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Nonhomogeneous Div-Curl Decompositions for Local Hardy Spaces on a Domain

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. We prove div-curl type lemmas for the local spaces of functions of bounded mean oscillation on Ω, $\mathrm{bmo}_r(\Omega)$ and $\mathrm{bmo}_z(\Omega)$, resulting in decompositions for the corresponding local Hardy spaces $h^1_z(\Omega)$ and $h^1_r(\Omega)$ into nonhomogeneous div-curl quantities.

1. Div-curl lemmas for Hardy spaces and BMO on $\mathbb{R}^n$

This article is an outgrowth, among many others, of the results of Coifman, Lions, Meyer and Semmes ([7]) which connected the div-curl lemma, part of the theory of compensated compactness developed by Tartar and Murat, to the theory of real Hardy spaces in $\mathbb{R}^n$ (see [10]). In particular, denote by $H^1(\mathbb{R}^n)$ the space of distributions (in fact L^1 functions) f on $\mathbb{R}^n$ satisfying

$$\mathcal{M}_\phi(f) \in L^1(\mathbb{R}^n) \tag{1}$$

for some fixed choice of Schwartz function ϕ with $\int \phi = 1$, with the maximal function $\mathcal{M}_\phi$ defined by

$$\mathcal{M}_\phi(f)(x) = \sup_{0<t<\infty} |f * \phi_t(x)|, \qquad \phi_t(\cdot) = t^{-n}\phi(t^{-1}\cdot).$$

One version of the results in [7] states that for exponents p, q with $1 < p < \infty$, $1/p + 1/q = 1$, and vector fields $\vec{V}$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, $\vec{W}$ in $L^q(\mathbb{R}^n, \mathbb{R}^n)$ with

$$\operatorname{div} \vec{V} = 0, \qquad \operatorname{curl} \vec{W} = 0$$

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This is the final form of the paper.

in the sense of distributions, the scalar (dot) product $f = \vec{V} \cdot \vec{W}$ belongs to $H^1(\mathbb{R}^n)$. Moreover, one can bound the H^1 norm (defined, say, as the L^1 norm of $\mathcal{M}_\phi(f)$) by $\|\vec{V}\|_{L^p}\|\vec{W}\|_{L^q}$.

While a local version of this result, in terms of H^1_{loc}, is given in [7], in order to obtain norm estimates we use instead the local Hardy space $h^1(\mathbb{R}^n)$. This was defined by Goldberg (see [11]) by replacing the maximal function in (1) by its "local" version

$$m_\phi(f)(x) = \sup_{0<t<1} |f * \phi_t(x)|. \tag{2}$$

Again the norm can be given by $\|m_\phi(f)\|_{L^1(\mathbb{R}^n)}$ and different choices of ϕ give equivalent norms. In addition, we can replace the number 1 in (2) by any finite constant without changing the space.

For this space the following nonhomogeneous versions of the div-curl lemma can be shown (these are special cases of Theorems 3 and 4 in [8]):

Theorem 1 ([8])**.** *Suppose $\vec{v}$ and $\vec{w}$ are vector fields on $\mathbb{R}^n$ satisfying*

$$\vec{V} \in L^p(\mathbb{R}^n)^n, \quad \vec{W} \in L^q(\mathbb{R}^n)^n, \qquad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(a) *Assume*

$$\operatorname{div} \vec{V} = f \in L^p(\mathbb{R}^n), \qquad \operatorname{curl} \vec{W} = 0 \tag{3}$$

in the sense of distributions. Then $\vec{V} \cdot \vec{W}$ belongs to the local Hardy space $h^1(\mathbb{R}^n)$ with

$$\|\vec{V} \cdot \vec{W}\|_{h^1(\mathbb{R}^n)} \leq C(\|\vec{V}\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)})\|\vec{W}\|_{L^q(\mathbb{R}^n)}. \tag{4}$$

(b) *If $M^{n\times n}$ denotes the space of n-by-n matrices over $\mathbb{R}$ and*

$$\operatorname{div} \vec{V} = 0, \quad \operatorname{curl} \vec{W} = B \in L^q(\mathbb{R}^n, M^{n\times n}) \tag{5}$$

in the sense of distributions, then $\vec{V} \cdot \vec{W}$ belongs to the local Hardy space $h^1(\mathbb{R}^n)$ with

$$\|\vec{V} \cdot \vec{W}\|_{h^1(\mathbb{R}^n)} \leq C\|\vec{V}\|_{L^p(\mathbb{R}^n)}\left[\|\vec{W}\|_{L^q(\mathbb{R}^n)} + \sum_{i,j}\|B_{ij}\|_{L^q(\mathbb{R}^n)}\right]. \tag{6}$$

Before continuing further, let us make clear what we mean by the divergence and curl of a vector field in the sense of distributions. Let Ω be an open subset of $\mathbb{R}^n$, and suppose $\vec{v} = (v_1, \ldots, v_n)$ with v_i locally integrable on Ω. For a locally integrable function f on Ω, one says that $\operatorname{div} \vec{v} = f$ in the sense of distributions on Ω if

$$\int_\Omega \vec{v} \cdot \vec{\nabla}\varphi = -\int_\Omega f\varphi \tag{7}$$

for all $\varphi \in C_0^\infty(\Omega)$ (i.e., smooth functions with compact support in Ω).

