned} \tag{32}$$

When $j \to \infty$, $I_j \to 0$ since $\sum_j 2^j \mu(\Omega_j) \approx \int \mathcal{M}_q(Nf)\, d\mu < \infty$. This also implies $\mathcal{M}_q(Nf)$ is finite $\mu$-a.e., hence $\bigcap \Omega_j = \emptyset$ so $II_j \to 0$, since $|\nabla f| \in L_1$.

When $j \to -\infty$, we want to show $\|\nabla g_j\|_1 \to 0$. Breaking $\nabla g$ up as in (28), we know that

$$\int\limits_{F^j} \left|\nabla g^j\right| = \int \mathbb{1}_{F^j} |\nabla f| \leqslant \int\limits_{\{Nf \leqslant 2^j\}} Nf \to 0, \tag{33}$$

since $Nf \in L_1$. For the other part we have, by (30),

$$\int\limits_{\Omega^j} \left|\nabla g^j\right| = \int \left|h(x)\right| \leqslant CK2^j \mu\left(\Omega^j\right) \to 0 \tag{34}$$

from the convergence of $\sum 2^j \mu(\Omega^j)$, as above.

Denoting $g^{j+1} - g^j$ by $\ell^j$, we have $\operatorname{supp} \ell^j \subset \Omega_j$ so using the partition of unity $\{\chi_k^j\}$ corresponding to the Whitney decomposition for $\Omega_j$, we can write $f = \sum_{j,k} \ell^j \chi_k^j$ in $\dot{W}_1^1$. Let us compute $\|\ell^j \chi_k^j\|_{\dot{W}_{q^*}^1}$. We have

$$\nabla\left(\ell^j \chi_k^j\right) = \left(\nabla \ell^j\right)\chi_k^j + \ell^j \nabla \chi_k^j.$$

From the estimate $\|\nabla g^j\|_\infty \leqslant C2^j$ it follows that $(\fint_{B_k^j} |\nabla \ell^j|^{q^*}\, d\mu)^{1/q^*} \leqslant C2^j$, while

$$\ell^j \nabla \chi_k^j = \Biggl( \sum_{i:B_k^j \cap B_i^j \neq \emptyset} \bigl(f - c_i^j\bigr)\chi_i^j - \sum_{l:B_k^j \cap B_l^{j+1} \neq \emptyset} \bigl(f - c_l^{j+1}\bigr)\chi_l^{j+1} \Biggr) \nabla \chi_k^j. \tag{35}$$

Observe that since $\Omega_{j+1} \subset \Omega_j$, for a fixed $k$, the balls $B_l^{j+1}$ with $B_k^j \cap B_l^{j+1} \neq \emptyset$ must have radii $r_l^{j+1} \leqslant c r_k^j$ for some constant $c$. Therefore $B_l^{j+1} \subset (B_k^j)' := (1+2c)B_k^j$. Moreover, by the properties of the Whitney balls, given $\lambda > 1$ we can take $c$ sufficiently large so that $(B_k^j)'$ contains $\lambda B_i^j$ for all $B_i^j$ intersecting $B_k^j$. Using this fact as well as (6) and (23), and proceeding in the same way as in the derivations of (25) and (29), we get

$$\begin{aligned}
\bigl(r_k^j\bigr)^{q^*} \int_{B_k^j} \bigl|\ell^j \nabla \chi_k^j\bigr|^{q^*} d\mu &\leqslant K^{q^*-1} \int_{B_k^j} \Bigl( \sum_i \mathbb{1}_{B_i^j} \bigl|f - c_i^j\bigr|^{q^*} + \sum_l \mathbb{1}_{B_l^{j+1}} \bigl|f - c_l^{j+1}\bigr|^{q^*} \Bigr) d\mu \\
&\leqslant K^{q^*-1} \sum_{i:\ B_k^j \cap B_i^j \neq \emptyset} \int_{B_i^j} \bigl|f - c_i^j\bigr|^{q^*} d\mu \\
&\quad + K^{q^*-1} \int_{(B_k^j)'} \sum_l \mathbb{1}_{B_l^{j+1}} \bigl|f - f_{(B_k^j)'} + f_{(B_k^j)'} - c_l^{j+1}\bigr|^{q^*} d\mu \\
&\lesssim K^{q^*-1} \sum_{i:\ B_k^j \cap B_i^j \neq \emptyset} \bigl(r_i^j 2^j\bigr)^{q^*} \mu\bigl(B_i^j\bigr) + K^{q^*} \bigl(r_k^j 2^j\bigr)^{q^*} \mu\bigl(\bigl(B_k^j\bigr)'\bigr) \\
&\lesssim K^{q^*} \bigl(r_k^j 2^j\bigr)^{q^*} \mu\bigl(\bigl(B_k^j\bigr)'\bigr).
\end{aligned} \tag{36}$$

Therefore

$$\Biggl( \fint_{(B_k^j)'} \bigl|\ell^j \nabla \chi_k^j\bigr|^{q^*} d\mu \Biggr)^{\frac{1}{q^*}} \leqslant CK2^j. \tag{37}$$

The $\ell^j \chi_k^j$'s seem to be a good choice for our atoms but unfortunately they do not satisfy the cancellation condition. If we wanted to get atoms with property $3'$ (see Remarks 2.12) instead of the vanishing moment condition 3, we could use (25) to bound the $L_1$ norm of $\ell^j \chi_k^j$, then normalize as below. However, if we want to obtain the vanishing moment condition, we need to consider instead the following decomposition of the $\ell^j$'s: $\ell^j = \sum_k \ell_k^j$ with

$$\ell_k^j = \bigl(f - c_k^j\bigr)\chi_k^j - \sum_l \bigl(f - c_l^{j+1}\bigr)\chi_l^{j+1} \chi_k^j + \sum_l c_{k,l} \chi_l^{j+1}, \tag{38}$$

where

$$c_{k,l} := \frac{1}{\chi_l^{j+1}(B_l^{j+1})} \int\limits_{B_l^{j+1}} \left(f - c_j^{l+1}\right) \chi_l^{j+1} \chi_k^j \, d\mu.$$

First, this decomposition holds since $\sum_k \chi_k^j = 1$ on the support of $\chi_l^{j+1}$ and $\sum_k c_{k,l} = 0$. Furthermore, the cancellation condition

$$\int\limits_M \ell_k^j \, d\mu = 0$$

follows from the fact that $\int_M (f - c_k^j) \chi_k^j \, d\mu = 0$ and the definition of $c_{k,l}$, which immediately gives $\int ((f - c_l^{j+1}) \chi_l^{j+1} \chi_k^j - c_{k,l} \chi_l^{j+1}) \, d\mu = 0$.

Noting that $\ell_k^j$ is supported in the ball $(B_k^j)'$ (see above), let us estimate $\|\nabla \ell_k^j\|_{L_{q^*}((B_k^j)')}$. Write

$$\begin{aligned}
\nabla \ell_k^j &= (\nabla f) \chi_k^j + \left(f - c_k^j\right) \nabla \chi_k^j - \sum_l \left(f - c_l^{j+1}\right) \nabla \chi_l^{j+1} \chi_k^j \\
&\quad - \sum_l \left(f - c_l^{j+1}\right) \chi_l^{j+1} \nabla \chi_k^j - (\nabla f) \mathbb{1}_{\Omega_{j+1}} \chi_k^j + \sum_l c_{k,l} \nabla \chi_l^{j+1} \\
&= \nabla f (1 - \mathbb{1}_{\Omega_{j+1}}) \chi_k^j + \left( \left(f - c_k^j\right) - \sum_l \left(f - c_l^{j+1}\right) \chi_l^{j+1} \right) \nabla \chi_k^j \\
&\quad - \sum_l \left(f - c_l^{j+1}\right) \nabla \chi_l^{j+1} \chi_k^j + \sum_l c_{k,l} \nabla \chi_l^{j+1}.
\end{aligned}$$

Since the first term, concerning the gradient of $f$, is supported in $B_k^j \cap F_{j+1}$, we can use Proposition 2.6, the definition of $F_{j+1}$ and the Lebesgue differentiation theorem to bound it, namely

$$\int\limits_{B_k^j} |\nabla f|^{q^*} \, d\mu \leqslant 2^{(j+1)q^*} \mu\left(B_k^j\right).$$

Recalling (35), we see that the estimate of the $L_{q^*}$ norm of the second term is given by (37). The third term can be handled by the pointwise estimate (30):

$$\left\| \sum_l \left(f - c_l^{j+1}\right) \nabla \chi_l^{j+1} \chi_k^j \right\|_{q^*} \leqslant C K 2^{j+1} \mu\left(B_k^j\right)^{1/q^*}.$$

For $\sum_l c_{k,l}\nabla\chi_l^{j+1}$, note first that $c_{k,l}=0$ when $B_k^j\cap B_l^{j+1}=\emptyset$ and $|c_{k,l}|\leqslant C2^j r_l^{j+1}$ thanks to (25). By the properties of the partition of unity, this gives $|c_{k,l}\nabla\chi_l^{j+1}|\leqslant C2^j$ for every $l$, and as the sum has at most $K$ terms at each point we get the pointwise bound

$$\Bigl|\sum_l c_{k,l}\nabla\chi_l^{j+1}\Bigr|\leqslant CK2^j,$$

from which it follows that

$$\Bigl\|\sum_l c_{k,l}\nabla\chi_l^{j+1}\Bigr\|_{q^*}\leqslant CK2^j\mu\bigl(\bigl(B_k^j\bigr)'\bigr)^{1/q^*}.$$

