A div-curl decomposition for the local Hardy space

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Abstract

A decomposition theorem for the local Hardy space of Goldberg, in terms of nonhomogeneous div-curl quantities, is proved via a dual result for the space bmo.

1 Introduction

Given vector fields $\mathbf{V} = (v_1, \ldots, v_n)$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, $\mathbf{W} = (w_1, \ldots, w_n)$ in $L^{p'}(\mathbb{R}^n, \mathbb{R}^n)$ with $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, the scalar (dot) product $\mathbf{V} \cdot \mathbf{W} = \sum v_i w_i$ will lie in $L^1(\mathbb{R}^n)$. Coifman, Lions, Meyer, and Semmes [CLMS] showed that if the following conditions hold in the sense of distributions:

$$\text{div } \mathbf{V} := \sum \frac{\partial v_i}{\partial x_i} = 0,$$

$$\text{curl } \mathbf{W} := \left(\frac{\partial w_j}{\partial x_i} - \frac{\partial w_i}{\partial x_j} \right)_{ij} = 0,$$

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then $\mathbf{V} \cdot \mathbf{W}$ belongs to the real Hardy space $H^1(\mathbb{R}^n)$, a proper subspace of L^1. Moreover,

$$\|\mathbf{V} \cdot \mathbf{W}\|_{H^1} \leq C\|\mathbf{V}\|_{L^p}\|\mathbf{W}\|_{L^{p'}}. \tag{1.1}$$

This "div-curl lemma" and other results of a similar nature illustrate the recent use of Hardy spaces for applications to nonlinear PDE and in the method of compensated compactness, originally going back to the work of Murat and Tartar.

Recall (see [FS]) that a function f belongs to $H^1(\mathbb{R}^n)$ if the maximal function $M_\varphi(f)$ belongs to $L^1(\mathbb{R}^n)$, where, for a fixed Schwartz function φ, $\int \varphi = 1$, we define

$$M_\varphi(f)(x) = \sup_{t>0} |f * \varphi_t(x)|, \quad \varphi_t(\cdot) = t^{-n}\varphi(t^{-1}\cdot).$$

Here the norm, defined by $\|f\|_{H^1} := \|M_\varphi\|_{L^1}$, depends on the choice of φ, but the space does not since different choices of φ give equivalent norms. Functions in the Hardy space enjoy both improved integrability and cancellation conditions compared to L^1 functions. In particular, if $f \in H^1$ then its integral must vanish.

The local real Hardy space $h^1(\mathbb{R}^n)$, defined by Goldberg [Go], is larger than H^1 and allows for more flexibility, since global cancellation conditions are not necessary. For example, the Schwartz space is contained in h^1 but not in H^1, and multiplication by cut-off functions preserves h^1 but not H^1, thus making it more suitable for working in domains and on manifolds. For membership of a function f in $h^1(\mathbb{R}^n)$, we use a "local" maximal function $m_\varphi(f)$, with

$$m_\varphi(f)(x) = \sup_{0<t<1} |f * \varphi_t(x)|,$$

and require $m_\varphi(f) \in L^1(\mathbb{R}^n)$. As for H^1, $\|f\|_{h^1} := \|m_\varphi\|_{L^1}$ defines a norm, and different choices of φ give equivalent norms, so we can choose φ with compact support, making clear the local nature of the maximal function.

In [D], sufficient nonhomogeneous conditions on the divergence and curl were found in order for the dot product $\mathbf{V} \cdot \mathbf{W}$, where $\mathbf{V}$ and $\mathbf{W}$ are vector fields as above, to be in $h^1(\mathbb{R}^n)$. In particular, as a corollary, the following special case was proved:

Proposition 1.1 ([D]) *Suppose* $\mathbf{V}$ *and* $\mathbf{W}$ *are vector fields on* $\mathbb{R}^n$ *satisfying*

$$\mathbf{V} \in L^p(\mathbb{R}^n)^n, \quad \mathbf{W} \in L^{p'}(\mathbb{R}^n)^n, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

If there exists a function f *in* $L^p(\mathbb{R}^n)$ *and a matrix-valued function* A *with components in* $L^{p'}(\mathbb{R}^n)$ *such that, in the sense of distributions,*

$$\operatorname{div} \mathbf{V} = f, \quad \operatorname{curl} \mathbf{W} = A,$$

then $\mathbf{V} \cdot \mathbf{W}$ *belongs to the local Hardy space* $h^1(\mathbb{R}^n)$ *with*

$$\|\mathbf{V} \cdot \mathbf{W}\|_{h^1} \leq C\,(\|\mathbf{V}\|_{L^p}\|\mathbf{W}\|_{L^{p'}} + \|f\|_{L^p}\|\mathbf{W}\|_{L^{p'}} + \|\mathbf{V}\|_{L^p}\sum_{i,j}\|A_{ij}\|_{L^{p'}}).$$