Similarly, if $\vec{w} = (w_1, \ldots, w_n)$ with w_i locally integrable on Ω , and B is an $n \times n$ matrix of locally integrable functions B_{ij} on Ω, we say $\operatorname{curl} \vec{w} = B$ in the sense of distributions on Ω if

$$\int_\Omega w_i \frac{\partial\varphi}{\partial x_j} - w_j \frac{\partial\varphi}{\partial x_i} = -\int_\Omega B_{ij}\varphi \tag{8}$$

for all $i, j \in \{1, \ldots, n\}$ and all $\varphi \in C_0^\infty(\Omega)$. If the components of $\vec{v}$ or $\vec{w}$ are sufficiently smooth, these definitions are equivalent to the classical notions of divergence and curl via integration by parts.

Recall that C. Fefferman [9] identified the dual of the real Hardy space H^1 with the space BMO of functions of bounded mean oscillation, introduced by John and Nirenberg [13]. In the local case, Goldberg [11] showed that the dual of $h^1(\mathbb{R}^n)$ can be identified with the space $\mathrm{bmo}(\mathbb{R}^n)$, the Banach space of locally integrable functions f which satisfy

$$\|f\|_{\mathrm{bmo}} := \sup_{|I|\le 1} \frac{1}{|I|}\int_I |f - f_I| + \sup_{|I|>1} \frac{1}{|I|}\int_I |f| < \infty. \tag{9}$$

Here I can be used to denote either balls or cubes with sides parallel to the axes, $|I|$ denotes Lebesgue measure (volume) and f_I is the mean of f over I, i.e., $(1/|I|)\int_I f$. As in the case of h^1, the upper-bound 1 on the size of the cubes in the definition can be replaced by any other finite nonzero constant, resulting in an equivalent norm.

In [5], the authors prove (in Theorem 2.2) a kind of dual version to the div-curl lemmas in Theorem 1, which is a local analogue of a result proved in [7] for BMO: for $G \in \mathrm{bmo}(\mathbb{R}^n)$,

$$\|G\|_{\mathrm{bmo}} \approx \sup_{\vec{V},\vec{W}} \int_{\mathbb{R}^n} G\vec{V}\cdot\vec{W}, \tag{10}$$

where the supremum is taken over all vector fields $\vec{V}, \vec{W}$ as above, satisfying (3), with $\|\vec{V}\|_{L^p}$, $\|f\|_{L^p}$ and $\|\vec{W}\|_{L^q}$ all bounded by 1. Here, and below, one must obviously consider only real-valued functions g in bmo.

Moreover, the same equation (10) holds if the vector fields, instead of (3), satisfy (5) with $\|B_{ij}\|_{L^q} \le 1$ for all $i, j \in \{1, \ldots, n\}$, as well as $\|\vec{V}\|_{L^p}, \|\vec{W}\|_{L^q} \le 1$.

As a consequence of these results, one is able to show (see [5, Theorem 3.1]) a decomposition of functions in $h^1(\mathbb{R}^n)$ into nonhomogeneous div-curl quantities $\vec{V}\cdot\vec{W}$ of the type found in Theorem 1, part (a) or part (b).

The goal of this paper is to prove analogues of (10) for functions in local bmo spaces on a domain Ω, and obtain decomposition results for the local Hardy spaces. This was done in the case of BMO and with homogeneous, L^2 div-curl quantities in [3], and independently by Lou [16]. In [1] homogeneous div-curl results on domains were stated under the assumption that one of the vector fields is a gradient, and extended to Hardy−Sobolev spaces. Related work may be found in [12, 17].

In the next section we introduce some definitions of Hardy spaces and BMO on domains, as well as explain the boundary conditions for equations (7) and (8). The statements and proofs of our results are contained in Section 3.

2. Preliminary definitions for a domain Ω

For the moment we will just assume Ω is an open subset of $\mathbb{R}^n$, but often we will restrict ourselves to a Lipschitz domain, i.e., one whose boundary is made up, piecewise, of Lipschitz graphs.