Thus

$$\bigl\|\nabla\ell_k^j\bigr\|_{q^*}\leqslant\gamma 2^j\mu\bigl(\bigl(B_k^j\bigr)'\bigr)^{1/q^*}. \tag{39}$$

We now set $a_k^j=\gamma^{-1}2^{-j}\mu((B_k^j)')^{-1}\ell_k^j$ and $\lambda_{j,k}=\gamma 2^j\mu((B_k^j)')$. Then $f=\sum_{j,k}\lambda_{j,k}a_k^j$, with $a_k^j$ being $(1,q^*)$ homogeneous Hardy–Sobolev atoms and

$$\begin{aligned}\sum_{j,k}|\lambda_{j,k}|&=\gamma\sum_{j,k}2^j\mu\bigl(\bigl(B_k^j\bigr)'\bigr)\\&\leqslant\gamma'\sum_{j,k}2^j\mu\bigl(\underline{B_k^j}\bigr)\\&\leqslant\gamma'\sum_j 2^j\mu\bigl(\bigl\{x\colon\ \mathcal{M}_q(Nf)(x)>2^j\bigr\}\bigr)\\&\leqslant C\int\mathcal{M}_q(Nf)\,d\mu\\&\leqslant C_q\|Nf\|_1\sim\|f\|_{\dot M_1^1}.\end{aligned}$$

We used that $\mu((B_k^j)')\sim\mu(\underline{B_k^j})$ thanks to $(D)$, and the fact that the $\underline{B_k^j}$ are disjoint. □

**Remark 3.6.** As pointed out in the proof following (37), we can get an atomic decomposition as in Proposition 3.4, but replacing the vanishing moment condition 3 of the atoms from Definition 2.11 by condition 3′ in Remarks 2.12. This does not assume a Poincaré inequality.

**Conclusion.** Let $M$ be a complete Riemannian manifold satisfying $(D)$. Then

1. for all $\frac{s}{s+1}<q<1$,

$$\dot M_1^1\subset \dot{HS}{}^1_{q^*,\mathrm{ato}}.$$

2. (Theorem 1.5) If moreover we assume $(P_1)$, then

$$\dot{M}_1^1 = \dot{HS}^1_{t,\mathrm{ato}}$$

for all $t > 1$.

## 4. The nonhomogeneous case

We begin by recalling the definitions of the nonhomogeneous versions of the spaces considered above.

**Definition 4.1.** (See [16].) Let $1 \leqslant p \leqslant \infty$. The Sobolev space $M_p^1$ is the set of all functions $u \in L_p$ such that there exists a measurable function $g \geqslant 0$, $g \in L_p$, satisfying

$$\bigl|u(x) - u(y)\bigr| \leqslant d(x, y)\bigl(g(x) + g(y)\bigr), \quad \mu\text{-a.e.} \tag{40}$$

That is, $M_p^1 = L_p \cap \dot{M}_p^1$. We equip $M_p^1$ with the norm

$$\|u\|_{M_p^1} = \|u\|_p + \inf_{g \text{ satisfies (40)}} \|g\|_p.$$

From Theorem 1.3, we deduce that for $1 \leqslant p \leqslant \infty$,

$$M_p^1 = \{f \in L_p\colon Nf \in L_p\}$$

with equivalent norm

$$\|f\|_{M_p^1} = \|f\|_p + \|Nf\|_p.$$

**Definition 4.2.** We define the Hardy–Sobolev space $\widetilde{M}_1^1$ as the set of all functions $u \in H_{1,\max}$ such that there exists a measurable function $g \geqslant 0$, $g \in L_1$, satisfying

$$\bigl|u(x) - u(y)\bigr| \leqslant d(x, y)\bigl(g(x) + g(y)\bigr) \quad \mu\text{-a.e.} \tag{41}$$

We equip $\widetilde{M}_1^1$ with the norm

$$\|u\|_{\widetilde{M}_1^1} = \bigl\|u^+\bigr\|_1 + \inf_{g \text{ satisfies (41)}} \|g\|_1.$$

We have $\widetilde{M}_1^1 = H_{1,\max} \cap \dot{M}_1^1$.

Again by Theorem 1.3,

$$\widetilde{M}_1^1 = \{f \in H_{1,\max}\colon Nf \in L_1\},$$

with equivalent norm

$$\|f\|_{\widetilde{M}_1^1} = \bigl\|f^+\bigr\|_1 + \|Nf\|_1.$$

By (12) and Corollary 2.7, we have

$$\widetilde{M}_1^1 \subset M_1^1 \subset W_1^1.$$

In [5], the authors also defined the nonhomogeneous atomic Hardy–Sobolev spaces. Let us recall their definition.

**Definition 4.3.** (See [5].) For $1 < t \leqslant \infty$, we say that a function $a$ is a nonhomogeneous Hardy–Sobolev $(1, t)$-atom if

1. $a$ is supported in a ball $B$,
2. $\|a\|_{W_t^1} := \|a\|_t + \|\nabla a\|_t \leqslant \mu(B)^{-\frac{1}{t'}}$,
3. $\int a\, d\mu = 0$.

They then define, for every $1 < t \leqslant \infty$, the nonhomogeneous Hardy–Sobolev space $HS_{t,\mathrm{ato}}^1$ as follows: $f \in HS_{t,\mathrm{ato}}^1$ if there exists a sequence of nonhomogeneous Hardy–Sobolev $(1, t)$-atoms $\{a_j\}_j$ such that $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j| < \infty$. This space is equipped with the norm

$$\|f\|_{HS_{t,\mathrm{ato}}^1} := \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all such decompositions.

We also recall the following comparison between these atomic Hardy–Sobolev spaces.

**Theorem 4.4.** *(See [5].) Let $M$ be a complete Riemannian manifold satisfying $(D)$ and a Poincaré inequality $(P_q)$ for some $q > 1$. Then $HS_{t,\mathrm{ato}}^1 \subset HS_{\infty,\mathrm{ato}}^1$ for every $t \geqslant q$ and therefore $HS_{t_1,\mathrm{ato}}^1 = HS_{t_2,\mathrm{ato}}^1$ for every $q \leqslant t_1, t_2 \leqslant \infty$.*

### *4.1. Atomic decomposition of $\widetilde{M}_1^1$ and comparison with $HS_{t,\mathrm{ato}}^1$*

As in the homogeneous case, under the Poincaré inequality $(P_1)$, $HS_{t,\mathrm{ato}}^1 \subset \widetilde{M}_1^1$:

**Proposition 4.5.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$ and $(P_1)$. Let $1 < t \leqslant \infty$ and $a$ be a nonhomogeneous Hardy–Sobolev $(1, t)$-atom. Then $a \in \widetilde{M}_1^1$ with $\|a\|_{\widetilde{M}_1^1} \leqslant C_t$, the constant depending only on $t$, the doubling constant and the constant appearing in $(P_1)$, but not on $a$. Consequently $HS_{t,\mathrm{ato}}^1 \subset \widetilde{M}_1^1$ with*

$$\|f\|_{\widetilde{M}_1^1} \leqslant C_t \|f\|_{HS_{t,\mathrm{ato}}^1}.$$

**Proof.** The proof follows analogously to that of Proposition 3.1, noting that in the nonhomogeneous case every Hardy–Sobolev $(1, t)$-atom $a$ is an $H_1$ atom and so by (14) is in $H_{1,\max}$ with norm bounded by a constant. □

Now for the converse, that is, to prove that $\widetilde{M}_1^1 \subset HS_{t,\mathrm{ato}}^1$, we establish, as in the homogeneous case, an atomic decomposition for functions $f \in \widetilde{M}_1^1$ using a Calderón–Zygmund decomposition for such functions.

**Proposition 4.6** *(Calderón–Zygmund decomposition). Let* $M$ *be a complete Riemannian manifold satisfying* $(D)$*. Let* $f \in \widetilde{M}_1^1$, $\frac{s}{s+1} < q < 1$ *and* $\alpha > 0$*. Then one can find a collection of balls* $\{B_i\}_i$*, functions* $b_i \in W_1^1$ *and a Lipschitz function* $g$ *such that the following properties hold*:

$$f = g + \sum_i b_i,$$

$$\left|g(x)\right| + \left|\nabla g(x)\right| \leqslant C\alpha \quad \text{for } \mu\text{-a.e } x \in M,$$

$$\operatorname{supp} b_i \subset B_i, \qquad \|b_i\|_1 \leqslant C\alpha\mu(B_i)r_i, \qquad \left\|b_i + |\nabla b_i|\right\|_q \leqslant C\alpha\mu(B_i)^{1/q},$$

$$\sum_i \mu(B_i) \leqslant \frac{C}{\alpha}\int \left(f^+ + Nf\right) d\mu,$$

$$\text{and} \quad \sum_i \chi_{B_i} \leqslant K.$$

*The constants* $C$ *and* $K$ *only depend on the constant in* $(D)$.