Note that the requirement on the divergence and the components of the curl to be functions in L^p and $L^{p'}$, respectively, was also used in the original div-curl lemma of Murat [Mu], and is a natural relaxation of the vanishing divergence and curl conditions. In fact, by the Hodge decomposition, this can be viewed as a combination of the following two theorems:

Theorem 1.2 ([D]) *Suppose* $\mathbf{V}$ *and* $\mathbf{W}$ *are vector fields on* $\mathbb{R}^n$ *satisfying*

$$\mathbf{V} \in L^p(\mathbb{R}^n)^n, \quad \mathbf{W} \in L^{p'}(\mathbb{R}^n)^n, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

and

$$\text{div } \mathbf{V} = f \in L^p(\mathbb{R}^n), \quad \text{curl } \mathbf{W} = 0$$

in the sense of distributions. Then $\mathbf{V} \cdot \mathbf{W}$ *belongs to the local Hardy space* $h^1(\mathbb{R}^n)$ *with*

$$\|\mathbf{V} \cdot \mathbf{W}\|_{h^1(\mathbb{R}^n)} \leq C \left(\|\mathbf{V}\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \right) \|\mathbf{W}\|_{L^{p'}(\mathbb{R}^n)}; \tag{1.2}$$

and

Theorem 1.3 ([D]) *Suppose* $\mathbf{V}$ *and* $\mathbf{W}$ *are vector fields on* $\mathbb{R}^n$ *satisfying*

$$\mathbf{V} \in L^p(\mathbb{R}^n)^n, \quad \mathbf{W} \in L^{p'}(\mathbb{R}^n)^n, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

If A *is a matrix with components in* $L^{p'}(\mathbb{R}^n)$ *and*

$$\text{div } \mathbf{V} = 0, \quad \text{curl } \mathbf{W} = A$$

in the sense of distributions, then $\mathbf{V} \cdot \mathbf{W}$ *belongs to the local Hardy space* $h^1(\mathbb{R}^n)$ *with*

$$\|\mathbf{V} \cdot \mathbf{W}\|_{h^1(\mathbb{R}^n)} \leq C \, \|\mathbf{V}\|_{L^p(\mathbb{R}^n)} \Big[\|\mathbf{W}\|_{L^{p'}(\mathbb{R}^n)} + \sum_{i,j} \|A_{ij}\|_{L^{p'}(\mathbb{R}^n)} \Big]. \tag{1.3}$$

It may seem that the conditions on the divergence and the curl in the theorems above are too strong, so the question arises as to whether one can characterize functions in the Hardy space in terms of such div-curl quantities. Such a characterization, providing a kind of converse to the div-curl lemma, was shown in [CLMS] for $H^1(\mathbb{R}^n)$:

Theorem 1.4 ([CLMS]) *Every function* $f \in H^1(\mathbb{R}^n)$ *can be written as*

$$f = \sum_{k=1}^{\infty} \lambda_k \mathbf{V}_k \cdot \mathbf{W}_k,$$

with $\{\lambda_k\} \in \ell^1$ *and* $\mathbf{V}_k, \mathbf{W}_k$ *vector fields with norm bounded by* 1 *in* $L^2(\mathbb{R}^n, \mathbb{R}^n)$*, and* $\text{div } \mathbf{V}_k = 0$*,* $\text{curl } \mathbf{W}_k = 0$ *in the sense of distributions on* $\mathbb{R}^n$*.*

This decomposition was proved in [CLMS], via functional analysis arguments, from the following dual result: for $g \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\|g\|_{\mathrm{BMO}} \approx \sup_{\mathbf{V},\mathbf{W}} \int_{\mathbb{R}^n} g\mathbf{V}\cdot\mathbf{W}, \tag{1.4}$$

where the supremum is taken over all vector fields $\mathbf{V}$, $\mathbf{W}$ in $L^2(\mathbb{R}^n,\mathbb{R}^n)$, $\|\mathbf{V}\|_{L^2}, \|\mathbf{W}\|_{L^2} \leq 1$, satisfying div $\mathbf{V} = 0$, curl $\mathbf{W} = 0$ in the sense of distributions on $\mathbb{R}^n$.

The goal of this paper is to prove an analogue, Theorem 2.2, of (1.4) for functions in $\mathrm{bmo}(\mathbb{R}^n)$, the dual of the local Hardy space $h^1(\mathbb{R}^n)$, and consequently a decomposition theorem, Theorem 3.1, for h^1 in terms of the div-curl quantities used in Theorem 1.2 or in Theorem 1.3.