Miyachi [19] defined Hardy spaces on Ω by letting $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, replacing the maximal function $\mathcal{M}$ in (1) by

$$\mathcal{M}_{\phi,\Omega}(f)(x) = \mathrm{M}_{\phi,\delta(x)}(f)(x) = \sup_{0<t<\delta(x)} |f * \phi_t(x)|,$$

for $f \in L^1_{\text{loc}}(\Omega)$, and requiring $\mathrm{M}_{\phi,\Omega}(f) \in L^1(\Omega)$. The space of such functions was later denoted by $H^1_r(\Omega)$ in [6], since when the boundary is sufficiently nice (say Lipschitz), $H^1_r(\Omega)$ can be identified with the quotient space of restrictions to Ω of functions in $H^1(\mathbb{R}^n)$ (see [6,19]). Moreover,

$$\|f\|_{H^1_r(\Omega)} := \|\mathrm{M}_{\phi,\Omega}(f)\|_{L^1(\Omega)} \approx \inf\{\|F\|_{H^1(\mathbb{R}^n)} : F|_\Omega = f\}.$$

The space $h^1_r(\Omega)$, corresponding to restrictions to Ω of functions in $h^1(\mathbb{R}^n)$, can be defined by replacing $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ in Miyachi's definition by $\delta(x) = \min\big(\delta, \operatorname{dist}(x, \partial\Omega)\big)$, for some fixed finite $\delta > 0$. Since different choices of δ give equivalent norms, when Ω is bounded one can choose $\delta > \operatorname{diam}(\Omega)$, so $h^1_r(\Omega)$ is the same as $H^1_r(\Omega)$ (with norm equivalence involving constants depending on Ω).

For Ω a Lipschitz domain, the dual of $h^1_r(\Omega)$ (see [19] for the case of H^1 and BMO, and [2]) can be identified with the subspace

$$\mathrm{bmo}_z(\Omega) = \big\{g \in \mathrm{bmo}(\mathbb{R}^n) : \operatorname{supp}(g) \subset \overline{\Omega}\big\}.$$

Analogously, one can consider the subspace

$$h^1_z(\Omega) = \big\{g \in h^1(\mathbb{R}^n) : \operatorname{supp}(g) \subset \overline{\Omega}\big\}.$$

This was originally done in [15] in the case of H^1 functions supported on a closed subset with certain geometric properties, and later in [6] for a Lipschitz domain and in [4] for a domain with smooth boundary, in connection with boundary value problems. For a bounded domain Ω, $H^1_z(\Omega)$ and $h^1_z(\Omega)$ do not coincide since functions in H^1_z must satisfy a vanishing moment condition over the whole domain Ω, while those in h^1_z do not.

The dual of $h^1_z(\Omega)$ can be identified with $\mathrm{bmo}_r(\Omega)$, defined by requiring the supremum in (9) to be taken only over cubes I contained in Ω. In fact, one can actually require the cubes to satisfy $2I \subset \Omega$. This space was originally studied, in the case of BMO, by Jones [14], who showed that when the boundary of Ω is sufficiently nice, $\mathrm{BMO}_r(\Omega)$ is the same as the quotient space of restrictions to Ω of functions in $\mathrm{BMO}(\mathbb{R}^n)$. This holds in particular when Ω is a Lipschitz domain, and is also true in the case of bmo_r, with

$$\|g\|_{\mathrm{bmo}_r(\Omega)} \approx \inf\{\|G\|_{\mathrm{bmo}(\mathbb{R}^n)} : G|_\Omega = g\}.$$

Note that when Ω is a bounded domain, every element of $\mathrm{BMO}_r(\Omega)$ is in $\mathrm{bmo}_r(\Omega)$, but the bmo_r norm depends also on the norm of the function in $L^1(\Omega)$.

Since elements of $h^1_z(\Omega)$ are controlled in norm up to the boundary, when discussing div-curl quantities in this space one needs to consider the "boundary values" of the vector fields $\vec{v}$ and $\vec{w}$. As these vector fields are only defined in $L^p(\Omega)$ and do not have traces on the boundary, the appropriate boundary conditions are expressed, as in the case of Dirichlet and Neumann boundary value problems, by specifying the class of test functions. In particular, for the equations

$$\operatorname{div} \vec{v} = f \qquad \text{and} \qquad \operatorname{curl} \vec{w} = B,$$

we now require (7) and (8) to hold in the case when the test functions do not have compact support in Ω. This is equivalent to saying that the vector fields

$$\vec{V} = \begin{cases} \vec{v} & \text{in } \Omega \\ \vec{0} & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{11}$$

and

$$\vec{W} = \begin{cases} \vec{w} & \text{in } \Omega \\ \vec{0} & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{12}$$

satisfy $\operatorname{div} \vec{V} = f$ and $\operatorname{curl} \vec{W} = B$ in the sense of distributions on $\mathbb{R}^n$, with f and B vanishing outside of Ω.