**Proof.** The proof follows the same steps as that of Proposition 3.3. We will only mention the changes that occur due to the nonhomogeneous norm. Let $f \in \widetilde{M}_1^1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. The first change is that we consider the open set

$$\Omega = \left\{x\colon \mathcal{M}_q\left(f^+ + Nf\right)(x) > \alpha\right\}.$$

We define, as in the homogeneous case, the partition of unity $\chi_i$ corresponding to the Whitney decomposition $\{\underline{B_i}\}_i$ of $\Omega$, the functions $b_i = (f - c_i)\chi_i$ with $c_i := \frac{1}{\chi_i(B_i)}\int_{B_i} f\chi_i\, d\mu$, and $g = f - \sum b_i$. In addition to the previous estimates (25)–(27) for $b_i$ and $\nabla b_i$, we need here to estimate $\|b_i\|_q$.

We begin by showing that for $x \in \Omega$,

$$|c_i| \leqslant C\alpha \tag{42}$$

for every $i \in I_x$. Set $\varphi_i = \gamma\frac{\chi_i}{\chi_i(B_i)}$. From the properties of $\chi_i$, in particular since $\chi_i(B_i) \approx \mu(B_i)$, we see that we can choose $\gamma$ (independent of $i$) so that $\varphi_i \in \mathcal{T}_1(y)$ and thus

$$|c_i| \leqslant \gamma^{-1} f^+(y) \quad \text{for all } y \in B_i.$$

Recall that the ball $\overline{B_i} = C_2 B_i$ has nonempty intersection with $F$. Taking $y_0 \in F \cap \overline{B_i}$, we get, by integrating the inequality above,

$$|c_i| \leqslant \gamma^{-1}\left(\fint_{B_i} \left(f^+\right)^q d\mu\right)^{1/q} \leqslant C\left(\fint_{\overline{B_i}} \left(f^+\right)^q d\mu\right)^{1/q} \leqslant C\mathcal{M}_q\left(f^+\right)(y_0) \leqslant C\alpha.$$

Combining this with (12), we have

$$\|b_i\|_q \leqslant \left(\int\limits_{B_i} |f - c_i|^q\right)^{\frac{1}{q}} \leqslant \left(\fint\limits_{\overline{B_i}} |f^+|^q \, d\mu\right)^{\frac{1}{q}} \mu(B_i)^{\frac{1}{q}} + |c_i| \mu(B_i)^{\frac{1}{q}} \leqslant C\alpha\mu(B_i)^{\frac{1}{q}}.$$

For $g$, we need to prove that $\|g\|_\infty \leqslant C\alpha$. We have

$$g = f\mathbb{1}_F + \sum_i c_i \chi_i. \tag{43}$$

For the first term we have $|f| \leqslant f^+ \leqslant \mathcal{M}_q(f^+)$ at all Lebesgue points and thus $|f\mathbb{1}_F| \leqslant \alpha$, $\mu$-a.e. For the second term, thanks to the bounded overlap property and (42), we get the desired estimate. □

**Proposition 4.7.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$. Let $f \in \widetilde{M}_1^1$. Then for all $\frac{s}{s+1} < q < 1$, there is a sequence of $(1, q^*)$ $(q^* = \frac{sq}{s-q})$ nonhomogeneous atoms $\{a_j\}_j$, and a sequence of scalars $\{\lambda_j\}_j$, such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } W_1^1, \quad \text{and} \quad \sum |\lambda_j| \leqslant C_q \|f\|_{\widetilde{M}_1^1}.$$

*Consequently, $\widetilde{M}_1^1 \subset HS_{q^*,\mathrm{ato}}^1$ with $\|f\|_{HS_{q^*,\mathrm{ato}}^1} \leqslant C_q \|f\|_{\widetilde{M}_1^1}$.*

**Proof.** Again, we will only mention the additional properties that one should verify in comparison with the proof of Proposition 3.4.

First let us see that (31) holds in the nonhomogeneous Sobolev space $W_1^1$. We already showed convergence in the homogeneous $\dot{W}_1^1$ norm so we only need to verify convergence in $L_1$. By (24)

$$\|g^j - f\|_1 \leqslant \sum_i \|b_i^j\|_1 \leqslant C \sum_i \int \mathbb{1}_{B_i^j} |f| \, d\mu \leqslant CK \int\limits_{\Omega_j} |f| \, d\mu \to 0, \tag{44}$$

as $j \to \infty$. Here we've used the properties of the $\chi_i^j$, the bounded overlap property of the $B_i^j$, the fact that $f \in L_1$ and that $\bigcap \Omega_j = \emptyset$ since $\mathcal{M}_q(f^+ + Nf)$ is finite $\mu$-a.e.

Taking now $j \to -\infty$, we write, by (43), (42), and the bounded overlap property

$$\int |g^j| \leqslant \int\limits_{F^j} |f| + \int \sum_i |c_i^j| \chi_i^j \leqslant \int\limits_{\{\mathcal{M}_q(f^+) \leqslant 2^j\}} \mathcal{M}_q(f^+) + CK2^j |\Omega^j| \to 0. \tag{45}$$

For the functions $\ell^j = g^{j+1} - g^j$, we have

$$\|\ell^j \chi_k^j\|_{q^*} \leqslant C2^j \mu(B_k^j)^{\frac{1}{q^*}}$$

since by Proposition 4.6, $\|g^j\|_\infty \leqslant C2^j$. This estimate also applies when we replace $\ell^j \chi_k^j$ by the moment-free “pre-atoms”

$$\begin{aligned}\ell_k^j &:= (f - c_k^j)\chi_k^j - \sum_l (f - c_l^{j+1})\chi_l^{j+1}\chi_k^j + \sum_l c_{k,l}\chi_l^{j+1} \\ &= f\Bigl(1 - \sum_l \chi_l^{j+1}\Bigr)\chi_k^j + c_k^j\chi_k^j + \sum_l c_l^{j+1}\chi_l^{j+1}\chi_k^j + \sum_l c_{k,l}\chi_l^{j+1}.\end{aligned}$$

The first term, involving $f$, is $f\mathbb{1}_{F_{j+1}}\chi_k^j$ which is bounded by $2^{j+1}$ since $|f| \leqslant f^+ \leqslant \mathcal{M}_q(f^+)$, $\mu$-a.e. For the second and third terms, we use (42) and the bounded overlap property of the $B_l^{j+1}$. Finally, that

$$|c_{k,l}| = \left| \frac{1}{\chi_l^{j+1}(B_l^{j+1})} \int_{B_l^{j+1}} (f - c_j^{l+1})\chi_l^{j+1}\chi_k^j \, d\mu \right| \leqslant c2^j$$

follows by arguing as in the proof of (42), since $\frac{\chi_l^{j+1}\chi_k^j}{\chi_l^{j+1}(B_l^{j+1})}$ can be considered as a multiple of some $\varphi \in \mathcal{T}_1(x)$ for every $x \in \overline{B_l^{j+1}}$, due to the fact that $|\nabla \chi_k^j| \lesssim (r_k^j)^{-1} \lesssim (r_l^{j+1})^{-1}$ when $B_l^{j+1} \cap B_k^j \neq \emptyset$.

Thus we obtain the stronger $L_\infty$ estimate

$$\|\ell_k^j\|_\infty \leqslant C2^j \tag{46}$$

from which we conclude, as $\ell_k^j$ is supported in the ball $(B_k^j)' = (1 + 2c)B_k^j$, that $\|\ell_k^j\|_{q^*} \leqslant C2^j\mu(B_k^j)^{\frac{1}{q^*}}$.

The rest of the proof is exactly the same as that of Proposition 3.4. □

Now we can state the converse inclusion from Theorem 2.10:

**Corollary 4.8.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$. Then*

$$H_{1,\max}(M) \subset H_{1,\mathrm{ato}}(M)$$

*with*

$$\|f\|_{H_{1,\mathrm{ato}}} \lesssim \|f^+\|_1,$$

*for any choice of $t$ in the definition of $H_1$ atoms, $1 < t \leqslant \infty$, with a constant independent of $t$.*

**Proof.** Assuming $f^+ \in L_1$ and letting

$$\Omega_j = \bigl\{x\colon \mathcal{M}_q\bigl(f^+\bigr)(x) > 2^j\bigr\},$$

we follow the steps outlined in the proofs of Propositions 4.6 and 4.7, which use only the maximal function $f^+$, while ignoring the estimates on the gradients from the proofs of Proposition 3.3 and 3.4, which are the only ones involving $Nf$. From the $L_\infty$ bound (46) we are able to obtain atoms satisfying the conditions of Definition 2.9 with $t = \infty$, hence for every other $t$ with uniform bounds. □

**Conclusion.** Let $M$ be a complete Riemannian manifold satisfying $(D)$. Then

1. for all $\frac{s}{s+1} < q < 1$,

$$\widetilde{M}_1^1 \subset HS^1_{q^*,\mathrm{ato}}.$$

2. If we moreover assume $(P_1)$, then

$$\widetilde{M}_1^1 = HS^1_{t,\mathrm{ato}}$$

for all $t > 1$.

### 4.2. *Atomic decomposition for the Sobolev space $M_1^1$*

For this we need to define new nonhomogeneous atomic spaces $LS^1_{t,\mathrm{ato}}$, where the $L$ is used to indicate that the atoms will now be in $L_1$ but not necessarily in $H_1$. Let us define our atoms.