2 The div-curl lemma for local BMO

In [Go] it was shown that the dual of $h^1(\mathbb{R}^n)$ can be identified with the space $\mathrm{bmo}(\mathbb{R}^n)$, consisting of locally integrable functions f with

$$\|f\|_{\mathrm{bmo}} := \sup_{|I|\leq 1} \frac{1}{|I|}\int_I |f - f_I| + \sup_{|I|>1} \frac{1}{|I|}\int_I |f| < \infty.$$

Here the supremum can be taken over balls or cubes with sides parallel to the axes, $|I|$ denotes Lebesgue measure (volume) and f_I is the mean of f over I, i.e. $\frac{1}{|I|}\int_I f$. The number 1 in the definition can be replaced by any other finite nonzero constant. Note that unlike the case of BMO, we do not need to consider this norm modulo constants.

Before stating the main result, let us introduce the following definition in order to simplify notation.

Definition 2.1 *Denote by $\mathcal{DC}^p_{1,0}$ the collection of all functions which can be written in the form $\mathbf{V}\cdot\mathbf{W}$, where $\mathbf{V},\mathbf{W}$ are vectors fields, $\mathbf{V} \in L^p(\mathbb{R}^n,\mathbb{R}^n)$, $\mathbf{W} \in L^{p'}(\mathbb{R}^n,\mathbb{R}^n)$, $\|\mathbf{V}\|_{L^p} \leq 1$, $\|\mathbf{W}\|_{L^{p'}} \leq 1$, satisfying, in the sense of distributions on $\mathbb{R}^n$,*

$$\mathrm{div}\,\mathbf{V} = f \in L^p(\mathbb{R}^n), \quad \|f\|_{L^p} \leq 1, \quad \textit{and} \quad \mathrm{curl}\,\mathbf{W} = 0. \tag{2.1}$$

The collection $\mathcal{DC}^p_{0,1}$ can be defined analogously by requiring the vector fields to satisfy, instead of (2.1), the conditions

$$\mathrm{div}\,\mathbf{V} = 0,\ (\mathrm{curl}\,\mathbf{W})_{ij} = A_{ij} \in L^{p'}(\mathbb{R}^n),\ \|A_{ij}\|_{L^{p'}} \leq 1,\ i,j \in \{1,\ldots,n\}. \tag{2.2}$$

Note that by Theorems 1.2 and 1.3, both $\mathcal{DC}^p_{1,0}$ and $\mathcal{DC}^p_{0,1}$ are subsets of $h^1(\mathbb{R}^n)$ for every $1 < p < \infty$.

Theorem 2.2 *If $g \in \mathrm{bmo}(\mathbb{R}^n)$, then for $1 < p < \infty$,*

$$\|g\|_{\mathrm{bmo}} \approx \sup_{f \in \mathcal{DC}^p_{1,0}} \int_{\mathbb{R}^n} gf, \tag{2.3}$$

and

$$\|g\|_{\mathrm{bmo}} \approx \sup_{f \in \mathcal{DC}^p_{0,1}} \int_{\mathbb{R}^n} gf, \tag{2.4}$$

with constants that depend only on p and the dimension.

Proof: Let $g \in \mathrm{bmo}(\mathbb{R}^n)$ (we are assuming that g is real-valued) and take $f \in \mathcal{DC}^p_{1,0} \cup \mathcal{DC}^p_{0,1}$. As stated above, by Theorems 1.2 and 1.3, f belongs to $h^1(\mathbb{R}^n)$ with norm bounded by a constant. The duality of $\mathrm{bmo}(\mathbb{R}^n)$ with $h^1(\mathbb{R}^n)$ then gives

$$\int_{\mathbb{R}^n} gf \le C\|g\|_{\mathrm{bmo}}.$$

It remains to prove the other direction, i.e.

$$\|g\|_{\mathrm{bmo}} \le C' \int_{\mathbb{R}^n} gf$$

where we take the supremum over $f \in \mathcal{DC}^p_{1,0}$ or $f \in \mathcal{DC}^p_{0,1}$, respectively.

We will use the definition of bmo with the supremum taken over balls.