When the boundary $\partial\Omega$ of Ω is sufficiently smooth, let $\vec{\mathbf{n}} = (\eta_1, \dots, \eta_n)$ denote the outward unit normal vector. If the vector fields are sufficiently smooth (so as to have a trace on the $\partial\Omega$), we can integrate by parts in (7) and (8). If φ does not have compact support in Ω, the boundary values of $\vec{v}$ must satisfy $\vec{\mathbf{n}} \cdot \vec{v} = 0$, and in the case of a bounded domain, this also entails $\int_\Omega f = 0$, while for $\vec{w}$ it must hold that on $\partial\Omega$

$$w_j \eta_i = w_i \eta_j,$$

meaning that $\vec{w}$ is colinear with $\vec{\mathbf{n}}$.

We will denote these conditions as follows. Let Ω be a Lipschitz domain and suppose f and the components of the vector fields $\vec{v}$ and $\vec{w}$ are locally integrable on Ω. As in the statement of the Neumann problem on Ω, write

$$\begin{cases} \operatorname{div} \vec{v} = f & \text{in } \Omega, \\ \int_\Omega f = 0 & \text{if } \Omega \text{ is bounded}, \\ \vec{\mathbf{n}} \cdot \vec{v} = 0 & \text{on } \partial\Omega \end{cases} \tag{13}$$

to indicate that (7) holds for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, and

$$\begin{cases} \operatorname{curl} \vec{w} = B & \text{in } \Omega, \\ \vec{\mathbf{n}} \times \vec{w} = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

to indicate that (8) holds for all $i, j \in \{1, \dots, n\}$ and all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

3. Div-curl lemmas for local Hardy spaces and BMO on a domain

In order to prove an analogue of (10) for $\mathrm{bmo}_z(\Omega)$, one needs the following versions of the nonhomogeneous div-curl lemma for $h^1_r(\Omega)$. The first is a special case of Theorem 7 in [8]:

Theorem 2 ([8]). *Suppose $\vec{v}$ and $\vec{w}$ are vector fields on an open set $\Omega \subset \mathbb{R}^n$, satisfying*

$$\vec{v} \in L^p(\Omega, \mathbb{R}^n), \quad \vec{w} \in L^q(\Omega, \mathbb{R}^n), \qquad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and

$$\operatorname{div} \vec{v} = f \in L^p(\Omega), \quad \operatorname{curl} \vec{w} = 0$$

in the sense of distributions on Ω. Then $\vec{v} \cdot \vec{w}$ belongs to the local Hardy space $h^1_r(\Omega)$ with

$$\|\vec{v} \cdot \vec{w}\|_{h^1_r(\Omega)} \le C(\|\vec{v}\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})\|\vec{w}\|_{L^q(\Omega)}. \tag{15}$$

The second is a domain version of Theorem 4 in [8], whose proof we shall adapt below:

Theorem 3. *Suppose $\vec{v}$ and $\vec{w}$ are vector fields on an open set $\Omega \subset \mathbb{R}^n$, satisfying*

$$\vec{v} \in L^p(\Omega, \mathbb{R}^n), \quad \vec{w} \in L^q(\Omega, \mathbb{R}^n), \qquad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and

$$\operatorname{div} \vec{v} = 0, \qquad \operatorname{curl} \vec{w} = B \in L^q(\Omega; M^{n\times n})$$

in the sense of distributions on Ω. Then $\vec{v}\cdot\vec{w}$ belongs to the local Hardy space $h^1_r(\Omega)$ with

$$\|\vec{v}\cdot\vec{w}\|_{h^1_r(\Omega)} \le C\|\vec{v}\|_{L^p(\Omega)}\big(\|\vec{w}\|_{L^q(\Omega)} + \sum_{i,j}\|B_{ij}\|_{L^q(\Omega)}\big). \tag{16}$$

Proof. Consider a point $x \in \Omega$ and a cube Q^x_l, centered at x and of sidelength $l > 0$, depending on x. We choose $l = \min\big(1, \operatorname{dist}(x, \partial\Omega)\big)/\sqrt{n}$, which guarantees Q^x_l lies inside Ω. Without loss of generality assume $Q^x_l = [0,l]^n$. Writing $\vec{v} = (v_1, \dots, v_n)$, and fixing i, we solve $-\Delta u_i = v_i$ with mixed boundary conditions: on the two faces $x_i = 0$ and $x_i = l$ we impose Neumann boundary values

$$\frac{\partial u_i}{\partial x_i} = 0,$$

and on the other faces (corresponding to $x_j = 0$ and $x_j = l$, $j \ne i$) Dirichlet boundary values $u_i = 0$. This can be done by expanding in multiple Fourier series (with even coefficients in x_i and odd coefficients in x_j, $j \ne i$). By the Marcinkiewicz multiplier theorem (see [18, Theorem 4]) the second derivatives of the solution u_i will be bounded in $L^\alpha(Q^x_l)$ by $\|v_i\|_{L^\alpha(Q^x_l)}$, for every $\alpha \le p$, $i = 1, \dots, n$. Note that by the homogeneity of the multipliers, the constants will be independent of l. Since we have taken $l \le 1$, we also get that $\|u_i\|_{W^{2,p}(Q^x_l)} \le C\|v_i\|_{L^p(Q^x_l)}$ with a constant independent of l.