**Definition 4.9.** For $1 < t \leqslant \infty$, we say that a function $a$ is an $LS^1_{t,\mathrm{ato}}$-atom if

1. $a$ is supported in a ball $B$;
2. $\|\nabla a\|_t \leqslant \mu(B)^{-\frac{1}{t'}}$; and
3. $\|a\|_1 \leqslant \min(1, r(B))$.

We then say that $f$ belongs to $LS^1_{t,\mathrm{ato}}$ if there exists a sequence of $LS^1_{t,\mathrm{ato}}$-atoms $\{a_j\}_j$ such that $f = \sum_j \lambda_j a_j$ in $W_1^1$, with $\sum_j |\lambda_j| < \infty$. This space is equipped with the norm

$$\|f\|_{LS^1_{t,\mathrm{ato}}} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all such decompositions.

**Remark 4.10.** As discussed previously, condition 3 in Definition 4.9 is a substitute for the cancellation condition 3 in Definition 2.11. Assuming a Poincaré inequality $(P_t)$, $LS^1_{t,\mathrm{ato}}$-atoms corresponding to small balls (with $r(B)$ bounded above) can be shown (see [11], Appendix B) to be elements of Goldberg's local Hardy space (defined by restricting the supports of the test functions in Definition 2.8 to balls of radii $r < R$ for some fixed $R$ – see [29], Section III.5.17), so that $LS^1_{t,\mathrm{ato}}$ is a subset of the "localized" space $H_{1,\mathrm{loc}}$.

As in the homogeneous case, under the Poincaré inequality $(P_1)$, $LS^1_{t,\mathrm{ato}} \subset M_1^1$:

**Proposition 4.11.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$ and $(P_1)$. Let $1 < t \leqslant \infty$ and a be an $LS^1_{t,\mathrm{ato}}$-atom. Then $a \in M_1^1$ with $\|a\|_{M_1^1} \leqslant C_t$, the constant C depending only on t, the doubling constant and the constant appearing in $(P_1)$, and independent of a.*

*Consequently* $LS^1_{t,\mathrm{ato}} \subset M^1_1$ *with*

$$\|f\|_{M^1_1} \leqslant C_t \|f\|_{LS^1_{t,\mathrm{ato}}}.$$

**Proof.** The proof follows analogously to that of Proposition 3.1, noting that we can use Remark 3.2 thanks to property 3 in Definition 4.9, and that this property also implies every atom $a$ is in $L_1$. □

Now for the converse, that is, to prove that $M^1_1 \subset LS^1_{t,\mathrm{ato}}$, we again establish an atomic decomposition for functions $f \in M^1_1$. In order to do that we must introduce an equivalent maximal function $f^\star$, which is a variant of the one originally defined by Calderón [6] and denoted by $N(f,x)$ (here we are only defining it in the special case $q=1$ and $m=1$, where for $x$ a Lebesgue point of $f$, the constant $P(x,y)$ in Calderón's definition is equal to $f(x)$, and we are allowing for the balls not to be centered at $x$).

**Definition 4.12.** Let $f \in L_{1,\mathrm{loc}}(M)$. Suppose $x$ is a Lebesgue point of $f$, i.e.

$$\lim_{r\to 0} \fint_{B(x,r)} \left|f(y) - f(x)\right| d\mu(y) = 0.$$

We define

$$f^\star(x) := \sup_{B:\, x\in B} \frac{1}{r(B)} \fint_B \left|f(y) - f(x)\right| d\mu(y).$$

Then $f^\star$ is defined $\mu$-almost everywhere.

We now show the equivalence of $f^\star$ and $Nf$. As discussed in the Introduction, the following Proposition was proved in [12] (see also [28]) in the Euclidean case.

**Proposition 4.13.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$. Then, there exist constants $C_1$, $C_2$ such that for all $f \in L_{1,\mathrm{loc}}(M)$,*

$$C_1 Nf \leqslant f^\star \leqslant C_2 Nf$$

*pointwise $\mu$-almost everywhere.*

**Proof.** Let $f \in L_{1,\mathrm{loc}}$ and $x$ be a Lebesgue point of $f$, so that there exists a sequence of balls $B_n = B(x, r_n)$ with $r_n \to 0$ and $f_{B_n} \to f(x)$. Given a ball $B$ containing $x$, take $n$ sufficiently large so that $B_n \subset B$. Since $x \in B$, there is a smallest $k \geqslant 1$ such that $2^k B_n = B(x, 2^k r_n) \supset B$, and for this $k$ we have $2^k r_n \leqslant 4r(B)$, so

$$\left|f_B - f_{B_n}(x)\right| \leqslant \fint_B |f - f_{2^k B_n}|\, d\mu + \sum_{j=1}^{k} |f_{2^j B_n} - f_{2^{j-1} B_n}|$$

$$\begin{aligned}
&\leqslant \frac{\mu(2^k B_n)}{\mu(B)} \fint_{2^k Bn} |f - f_{2^k B_n}| \, d\mu + \sum_{j=1}^{k} \frac{\mu(2^j B_n)}{\mu(2^{j-1} B_n)} \fint_{2^j B_n} |f - f_{2^j B_n}| \, d\mu \\
&\leqslant 2C_{(D)}^2 \sum_{j=1}^{k} 2^j r_n Nf(x) \\
&\leqslant 16 C_{(D)}^2 r(B) Nf(x).
\end{aligned}$$

Taking the limit as $n \to \infty$, we see that $|f_B - f(x)| \leqslant Cr(B)Nf(x)$ so that

$$\fint_B \big|f(y) - f(x)\big| \, d\mu(y) \leqslant \fint_B \big|f(y) - f_B\big| \, d\mu(y) + \big|f_B - f(x)\big| \leqslant Cr(B)Nf(x).$$

Dividing by $r(B)$ and taking the supremum over all balls $B$ containing $x$, we conclude that $f^\star(x) \leqslant CNf(x)$.

For the converse, again take any Lebesgue point $x$ and let $B$ be a ball containing $x$. Writing $|f(y) - f_B| \leqslant |f(y) - f(x)| + |\fint_B f - f(x)|$, we have

$$\fint_B \big|f(y) - f_B\big| \, d\mu(y) \leqslant 2 \fint_B \big|f(y) - f(x)\big| \, d\mu(y) \leqslant 2r(B) f^\star(x).$$

Taking the supremum over all balls $B$ containing $x$, we deduce that $Nf(x) \leqslant 2f^\star(x)$. □

**Proposition 4.14** *(Calderón–Zygmund decomposition). Let $M$ be a complete Riemannian manifold satisfying $(D)$. Let $f \in M_1^1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Then one can find a collection of balls $\{B_i\}_i$, functions $b_i \in W_1^1$ and a Lipschitz function $g$ such that the following properties hold:*

$$f = g + \sum_i b_i,$$

$$\big|g(x)\big| + \big|\nabla g(x)\big| \leqslant C\alpha \quad \text{for } \mu\text{-a.e. } x \in M, \tag{47}$$

$$\operatorname{supp} b_i \subset B_i, \qquad \|b_i\|_1 \leqslant C\alpha\mu(B_i) r_i, \qquad \big\|b_i + |\nabla b_i|\big\|_q \leqslant C\alpha\mu(B_i)^{\frac{1}{q}}, \tag{48}$$

$$\sum_i \mu(B_i) \leqslant \frac{C_q}{\alpha} \int \big(|f| + Nf\big) \, d\mu, \tag{49}$$

$$\text{and} \quad \sum_i \chi_{B_i} \leqslant K. \tag{50}$$

*The constants $C$ and $K$ only depend on the constant in $(D)$.*

**Proof.** The proof follows the same steps as that of Propositions 3.3 and 4.6. Again we will only mention the changes that occur. Let $f \in M_1^1$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. By Proposition 4.13, we have $f^\star \in L_1$ with norm equivalent to $\|Nf\|_1$. Thus if we consider the open set

$$\Omega = \big\{x \colon \mathcal{M}_q\big(|f| + f^\star\big)(x) > \alpha\big\},$$

its Whitney decomposition $\{B_i\}_i$, and the corresponding partition of unity $\{\chi_i\}_i$, we get immediately (50) and (49) by the bounded overlap property and the boundedness of the maximal function in $L_{1/q}$.

We again define $b_i = (f - c_i)\chi_i$ but this time we set $c_i = f(x_i)$ for some $x_i \in \overline{B_i}$ chosen as follows. Recall that $\overline{B_i} = 4B_i$ contains some point $y$ of $F = M \setminus \Omega$ so that

$$\fint_{\overline{B_i}} |f|^q \leqslant \mathcal{M}_q(f)^q(y) \leqslant \alpha^q \tag{51}$$

as well as

$$\fint_{\overline{B_i}} \bigl(f^\star\bigr)^q \leqslant \mathcal{M}_q\bigl(f^\star\bigr)^q(y) \leqslant \alpha^q. \tag{52}$$

Let

$$E_i = \bigl\{x \in \overline{B_i}\colon\ x \text{ is a Lebesgue point of } f \text{ and } |f|^q, \text{ and } \bigl|f(x)\bigr| \leqslant 2\alpha\bigr\}.$$

We claim that

$$\mu(E_i) \geqslant \bigl(1 - 2^{-q}\bigr)\mu(\overline{B_i}).$$

Otherwise we would have $\mu(\overline{B_i} \setminus E_i) > 2^{-q}\mu(\overline{B_i})$ and so, since $f$ and $|f|^q$ are locally integrable and the set of points which are not their Lebesgue points has measure zero,

$$\int_{\overline{B_i} \setminus E_i} |f|^q \geqslant (2\alpha)^q \mu(\overline{B_i} \setminus E_i) > \alpha^q \mu(\overline{B_i}),$$

contradicting (51).

Now we claim that for an appropriate constant $c_q$ (to be chosen independent of $i$ and $\alpha$), there exists a point $x_i \in E_i$ with

$$f^\star(x_i) \leqslant c_q \alpha. \tag{53}$$

Again, suppose not. Then we have, by (52),

$$(c_q\alpha)^q \mu(E_i) \leqslant \int_{E_i} \bigl(f^\star\bigr)^q \, d\mu \leqslant \alpha^q \mu(\overline{B_i}),$$

implying that $\mu(E_i) \leqslant c_q^{-q}\mu(\overline{B_i})$. Taking $c_q > (1 - 2^{-q})^{-1/q}$, we get a contradiction.

Thanks to our choice of $x_i$, we now have

$$|c_i| = \bigl|f(x_i)\bigr| \leqslant 2\alpha$$

and

$$\|b_i\|_1 \leqslant C \int\limits_{B_i} \big|f(y) - f(x_i)\big|\,d\mu(y) \leqslant C\mu(B_i) r_i f^\star(x_i) \leqslant C c_q r_i \alpha \mu(B_i).$$

Moreover for $\|b_i\|_q$, one has, by (51),

$$\|b_i\|_q \leqslant C\Bigg(\int\limits_{B_i} |f - c_i|^q\,d\mu\Bigg)^{\frac{1}{q}} \leqslant C\Bigg(\int\limits_{B_i} |f|^q\,d\mu\Bigg)^{\frac{1}{q}} + C2\alpha\mu(B_i)^{\frac{1}{q}} \leqslant C\alpha\mu(B_i)^{\frac{1}{q}}.$$

Finally, for $\nabla b_i$, we can estimate the $L_1$ norm by

$$\begin{aligned}
\|\nabla b_i\|_1 &\leqslant \big\|(f - c_i)\nabla\chi_i|\big\|_1 + \big\|(\nabla f)\chi_i\big\|_1 \\
&\leqslant \int\limits_{B_i} \big|f(x) - f(x_i)\big|\big|\nabla\chi_i(x)\big|\,d\mu(x) + \int\limits_{B_i} |\nabla f|\,d\mu \\
&\leqslant C\mu(B_i) f^\star(x_i) + \int\limits_{B_i} |\nabla f|\,d\mu \\
&\leqslant C c_q \alpha\mu(B_i) + \int\limits_{B_i} |\nabla f|\,d\mu, \qquad (54)
\end{aligned}$$

showing (since $|\nabla f|$ in $L_1$ by Proposition 2.6) that $b_i \in W_1^1$, and the $L_q$ norm by

$$\begin{aligned}
\|\nabla b_i\|_q^q &\leqslant \big\|(f - c_i)\nabla\chi_i|\big\|_q^q + \big\|(\nabla f)\chi_i\big\|_q^q \\
&\leqslant \mu(B_i)^{1-q}\Bigg(\int\limits_{B_i} \big|f(x) - f(x_i)\big|\big|\nabla\chi_i(x)\big|\,d\mu(x)\Bigg)^q + \int\limits_{B_i} |\nabla f|^q\,d\mu \\
&\leqslant C\mu(B_i) f^\star(x_i)^q + \int\limits_{\overline{B_i}} |Nf|^q\,d\mu \\
&\leqslant C(c_q\alpha)^q\mu(B_i) + \int\limits_{\overline{B_i}} \big|f^\star\big|^q\,d\mu \\
&\leqslant C\alpha^q\mu(B_i),
\end{aligned}$$

where we used Propositions 2.6 and 4.13, and (52). Taking the $1/q$-th power on both sides, we get (48).

It remains to prove (47). First note that $\|g\|_\infty \leqslant C\alpha$ since

$$g = f\mathbb{1}_F + \sum_i c_i\chi_i$$

and for the first term, by the Lebesgue differentiation theorem, we have $|f\mathbb{1}_F| \leqslant \mathcal{M}_q(f)\mathbb{1}_F \leqslant \alpha$, $\mu$-a.e., while for the second term, thanks to the bounded overlap property and $|c_i| \leqslant 2\alpha$, we get the desired estimate.

Now for the gradient, we write, as in (28),

$$\nabla g = \mathbb{1}_F(\nabla f) - \sum_i \bigl(f - f(x_i)\bigr)\nabla\chi_i.$$

Again we have, by Propositions 2.6 and 4.13, that $\mathbb{1}_F(|\nabla f|) \leqslant C\mathbb{1}_F(Nf) \leqslant C\mathbb{1}_F(f^\star) \leqslant C\alpha$, $\mu$-a.e. Let

$$h = \sum_i \bigl(f - f(x_i)\bigr)\nabla\chi_i.$$

We will show $|h(x)| \leqslant C\alpha$ for all $x \in M$. Note first that the sum defining $h$ is locally finite on $\Omega$ and vanishes on $F$. Then take $x \in \Omega$ and a Whitney ball $B_k$ containing $x$. As before, since $\sum_i \nabla\chi_i(x) = 0$, we can replace $f(x)$ in the sum by any constant so

$$h(x) = \sum_{i\in I_x} \bigl(f(x_k) - f(x_i)\bigr)\nabla\chi_i(x).$$

Recall that for all $i, k \in I_x$, by the construction of the Whitney collection, the balls $B_i$ and $B_k$ have equivalent radii and $B_i \subset 7B_k$. Thus

$$\begin{aligned}\bigl|f(x_k) - f(x_i)\bigr| &\leqslant \bigl|f_{7B_k} - f(x_k)\bigr| + \bigl|f_{7B_k} - f(x_i)\bigr| \\ &\leqslant \fint_{7B_k} \bigl|f - f(x_k)\bigr|\,d\mu + \fint_{7B_k} \bigl|f - f(x_i)\bigr|\,d\mu \\ &\leqslant 7r_k\bigl(f^\star(x_k) + f^\star(x_i)\bigr) \leqslant 14 r_k c_q \alpha, \end{aligned} \tag{55}$$

by (53). Therefore we again get the estimate (30). □

**Proposition 4.15.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$. Let $f \in M_1^1$. Then for all $\frac{s}{s+1} < q < 1$, there is a sequence of $LS^1_{q^*,\mathrm{ato}}$-atoms $\{a_j\}_j$ ($q^* = \frac{sq}{s-q}$), as in Definition 4.9, and a sequence of scalars $\{\lambda_j\}_j$, such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } W_1^1, \quad \text{and} \quad \sum |\lambda_j| \leqslant C_q \|f\|_{M_1^1}.$$

*Consequently, $M_1^1 \subset HS^1_{q^*,\mathrm{ato}}$ with $\|f\|_{LS^1_{q^*,\mathrm{ato}}} \leqslant C_q\|f\|_{M_1^1}$.*

**Proof.** Here as well we will only mention the additional properties that one should verify in comparison with Proposition 3.4 and 4.7. We use the Calderón–Zygmund decomposition (Proposition 4.14) above with $\Omega^j$ corresponding to $\alpha = 2^j$, and denote the resulting functions by $g^j$ and $b_i^j$, recalling that for the definition of the constant $c_i^j$ we have $c_i^j = f(x_i^j)$ for a specially chosen point $x_i^j \in \overline{B_i^j}$.

First let us see that $g^j \to f$ in $W_1^1$. For the convergence in $L_1$ we just repeat (44) and (45) from the nonhomogeneous case, replacing $f^+$ by $|f|$. For the convergence in $\dot{W}_1^1$, we can estimate $\sum_i \|\nabla b_i^j\|_1$ exactly as in (32), using (54) instead of (26), and replacing $Nf$ by $f^\star$ and $\mathcal{M}_q(Nf)$ by $\mathcal{M}_q(|f|+f^\star)$. This gives $\nabla g^j \to \nabla f$ in $L_1$ as $j\to\infty$. For the convergence of $\nabla g^j$ to 0 as $j\to-\infty$, we imitate (33) and (34), using (28) and (30) with $f^\star$ and our new choice of $c_i^j$.