Set $\vec{U} = (u_1, \dots, u_n)$ and consider the function $\operatorname{div}\vec{U} \in W^{1,p}(Q^x_l)$. This function satisfies

$$\Delta(\operatorname{div}\vec{U}) = -\operatorname{div}\vec{v} = 0$$

in the sense of distributions on Q^x_l, since $Q^x_l \subset \Omega$, and moreover

$$\operatorname{div}\vec{U} = \sum \frac{\partial u_i}{\partial x_i} = 0$$

on the boundary, by the choice of boundary conditions above. By the uniqueness of the solution of the Dirichlet problem in $W^{1,p}_0(Q^x_l)$, we must have $\operatorname{div}\vec{U} = 0$ on Q^x_l. Let A be the matrix $\operatorname{curl}\vec{U}$, i.e.,

$$A_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}.$$

These are functions in $W^{1,p}(Q^x_l)$ with first derivatives bounded in the $L^\alpha(Q^x_l)$-norm by $\|v_i\|_{L^\alpha(Q^x_l)}$, for every $\alpha \le p$.

Now writing $\vec{A}_j$ for the jth column of the matrix A, we have, in the sense of distributions on Q^x_l,

$$\operatorname{div}\vec{A}_j = \sum_{i=1}^n \left(\frac{\partial^2 u_i}{\partial x_i \partial x_j} - \frac{\partial^2 u_j}{\partial x_i^2}\right) = \frac{\partial}{\partial x_j}\operatorname{div}\vec{U} - \Delta u_j = v_j, \tag{17}$$

for each $j = 1, \dots, n$. Taking the dot product with $\vec{w}$ and recalling that we identify $\operatorname{curl} \vec{w}$, in the sense of distributions on Ω, with a matrix B whose components are in $L^q(\Omega)$, we have

$$\begin{aligned}\vec{v} \cdot \vec{w} = \sum_{j=1}^{n} (\operatorname{div} \vec{A}_j) w_j &= \sum_{j=1}^{n} \operatorname{div}(\vec{A}_j w_j) - \sum_{i,j} A_{ij} \frac{\partial w_j}{\partial x_i} \\ &= \sum_{j=1}^{n} \operatorname{div}(\vec{A}_j w_j) + \sum_{i<j} A_{ij} B_{ij},\end{aligned}$$

again in the sense of distributions on Q_l^x.

Take $\phi \in C^\infty$ with support in $B\big(0, 1/(2\sqrt{n})\big)$ and $\int \phi = 1$, and define, for $0 < t \leq \min\big(1, \operatorname{dist}(x, \partial\Omega)\big)$, ϕ_t^x by $\phi_t^x(y) = t^{-n} \phi\big(t^{-1}(x-y)\big)$. Since $l = \min\big(1, \operatorname{dist}(x, \partial\Omega)\big)/\sqrt{n}$ we have

$$\operatorname{supp}(\phi_t^x) \subset B(x, t/2\sqrt{n}) \subset Q_l^x \subset \Omega.$$

Denote $B(x, t/2\sqrt{n})$ by $\widetilde{B_t^x}$.

We integrate $\vec{v} \cdot \vec{w}$ against ϕ_t^x, noting that equation (17) holds even if we change $\vec{A}_j$ by adding a vector field which is constant on Q_l^x. In particular we modify each A_{ij} by subtracting its average $(A_{ij})_{\widetilde{B_t^x}}$ over $\widetilde{B_t^x}$. Integration by parts yields:

$$\begin{aligned}\int \phi_t^x (\vec{v} \cdot \vec{w}) = -\sum_{i,j} \int t^{-(n+1)} \frac{\partial \phi}{\partial y_i} \big(t^{-1}(x-y)\big) \big(A_{ij}(y) - (A_{ij})_{\widetilde{B_t^x}}\big) w_j(y) \, \mathrm{d}y \\ + \sum_{i<j} \int t^{-n} \phi\big(t^{-1}(x-y)\big) \big(A_{ij} - (A_{ij})_{\widetilde{B_t^x}}\big) B_{ij}.\end{aligned}$$

Take α, β with $1 < \alpha < p$, $1 < \beta < q$ and $1/\alpha + 1/\beta = 1 + 1/n$. The Sobolev–Poincaré inequality in $\widetilde{B_t^x}$, together with the fact that $t \leq 1$, gives (see the proof of Lemma II.1 in [7]):

$$\begin{aligned}|\phi_t * (\vec{v} \cdot \vec{w})(x)| &\leq C \|\vec{\nabla}\phi\|_\infty \sum_{i,j} \left(t^{-n} \int_{\widetilde{B_t^x}} |\vec{\nabla} A_{ij}|^\alpha \right)^{1/\alpha} \left(t^{-n} \int_{\widetilde{B_t^x}} |\vec{w}|^\beta \right)^{1/\beta} \\ &\quad + C \|\phi\|_\infty \sum_{i,j} \left(t^{-n} \int_{\widetilde{B_t^x}} |\vec{\nabla} A_{ij}|^\alpha \right)^{1/\alpha} \left(t^{-n} \int_{\widetilde{B_t^x}} |B_{ij}|^\beta \right)^{1/\beta} \\ &\leq C_\phi \mathrm{M}(|\vec{v}|^\alpha)(x)^{1/\alpha} \left[\mathrm{M}(|\vec{w}|^\beta)(x)^{1/\beta} + \sum_{i,j} \mathrm{M}(|B_{ij}|^\beta)(x)^{1/\beta} \right].\end{aligned}$$

Here the Hardy–Littlewood maximal function on $\mathbb{R}^n$, denoted by M, is applied to the functions $|\vec{v}|^\alpha, |\vec{w}|^\beta$ and $|B_{ij}|^\beta$ by extending them by zero outside Ω. The constant depends on the choice of ϕ but not on t or x.

Recalling that the maximal function is bounded on $L^r(\mathbb{R}^n)$, $r > 1$, we conclude that:

$$\int_\Omega \sup_{0<t<\operatorname{dist}(x,\partial\Omega)} |\phi_t * (\vec{v}\cdot\vec{w})(x)|\, \mathrm{d}x$$
$$\le C_\phi \left(\int_\Omega \left(\mathrm{M}(|\vec{v}|^\alpha)(x)\right)^{p/\alpha} \mathrm{d}x \right)^{1/p}$$
$$\times \left[\left(\int_\Omega \left(\mathrm{M}(|\vec{w}|^\beta)(x)\right)^{q/\beta} \mathrm{d}x \right)^{1/q} + \sum_{i,j} \left(\int_\Omega \left(\mathrm{M}(|B_{ij}|^\beta)(x)\right)^{q/\beta} \mathrm{d}x \right)^{1/q} \right]$$
$$\le C_\phi \|\vec{v}\|_{L^p(\Omega)} \Big[\|\vec{w}\|_{L^q(\Omega)} + \sum_{i,j} \|B_{ij}\|_{L^q(\Omega)} \Big].$$

This shows $\vec{v}\cdot\vec{w} \in h^1_r(\Omega)$, and (16) holds. □

Lemma 4. *Suppose $\vec{v}$ and $\vec{w}$ are vector fields on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, satisfying the hypotheses of either Theorem 2 or Theorem 3, but with the conditions on the divergence and the curl satisfied in the stronger sense of (13) and (14). Then $\vec{v}\cdot\vec{w} \in h^1_z(\Omega)$ with norm bounded as in (15) or (16).*

Proof. Given such vector fields $\vec{v}$ and $\vec{w}$ on Ω, define the zero extensions $\vec{V}$ and $\vec{W}$ as in (11) and (12). The L^p and L^q norms of $\vec{V}$ and $\vec{W}$ are the same as those of $\vec{v}$ and $\vec{w}$ on Ω. Moreover, conditions (13) and (14) guarantee that $\vec{V}$ and $\vec{W}$ satisfy (3) (respectively (5)) in the sense of distributions on $\mathbb{R}^n$. Therefore, by using Theorem 1, part (a) (respectively part (b)), we can conclude that $\vec{V}\cdot\vec{W} \in h^1(\mathbb{R}^n)$ with the appropriate bound on its norm. But $\vec{V}\cdot\vec{W}$ is equal to zero outside Ω and is $\vec{v}\cdot\vec{w}$ on Ω, hence this is a function in $h^1_z(\Omega)$. The h^1_z norm is the same as the h^1 norm and the bounds can be given in terms of the L^p and L^q norms of the relevant quantities on Ω. □

Now we can proceed to state and prove the local bmo versions of the div-curl lemma on a Lipschitz domain:

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain.*

(a) *If $g \in \mathrm{bmo}_z(\Omega)$, then*

$$\|g\|_{\mathrm{bmo}_z} \approx \sup_{\vec{v},\vec{w}} \int_\Omega g\, \vec{v}\cdot\vec{w}, \tag{18}$$

where the supremum is taken over all vector fields $\vec{v} \in L^p(\Omega,\mathbb{R}^n)$, $\vec{w} \in L^q(\Omega,\mathbb{R}^n)$, $\|\vec{v}\|_{L^p(\Omega)} \le 1$, $\|\vec{w}\|_{L^q(\Omega)} \le 1$, satisfying (3) in the sense of distributions on Ω, with $\|f\|_{L^p(\Omega)} \le 1$.