We define the functions $\ell^j = g^{j+1}-g^j$ as in Proposition 3.4 but this time we just use

$$\ell_k^j := \ell^j\chi_k^j$$

for the "pre-atoms", since we no longer need to have the moment condition $\int \ell_k^j = 0$ (see Remark 3.6). From the $L_\infty$ bounds (47) on $g^j$ and $\nabla g^j$ in Proposition 4.14, we immediately get

$$\big\|\ell_k^j\big\|_1 \leqslant C2^j\mu\big(B_k^j\big)$$

and $\||\nabla\ell^j|\chi_k^j\|_{q^*} \leqslant C2^j\mu(B_k^j)^{1/q^*}$. We need a similar estimate on $\|\ell^j|\nabla\chi_k^j|\|_{q^*}$ in order to bound $\|\nabla\ell_k^j\|_{q^*}$. As in (36), write

$$r_k^j\bigg(\fint_{B_k^j}\big|\ell^j\nabla\chi_k^j\big|^{q^*}\,d\mu\bigg)^{1/q^*} \leqslant C\bigg(\fint_{B_k^j}\bigg(\sum_i \mathbb{1}_{B_i^j}\big|f-c_i^j\big| + \sum_l \mathbb{1}_{B_l^{j+1}}\big|f-c_l^{j+1}\big|\bigg)^{q^*}\,d\mu\bigg)^{1/q^*}$$

Expanding $|f-c_i^j| = |f-f_{B_k^j}+f_{B_k^j}-c_k^j+c_k^j-c_i^j|$ and using the bounded overlap property of the balls, the Sobolev–Poincaré inequality (6), Proposition 4.13, and properties (53) and (55) of the constants $c_i^j = f(x_i^j)$, we have for the integral of the first sum on the right-hand side:

$$\begin{aligned}
&\bigg(\fint_{B_k^j}\bigg(\sum_i \mathbb{1}_{B_i^j}\big|f-c_i^j\big|\bigg)^{q^*}\,d\mu\bigg)^{1/q^*}\\
&\quad\leqslant K\bigg(\fint_{B_k^j}|f-f_{B_k^j}|^{q^*}\,d\mu\bigg)^{1/q^*} + K\big|f_{B_k^j}-c_k^j\big| + \bigg(\fint_{B_k^j}\bigg(\sum_{B_i^j\cap B_k^j\neq\emptyset}\mathbb{1}_{B_i^j}\big|c_k^j-c_i^j\big|\bigg)^{q^*}\,d\mu\bigg)^{1/q^*}\\
&\quad\leqslant CKr_k^j\bigg(\fint_{\overline{B_k^j}}(Nf)^q\bigg)^{1/q} + Kr_k^j f^\star\big(x_k^j\big) + CKr_k^j2^j\\
&\quad\leqslant CKr_k^j2^j.
\end{aligned}$$

The analogous estimate holds for the integral of the second sum, in $l$, since as pointed out previously, when $B_l^{j+1}\cap B_k^j \neq \emptyset$ we have that $r_l^{j+1} \leqslant cr_k^j$. This gives

$$\big\|\nabla\ell_k^j\big\|_{q^*} \leqslant \gamma 2^j\mu\big(\big(B_k^j\big)'\big)^{1/q^*},$$

as desired. The rest of the proof follows in the same way as that of Propositions 3.4 and 4.7. □

**Conclusion.** Let $M$ be a complete Riemannian manifold satisfying $(D)$. Then

1. for all $\frac{s}{s+1} < q < 1$,

$$M_1^1 \subset LS^1_{q^*,\mathrm{ato}}.$$

2. If moreover we assume $(P_1)$, then

$$M_1^1 = LS^1_{t,\mathrm{ato}}$$

for all $t > 1$.

## 5. Comparison between $\dot{M}_1^1$ and Hardy–Sobolev spaces defined in terms of derivatives

### 5.1. Using a maximal function definition

In the Euclidean case, the homogeneous Hardy–Sobolev space $\dot{HS}^1$ consists of all locally integrable functions $f$ such that $\nabla f \in H_1(\mathbb{R}^n)$ (i.e. the weak partial derivatives $D_j f = \frac{\partial f}{\partial x_j}$ belong to the real Hardy space $H_1(\mathbb{R}^n)$). In [28], it was proved that this space is nothing else than $\{f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)\colon\ Nf \in L_1\}$, which also coincides with the Sobolev space $\dot{M}_1^1$ [22].

Does this theory extends to the case of Riemannian manifolds? If this is the case, which hypotheses should one assume on the geometry of the manifold? We proved an atomic characterization of $\dot{M}_1^1$ but we would like to clarify the relation with Hardy–Sobolev spaces defined using maximal functions.

**Definition 5.1.** We define the (maximal) homogeneous Hardy–Sobolev space $\dot{HS}^1_{\max}$ as follows:

$$\dot{HS}^1_{\max} := \bigl\{f \in L_{1,\mathrm{loc}}(M)\colon\ (\nabla f)^+ \in L_1\bigr\}$$

where $\nabla f$ is the distributional gradient, as defined in (7), and the corresponding maximal function is defined, analogously to (10), by

$$(\nabla f)^+(x) := \sup\left|\int f\bigl(\langle \nabla\varphi, \boldsymbol{\Phi}\rangle + \varphi \operatorname{div} \boldsymbol{\Phi}\bigr)\, d\mu\right|,$$

where the supremum is taken over all pairs $\varphi \in \mathcal{T}_1(x)$, $\boldsymbol{\Phi} \in C_0^1(M, TM)$ such that

$$\|\boldsymbol{\Phi}\|_\infty \leqslant 1 \quad \text{and} \quad \|\operatorname{div} \boldsymbol{\Phi}\|_\infty \leqslant \frac{1}{r}$$

for the radius $r$ of the same ball $B$ containing $x$ for which $\varphi$ satisfies (11). We equip this space with the semi-norm

$$\|f\|_{\dot{HS}^1_{\max}} = \bigl\|(\nabla f)^+\bigr\|_1.$$

Note that in case both $\varphi$ and $\boldsymbol{\Phi}$ are smooth, the quantity $\langle \nabla\varphi, \boldsymbol{\Phi}\rangle + \varphi \operatorname{div} \boldsymbol{\Phi}$ represents the divergence of the product $\varphi\boldsymbol{\Phi}$, so the definition coincides with that of the maximal function $M^{(1)}f$ given in [3] for the case of domains in $\mathbb{R}^n$, but here we want to allow for the case of Lipschitz $\varphi$.

**Proposition 5.2.** *Let $f \in \dot{H}S^1_{\max}$. Then $\nabla f$, initially defined by* (7)*, is given by an $L_1$ function and satisfies*

$$|\nabla f| \leqslant C(\nabla f)^+, \quad \mu\text{-a.e.}$$

*Consequently,*

$$\dot{H}S^1_{\max} \subset \dot{W}^1_1$$

*with*

$$\|f\|_{\dot{W}^1_1} \leqslant C\|f\|_{\dot{H}S^1_{\max}}.$$

**Proof.** We follow the ideas in the proof of Proposition 2.6. Let $\Omega$ be any open subset of $M$ and consider the total variation of $u$ on $\Omega$, defined by

$$|Df|(\Omega) := \sup\bigl|\langle \nabla f, \boldsymbol{\Phi}\rangle\bigr|,$$

where the supremum is taken over all vector fields $\boldsymbol{\Phi} \in C^1_0(\Omega, TM)$ with $\|\boldsymbol{\Phi}\|_\infty \leqslant 1$. For such a vector field $\boldsymbol{\Phi}$, take $r > 0$ sufficiently small so that $\|\operatorname{div}\boldsymbol{\Phi}\|_\infty \leqslant r^{-1}$ and $\operatorname{dist}(\operatorname{supp}(\boldsymbol{\Phi}), M \setminus \Omega) > 12r$. As in the proof of Proposition 2.6, take a collection of balls $B_i = B(x_i, r)$ with $6B_i$ having bounded overlap (with a constant $K$ independent of $r$), covering $M$, and a Lipschitz partition of unity $\{\varphi_i\}_i$ subordinate to $\{6B_i\}_i$, with $0 \leqslant \varphi_i \leqslant 1$ and $|\nabla\varphi_i| \leqslant r^{-1}$. Then for all $x \in B_i$, $\varphi_i/\mu(B_i) \in \mathcal{T}_1(x)$, so

$$\left|\int f\bigl[\langle \nabla\varphi_i, \boldsymbol{\Phi}\rangle + \varphi_i \operatorname{div}\boldsymbol{\Phi}\bigr]\,d\mu\right| \leqslant (\nabla f)^+(x)\mu(B_i).$$

Hence

$$\left|\int f\bigl[\langle \nabla\varphi_i, \boldsymbol{\Phi}\rangle + \varphi_i \operatorname{div}\boldsymbol{\Phi}\bigr]\,d\mu\right| \leqslant \int_{B_i} (\nabla f)^+(x)\,d\mu.$$

Summing up over $i$ such that $6B_i \subset \Omega$, by the choice of $r$ we still get $\sum \varphi_i = 1$ on the support of $\boldsymbol{\Phi}$, hence $\sum \nabla\varphi_i = 0$, so using the bounded overlap of the balls we have

$$\left|\int f \operatorname{div}\boldsymbol{\Phi}\,d\mu\right| \leqslant \sum_{\{i:\, 6B_i\subset\Omega\}} \int_{B_i} (\nabla f)^+\,d\mu \leqslant K\int_{\Omega} (\nabla f)^+\,d\mu \leqslant K\bigl\|(\nabla f)^+\bigr\|_1 < \infty.$$

The rest of the proof proceeds as in the proof of Proposition 2.6, replacing $Nf$ by $(\nabla f)^+$. □

**Proposition 5.3.** *Let* $f \in L_{1,\mathrm{loc}}$. *Then at every point of* $M$,

$$(\nabla f)^+ \leqslant Nf.$$

*Consequently,*

$$\dot{M}_1^1 \subset \dot{HS}^1_{\max}$$

*with*

$$\|f\|_{\dot{HS}^1_{\max}} \leqslant C\|f\|_{\dot{M}_1^1}.$$

**Proof.** Let $f \in L_{1,\mathrm{loc}}$ and $x \in M$. Take $\varphi \in \mathcal{T}_1(x)$, $\boldsymbol{\Phi} \in C_0^1(M, TM)$ as in Definition 5.1. Then

$$\int \bigl(\langle \nabla\varphi, \boldsymbol{\Phi}\rangle + \varphi \operatorname{div} \boldsymbol{\Phi}\bigr)\, d\mu = 0$$

so we can write

$$\begin{aligned} \left|\int f\bigl(\langle \nabla\varphi, \boldsymbol{\Phi}\rangle + \varphi \operatorname{div} \boldsymbol{\Phi}\bigr)\, d\mu\right| &= \left|\int (f - f_B)\bigl(\langle \nabla\varphi, \boldsymbol{\Phi}\rangle + \varphi \operatorname{div} \boldsymbol{\Phi}\bigr)\, d\mu\right| \\ &\leqslant \frac{1}{r\mu(B)} \int |f - f_B|\, d\mu \\ &\leqslant Nf(x). \qquad \square \end{aligned}$$

We would like to prove the reverse inclusion. However, this would require some tools such as Lemma 6 in [22] or Lemma 10 in [3] (solving $\operatorname{div} \Psi = \phi$ with $\Psi$ having compact support) which are particular to $\mathbb{R}^n$.

Another possible maximal function we can use, following the ideas in [21] (see Section 4.1), is given by

**Definition 5.4.**

$$\mathcal{M}^*(\nabla f)(x) := \sup_j |\nabla f_{r_j}|$$

with the “discrete convolution” $f_{r_j}$ defined as in (8), corresponding to an enumeration of the positive rationals $\{r_j\}_j$, where for each $j$ we have a covering of $M$ by balls $\{B_i^j\}_i$ of radius $r_j$, and a partition of unity $\varphi_i^j$ subordinate to this covering.

We have already shown in the proof of Proposition 2.6 (see (9)) that

**Lemma 5.5.** *Let* $f \in L_{1,\mathrm{loc}}$. *Then at* $\mu$*-almost every point of* $M$,

$$\mathcal{M}^*(\nabla f) \leqslant Nf.$$

### 5.2. Derivatives of molecular Hardy spaces

As noted in the previous section, on a manifold, obtaining a decomposition with atoms of compact support from a maximal function definition is not obvious. In [4], the authors considered instead Hardy spaces generated by molecules. We begin by recalling their definition of $H_{\mathrm{mol},1}(\Lambda^1 T^*M)$ (a special case with $N = 1$ of $H^1_{\mathrm{mol},N}(\Lambda T^*M)$ in Definition 6.1 of [4], where we have dropped the superscript 1 for convenience). If in addition the heat kernel on $M$ satisfies Gaussian upper bounds, this space coincides with the space $H^1(\Lambda T^*M)$, which also has a maximal function characterization (see [4], Theorem 8.4).

A sequence of non-negative Lipschitz functions $\{\chi_k\}_k$ is said to be (a partition of unity) adapted to a ball $B$ of radius $r$ if $\operatorname{supp}\chi_0 \subset 4B$, $\operatorname{supp}\chi_k \subset 2^{k+2}B \setminus 2^{k-1}B$ for all $k \geqslant 1$,

$$\|\nabla\chi_k\|_\infty \leqslant C2^{-k}r^{-1} \tag{56}$$

and

$$\sum_k \chi_k = 1 \quad \text{on } M.$$

A 1-form $a \in L^2(\Lambda^1 T^*M)$ is called a 1-molecule if $a = db$ for some $b \in L_2(M)$ and there exists a ball $B$ with radius $r$, and a partition of unity $\{\chi_k\}_k$ adapted to $B$, such that for all $k \geqslant 0$

$$\|\chi_k a\|_{L^2(\Lambda^1 T^*M)} \leqslant 2^{-k}\bigl(\mu\bigl(2^k B\bigr)\bigr)^{-1/2} \tag{57}$$

and

$$\|\chi_k b\|_2 \leqslant 2^{-k} r\bigl(\mu\bigl(2^k B\bigr)\bigr)^{-1/2}.$$

Summing in $k$, this implies that $\|a\|_{L^2(\Lambda^1 T^*M)} \leqslant 2(\mu(B))^{-1/2}$ and $\|b\|_{L^2} \leqslant 2r(\mu(B))^{-1/2}$. Moreover, there exists a constant $C'$, depending only on the doubling constant in $(D)$, such that

$$\|\mathbb{1}_{2^{k+2}B\setminus 2^{k-1}B}b\|_2 \leqslant \left\|\sum_{l=k-3}^{k+3}\chi_l b\right\|_2 \leqslant C' r 2^{-k}\bigl(\mu\bigl(2^{k+2}B\bigr)\bigr)^{-1/2}. \tag{58}$$

**Definition 5.6.** (See [4].) We say that $f \in H_{\mathrm{mol},1}(\Lambda^1 T^*M)$ if there is a sequence $\{\lambda_j\}_j \in \ell^1$ and a sequence of 1-molecules $\{a_j\}_j$ such that

$$f = \sum_j \lambda_j a_j$$

in $L_1(\Lambda^1 T^*M)$, with the norm defined by

$$\|f\|_{H_{\mathrm{mol},1}(\Lambda^1 T^*M)} = \inf \sum_j |\lambda_j|.$$

Here the infimum is taken over all such decompositions. The space $H_{\mathrm{mol},1}(\Lambda^1 T^*M)$ is a Banach space.

**Proposition 5.7.** *Let $M$ be a complete Riemannian manifold satisfying $(D)$ and $(P_1)$. We then have*

$$H_{\mathrm{mol},1}\big(\Lambda^1 T^*M\big) = d\big(\dot{H}S^1_{2,\mathrm{ato}}(M)\big). \tag{59}$$

*Moreover*

$$\|g\|_{H_{\mathrm{mol},1}(\Lambda^1 T^*M)} \sim \inf_{df=g} \|f\|_{\dot{H}S^1_{2,\mathrm{ato}}(M)}.$$

*Consequently, in this case we have an atomic decomposition for $H_{\mathrm{mol},1}(\Lambda^1 T^*M)$ (this was already proved in [4], after Theorem 8.4).*

**Remark 5.8.** As pointed out in Remarks 3.2 and 3.6, we can define the atomic Hardy–Sobolev space $\dot{H}S^1_{2,\mathrm{ato}}(M)$ by using $(1,2)$-atoms satisfying condition $3''$ of Remarks 2.12 instead of condition 3 of Definition 2.11. As will be seen from the proof below, if we restrict ourselves to this kind of atoms we do not require the hypothesis $(P_1)$ for (59). Under the assumption $(P_1)$, we actually get the stronger conclusion

$$H_{\mathrm{mol},1}\big(\Lambda^1 T^*M\big) = d\big(\dot{H}S^1_{2,\mathrm{ato}}\big) = d\big(\dot{H}S^1_{t,\mathrm{ato}}\big) = d\big(\dot{M}^1_1\big)$$

for all $t > 1$.

**Proof.** Take $f \in \dot{H}S^1_{2,\mathrm{ato}}$. There exists a sequence $\{\lambda_j\}_j \in \ell^1$ and $(1,2)$-atoms $b_j$ such that $f = \sum_j \lambda_j b_j$ in $\dot{W}^1_1$. This means $\sum_j \lambda_j \nabla b_j$ converges in $L_1$ to $\nabla f$, and by the isometry between the vector fields and the 1-forms, we have $df = \sum_j \lambda_j db_j$ in $L_1(\Lambda^1 T^*M)$.

We claim that $a_j = db_j$ are 1-molecules. Indeed, fix $j$, take $B_j$ to be the ball containing the support of $b_j$ and let $\{\chi_j^k\}_k$ be a partition of unity adapted to $B_j$. Then

$$\big\|\chi_j^0 a_j\big\|_2 \leqslant \|db_j\|_2 = \|\nabla b_j\|_2 \leqslant \frac{1}{\mu(B_j)^{\frac{1}{2}}}$$

and by condition $3''$ of Remarks 2.12 (alternatively condition 3 of Definition 2.11 and $(P_1)$) we get

$$\big\|\chi_j^0 b_j\big\|_2 \leqslant \|b_j\|_2 \leqslant r_j \frac{1}{\mu(B_j)^{\frac{1}{2}}}.$$

For $k \geqslant 1$, there is nothing to do since $\operatorname{supp} b_j \subset B_j$ and $\operatorname{supp} \chi_j^k \subset 2^{k+2}B_j \setminus 2^{k-1}B_j \subset (B_j)^c$. Consequently, $df \in H_{\mathrm{mol},1}(\Lambda^1 T^*M)$ with $\|df\|_{H_{\mathrm{mol},1}(\Lambda^1 T^*M)} \leqslant \sum_j |\lambda_j|$. Taking the infimum over all such decompositions, we get $\|df\|_{H_{\mathrm{mol},1}(\Lambda^1 T^*M)} \leqslant \|f\|_{\dot{H}S^1_{2,\mathrm{ato}}}$.