(b) *If $g \in \mathrm{bmo}_z(\Omega)$, then equation (18) holds with the supremum now taken over all vector fields $\vec{v} \in L^p(\Omega,\mathbb{R}^n)$, $\vec{w} \in L^q(\Omega,\mathbb{R}^n)$, $\|\vec{v}\|_{L^p(\Omega)} \le 1$, $\|\vec{w}\|_{L^q(\Omega)} \le 1$, satisfying (5) in the sense of distributions on Ω, with $\|B_{ij}\|_{L^q}(\Omega) \le 1$ for all $i,j \in 1,\dots,n$.*

(c) *If $g \in \mathrm{bmo}_r(\Omega)$ then*

$$\|g\|_{\mathrm{bmo}_r} \approx \sup_{\vec{v},\vec{w}} \int_\Omega g\, \vec{v}\cdot\vec{w},$$

the supremum being taken over all vector fields $\vec{v}$ and $\vec{w}$ as in part (a) or in part (b), but satisfying the stronger conditions (13) and (14).

Proof. Let $g \in \mathrm{bmo}_z(\Omega)$ (real-valued) and take $\vec{v}, \vec{w}$ as in the hypotheses of part (a) (respectively part (b)). By Theorem 2 (resp. Theorem 3), the dot product $\vec{v} \cdot \vec{w}$ belongs to $h^1_r(\Omega)$ with norm bounded by a constant. The duality of $\mathrm{bmo}_z(\Omega)$ with $h^1_r(\Omega)$ then gives

$$\int_\Omega g\, \vec{v} \cdot \vec{w} \leq C \|g\|_{\mathrm{bmo}_z}.$$

Conversely, by the nature of $\mathrm{bmo}_z(\Omega)$, the extension G of g to $\mathbb{R}^n$ by setting it zero outside Ω is in $\mathrm{BMO}(\mathbb{R}^n)$ with $\|G\|_{\mathrm{bmo}} \approx \|g\|_{\mathrm{bmo}_z}$. Hence, by (10), one has

$$\|g\|_{\mathrm{bmo}_z} \approx \sup_{\vec{V}, \vec{W}} \int_{\mathbb{R}^n} G\, \vec{V} \cdot \vec{W} = \sup_{\vec{V}, \vec{W}} \int_\Omega g\, \vec{V}\big|_\Omega \cdot \vec{W}\big|_\Omega,$$

where the supremum is taken over all vector fields $\vec{V} \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $\vec{W} \in L^q(\mathbb{R}^n, \mathbb{R}^n)$, $\|\vec{V}\|_{L^p} \leq 1$, $\|\vec{W}\|_{L^q} \leq 1$, satisfying (3) (resp. (5)) in the sense of distributions on $\mathbb{R}^n$. The restrictions $\vec{v} = \vec{V}\big|_\Omega$, $\vec{w} = \vec{W}\big|_\Omega$ satisfy the same conditions in Ω, proving the inequality $\lesssim$ in (18).

If $g \in \mathrm{bmo}_r(\Omega)$ and $\vec{v}, \vec{w}$ are as in part (c), by Lemma 4 $\vec{v} \cdot \vec{w} \in h^1_z(\Omega)$ with norm bounded by a constant, so the duality of bmo_r and h^1_z implies

$$\int_\Omega g\, \vec{v} \cdot \vec{w} \leq C \|g\|_{\mathrm{bmo}_r} \|\vec{v} \cdot \vec{w}\|_{h^1_z} \leq C \|g\|_{\mathrm{bmo}_r}.$$

This shows that

$$\sup_{\vec{v}, \vec{w}} \int_\Omega g\, \vec{v} \cdot \vec{w} \leq C \|g\|_{\mathrm{bmo}_r}.$$

It remains to prove the other direction, i.e.,

$$\|g\|_{\mathrm{bmo}_r} \leq C' \sup_{\vec{v}, \vec{w}} \int_\Omega g\, \vec{v} \cdot \vec{w}.$$

The left-hand side is given by

$$\sup_{\substack{Q \subset \Omega \\ |Q| \leq 1}} \frac{1}{|Q|} \int_Q |g(x) - g_Q| \,\mathrm{d}x + \sup_{\substack{Q \subset \Omega \\ |Q| > 1}} \frac{1}{|Q|} \int_Q |g(x)| \,\mathrm{d}x.$$

As explained in the proof of Theorem 2.1 in [3] (for the case of $\mathrm{BMO}_r(\Omega)$ but the same arguments apply to $\mathrm{bmo}_r(\Omega)$), it suffices to take the supremum over cubes Q satisfying $\widetilde{Q} = 2Q \subset \Omega$ (or with some constant C_Ω replacing 2). In that case it just remains to point out that in the proof of estimate (10) in [5] (see the proof of Theorem 2.2., Case I), it was shown that for a ball $B \subset \mathbb{R}^n$ with radius bounded by 1,

$$\left(\frac{1}{|B|} \int_B |g(x) - g_B|^2 \,\mathrm{d}x \right)^{1/2} \leq C_n \sup \int g\, \vec{v} \cdot \vec{w},$$

where the supremum is taken over all vector fields $\vec{v}, \vec{w} \in C_0^\infty(\widetilde{B})$ with $\|\vec{v}\|_{L^p} \leq 1$, $\|\vec{w}\|_{L^q} \leq 1$ and $\operatorname{div} \vec{v} = 0, \operatorname{curl} \vec{w} = 0$. There we took $\widetilde{B} = 2B$ but the argument immediately applies to $\widetilde{B} = C_\Omega B$ for some $C_\Omega > 1$. Note that if $\widetilde{B} \subset \Omega$, such vector fields will vanish on the boundary $\partial\Omega$ and thus satisfy the boundary conditions (13) and (14).

Similarly, for a ball $B \subset \mathbb{R}^n$ with radius larger than 1, we showed in [5] (see the proof of Theorem 2.2., Case I) that

$$\left(\frac{1}{|B|}\int_B |g(x)|^2\,\mathrm{d}x\right)^{1/2} \le C_n \sup \int g\,\vec{v}\cdot\vec{w},$$

where this time the supremum can be taken over all vector fields $\vec{v}, \vec{w} \in C_0^\infty(\widetilde{B})$ with $\|\vec{v}\|_{L^p} \le 1$, $\|\vec{w}\|_{L^q} \le 1$ satisfying the relaxed div-curl conditions (3), or alternatively the supremum can be taken over such vector fields satisfying (5). Again such vector fields will automatically satisfy (13) and (14) — the boundary conditions follow from the vanishing on the boundary and the condition $\int_\Omega \operatorname{div}\vec{v} = 0$, in the case of bounded Ω, follows from the divergence theorem since we are now dealing with smooth functions. □

Finally we arrive at the desired nonhomogeneous div-curl decompositions for the local Hardy spaces on Ω. These are corollaries of Theorem 5 and follow from the duality between bmo_z and h_r^1 (respectively bmo_r and h_z^1) by using Lemmas III.1 and III.2 in [7]:

Theorem 6. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $1 < p < \infty$, $1/p+1/q = 1$.*

(a) *For a function f in $h_r^1(\Omega)$, there exists a sequence of scalars $\{\lambda_k\}$ with $\sum_{k=1}^\infty |\lambda_k| < \infty$, and sequences of vector fields $\{\vec{v}_k\}$ in $L^p(\Omega, \mathbb{R}^n)$, $\{\vec{w}_k\}$ in $L^q(\Omega, \mathbb{R}^n)$ with $\|\vec{v}_k\|_{L^p}$, $\|\vec{w}_k\|_{L^q} \le 1$ for all k, satisfying, for each k, condition* (3) *in the sense of distributions on Ω, so that*

$$f = \sum_{k=1}^\infty \lambda_k \vec{v}_k \cdot \vec{w}_k.$$

(b) *The same result holds as in part* (a) *but with $\vec{v}_k$ and $\vec{w}_k$ satisfying* (5) *instead of* (3), *for each k.*

(c) *For a function $f \in h_z^1(\Omega)$, there exists a sequence of scalars $\{\lambda_k\}$ with $\sum_{k=1}^\infty |\lambda_k| < \infty$, and sequences of vector fields $\{\vec{v}_k\}$ and $\{\vec{w}_k\}$, as in part* (a) *or part* (b), *but satisfying the div-curl conditions in the stronger sense of* (13) *for each $\vec{v}_k$ and* (14) *for each $\vec{w}_k$, so that*

$$f = \sum_{k=1}^\infty \lambda_k \vec{v}_k \cdot \vec{w}_k.$$

Remark. As pointed out in Section 2, when the domain Ω is bounded the "local" Hardy space $h_r^1(\Omega)$ coincides with $H_r^1(\Omega)$ and similarly for $\mathrm{BMO}_z(\Omega)$ and $\mathrm{bmo}_z(\Omega)$. By taking the constants in the definitions and proofs sufficiently large (depending on the size of Ω), we do not need to deal with the case of "large" balls or cubes, so everything reverts to the homogeneous case. As previously mentioned, this case was dealt with in [3] and [16], but only for $p = q = 2$, so the current results are a generalization of the older ones.

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