Now for the converse, let $g \in H_{\mathrm{mol},1}(\Lambda^1 T^* M)$. Write

$$g = \sum_j \lambda_j a_j := \sum_j \lambda_j d b_j$$

where $\sum_j |\lambda_j| < \infty$, for every $j$, $a_j$ is a 1-molecule associated to a ball $B_j$, and the convergence is in $L_1$. Let $\{\chi_j^k\}_k$ be the partition of unity adapted to $B_j$. Then

$$g = \sum_j \lambda_j \sum_k d b_j \chi_j^k = \sum_j \lambda_j d\biggl(\sum_k b_j \chi_j^k\biggr) = \sum_j \lambda_j \sum_k d\bigl(b_j \chi_j^k\bigr)$$

since the sum is locally finite and $\sum_k \chi_j^k = 1$.

We claim that for every $j$, $k$, $\beta_j^k := 2^{k-1} \gamma b_j \chi_j^k$, with $\gamma$ a constant to be determined, satisfies properties 1, 2 and 3″ (see Definition 2.11 and Remarks 2.12) of a $(1,2)$-homogeneous Hardy–Sobolev atom. Indeed, $\beta_j^k$ is supported in the ball $2^{k+2} B_j$ with

$$\|\beta_j^k\|_2 \leqslant 2^{k-1} \gamma \frac{2^{-k} r_j}{\mu(2^k B_j)^{\frac{1}{2}}} \leqslant \frac{2^{k+2} r_j}{\mu(2^{k+2} B_j)^{\frac{1}{2}}}$$

for an appropriate choice of $\gamma$ depending only on the doubling constant in $(D)$. Furthermore, by (57), (56), and (58),

$$\begin{aligned}
\|\nabla \beta_j^k\|_2 &= 2^{k-1} \gamma \bigl\|d\bigl(b_j \chi_j^k\bigr)\bigr\|_2 \\
&\leqslant 2^{k-1} \gamma \bigl(\|a_j \chi_j^k\|_2 + \|b_j d \chi_j^k\|_2\bigr) \\
&\leqslant 2^{k-1} \gamma \bigl(2^{-k} \bigl(\mu\bigl(2^k B_j\bigr)\bigr)^{-1/2} + C 2^{-k} r_i^{-1} \|\mathbb{1}_{2^{k+2} B_j \setminus 2^{k-1} B_j} b_j \|_2\bigr) \\
&\leqslant \mu\bigl(2^{k+2} B_j\bigr)^{-1/2}.
\end{aligned}$$

Here we again chose $\gamma$ conveniently, depending only on the doubling constant, and used the fact that $k \geqslant 0$.

Since $\sum_{j,k} |\lambda_j| \gamma^{-1} 2^{1-k} \leqslant 4\gamma^{-1} \sum_j |\lambda_j| < \infty$, the sum $f := \sum_j \lambda_j \sum_k \gamma^{-1} 2^{1-k} \beta_j^k$ defines an element of $\dot{H}S^1_{2,\mathrm{ato}}$, with the convergence being in $\dot{W}^1_1$. This means that in $L_1$ we have

$$df = d\biggl(\sum_{j,k} \lambda_j \bigl(b_j \chi_j^k\bigr)\biggr) = \sum_j \lambda_j \sum_k d\bigl(b_j \chi_j^k\bigr) = g.$$

Therefore $g = df = d(\sum_{j,k} \lambda_j (b_j \chi_j^k))$, with $\|f\|_{\dot{H}S^1_{2,\mathrm{ato}}} \leqslant 4\gamma^{-1} \sum_j |\lambda_j|$. Taking the infimum over all such decompositions of $g$, we see that

$$\inf_{df = g} \|f\|_{\dot{H}S^1_{2,\mathrm{ato}}} \leqslant 4\gamma^{-1} \|g\|_{H_{\mathrm{mol},1}(\Lambda^1 T^* M)}. \qquad \square$$

**Corollary 5.9.** *In the Euclidean case, we then obtain*

$$H_{\mathrm{mol},1}\left(\mathbb{R}^n, \Lambda^1\right) = \mathcal{H}^1_d\left(\mathbb{R}^n, \Lambda^1\right) = d\left(\dot{M}^1_1\right) = d\left(\dot{HS}^1_{t,\mathrm{ato}}\right)$$

*for all* $t > 1$. (*For details on* $\mathcal{H}^1_d(\mathbb{R}^n, \Lambda^1)$, *see* [24].)

## References

[1] L. Ambrosio, M. Miranda Jr., D. Pallara, Special functions of bounded variation in doubling metric measure spaces, in: Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, in: Quad. Mat., vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 1–45.

[2] P. Auscher, T. Coulhon, Riesz transform on manifolds and Poincaré inequalities, Ann. Sc. Norm. Super. Pisa 5 (2005) 531–555.

[3] P. Auscher, E. Russ, P. Tchamitchian, Hardy–Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$, J. Funct. Anal. 218 (2005) 54–109.

[4] P. Auscher, A. McIntosh, E. Russ, Hardy spaces of differential forms on Riemannian manifolds, J. Geom. Anal. 18 (2008) 192–248.

[5] N. Badr, F. Bernicot, Abstract Hardy–Sobolev spaces and interpolation, J. Funct. Anal. 259 (2010) 1169–1208.

[6] A.P. Calderón, Estimates for singular integral operators in terms of maximal functions, in: Collection of Articles Honoring the Completion by Antoni Zygmund of 50 Years of Scientific Activity, VI, Studia Math. 44 (1972) 563–582.

[7] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (1999) 428–517.

[8] Y.-K. Cho, J. Kim, Atomic decomposition on Hardy–Sobolev spaces, Studia Math. 177 (2006) 25–36.

[9] R. Coifman, G. Weiss, Analyse harmonique sur certains espaces homogènes, Lecture Notes in Math., Springer-Verlag, 1971.

[10] R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977) 569–645.

[11] G. Dafni, Hardy spaces on strongly pseudoconvex domains in $\mathbb{C}^n$ and domains of finite type in $\mathbb{C}^2$, PhD thesis, Princeton University, 1993.

[12] R.A. DeVore, R.C. Sharpley, Maximal functions measuring smoothness, Mem. Amer. Math. Soc. 47 (1984).

[13] B. Franchi, P. Hajłasz, P. Koskela, Definitions of Sobolev classes on metric spaces, Ann. Inst. Fourier 49 (1999) 1903–1924.

[14] A.E. Gatto, C. Segovia, J.R. Jiménez, On the solution of the equation $\Delta^m F = f$ for $f \in H^p$, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vols. I, II, Chicago, IL, 1981, in: Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 394–415.

[15] P. Hajłasz, Sobolev spaces on an arbitrary metric space, Potential Anal. 5 (1996) 403–415.

[16] P. Hajłasz, Sobolev spaces on metric-measure spaces, in: Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces, Paris, 2002, in: Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, pp. 173–218.

[17] P. Hajłasz, J. Kinnunen, Hölder quasicontinuity of Sobolev functions on metric spaces, Rev. Mat. Iberoamericana 14 (1998) 601–622.

[18] P. Hajłasz, P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. 145 (2000) 1–101.

[19] J. Heinonen, Lectures on Analysis on Metric Spaces, Universitext, Springer-Verlag, New York, 2001.

[20] S. Janson, On functions with derivatives in $H_1$, in: Harmonic Analysis and Partial Differential Equations, El Escorial, 1987, in: Lecture Notes in Math., vol. 1384, Springer-Verlag, Berlin, 1989, pp. 193–201.

[21] J. Kinnunen, H. Tuominen, Pointwise behaviour of $M^{1,1}$ Sobolev functions, Math. Z. 257 (2007) 613–630.

[22] P. Koskela, E. Saksman, Pointwise characterizations of Hardy–Sobolev functions, Math. Res. Lett. 15 (2008) 727–744.

[23] P. Koskela, D. Yang, Y. Zhou, A characterization of Hajłasz–Sobolev and Triebel–Lizorkin spaces via grand Littlewood–Paley functions, J. Funct. Anal. 258 (2010) 2637–2661.

[24] Z. Lou, A. McIntosh, Hardy spaces of exacts forms on $\mathbb{R}^n$, Trans. Amer. Math. Soc. 357 (2005) 1469–1496.

[25] Z. Lou, S. Yang, An atomic decomposition for the Hardy–Sobolev space, Taiwanese J. Math. 11 (2007) 1167–1176.

[26] R.A. Macias, C. Segovia, A decomposition into atoms of distributions on spaces of homogeneous type, Adv. Math. 33 (1979) 271–309.

[27] M. Miranda Jr., D. Pallara, F. Paronetto, M. Preunkert, Heat semigroup and functions of bounded variation on Riemannian manifolds, J. Reine Angew. Math. 613 (2007) 99–119.

[28] A. Miyachi, Hardy–Sobolev spaces and maximal functions, J. Math. Soc. Japan 42 (1990) 73–90.

[29] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Math. Ser., vol. 43, Princeton University Press, Princeton, NJ, 1993.

[30] R. Strichartz, $H^p$ Sobolev spaces, Colloq. Math. LX/LXI (1990) 129–139.

[31] A. Uchiyama, A maximal function characterization of $H^p$ on the space of homogeneous type, Trans. Amer. Math. Soc. 262 (1980) 579–592.

[32] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Scott, Foresman and Company, Glenview, IL, 1971.