Case I: If B is a small ball, say with radius bounded by 1 (although in fact the proof is independent of the radius), one can use the following estimate from the proof of Theorem III.2 in [CLMS]:

$$\Big(\frac{1}{|B|}\int_B \big|g(x) - g_B\big|^2 dx\Big)^{1/2} \le C_n \sup \int g\, \mathbf{V}\cdot\mathbf{W}, \tag{2.5}$$

where the supremum is taken over all vector fields $\mathbf{V}, \mathbf{W} \in C_0^\infty(\widetilde{B})$ (here and below $\widetilde{B}$ denotes the ball with the same center and twice the radius of B) with $\|\mathbf{V}\|_{L^p} \le 1, \|\mathbf{W}\|_{L^{p'}} \le 1$ and $\operatorname{div}\mathbf{V} = 0, \operatorname{curl}\mathbf{W} = 0$. Note that the result in [CLMS] is stated for a cube Q (instead of a ball B) and for $p = p' = 2$, but can be modified to the case of a ball and for $p \neq 2$ as follows.

The key inequality used in the proof in [CLMS],

$$\|g - g_B\|_{L^2(B)} \le C \sum_{i=1}^n \left\|\frac{\partial g}{\partial x_i}\right\|_{W^{-1,2}(B)}, \tag{2.6}$$

is valid for any Lipschitz domain (see for example [GR], Section I.2.1, Corollary 2.1). This inequality holds in the homogeneous sense, modulo constants. Therefore, while we denote by $W^{-1,2}(B)$ the dual of the Sobolev space $W^{1,2}_0(B)$, which is the closure of $C_0^\infty(B)$ under the norm $\|\varphi\|_{W^{1,2}(B)} = \|\varphi\|_{L^2(B)} + \|\nabla\varphi\|_{L^2(B)}$, to obtain the right-hand-side of (2.6) we test against test functions only bounded in the homogeneous Sobolev norm $\|u\|_{\dot{W}^{1,2}(B)} = \|\nabla u\|_{L^2(B)}$. For a fixed ball the homogeneous and nonhomogeneous norms on $C_0^\infty(B)$ functions are equivalent, but here we use only the homogeneous norm in order for the constant to be independent of the radius of the ball.

To estimate $\left\|\frac{\partial g}{\partial x_i}\right\|_{W^{-1,2}(B)}$, $i = 1, \ldots, n$, we fix a $j \in \{1, \ldots, n\} \setminus \{i\}$ and define the vector fields $\mathbf{V}$ and $\mathbf{W}$, for the case $p \leq 2$, as in [CLMS]: given $u \in C_0^\infty(B)$ with $\|\nabla u\|_{L^2} \leq 1$, let

$$\mathbf{V} = |B|^{1/2-1/p} \left(\frac{\partial u}{\partial x_i} \mathbf{e}_j - \frac{\partial u}{\partial x_j} \mathbf{e}_i \right), \tag{2.7}$$

where $\mathbf{e}_i$ denotes the ith coordinate vector in $\mathbb{R}^n$, and let

$$\mathbf{W} = \gamma |B|^{-1/p'} \nabla \left((x_j - x_j^0) \eta \Big(\frac{x - x^0}{R} \Big) \right), \tag{2.8}$$

where x^0 and R are the center and radius of B, respectively. Here η is a fixed smooth function supported in $B(0,2)$ and identically equal to 1 on $B(0,1)$.

Note that since $p \leq 2$, the support and bound on the L^2 norm of ∇u imply $\|\mathbf{V}\|_{L^p} \leq 1$, while γ can be chosen, depending only on η, n and p, so that $\|\mathbf{W}\|_{L^{p'}} \leq 1$. This gives us the desired properties for $\mathbf{V}$ and $\mathbf{W}$, and moreover,

$$\mathbf{V} \cdot \mathbf{W} = \gamma |B|^{-1/2} \frac{\partial u}{\partial x_i}. \tag{2.9}$$

When $p \geq 2$, we need to change the definitions of $\mathbf{V}$ and $\mathbf{W}$. As above, we take $u \in C_0^\infty(B)$ with $\|\nabla u\|_{L^2} \leq 1$, but now we set

$$\mathbf{W} = |B|^{1/2-1/p'} \nabla u \tag{2.10}$$

and (again taking $j \in \{1, \ldots, n\} \setminus \{i\}$)

$$\mathbf{V} = \gamma' |B|^{-1/p} \left\{ \frac{\partial(\eta_B x_j)}{\partial x_j} \mathbf{e}_i - \frac{\partial(\eta_B x_j)}{\partial x_i} \mathbf{e}_j \right\}, \tag{2.11}$$

with η_B defined by $\eta_B(x) = \eta\left(\frac{x-x^0}{R}\right)$ for η as above. Clearly $\operatorname{div} \mathbf{V} = 0$ and $\operatorname{curl} \mathbf{W} = 0$, and by the conditions on u, the fact that $p \geq 2$, and the choice of γ', we can make $\|\mathbf{V}\|_{L^p} \leq 1$ and $\|\mathbf{W}\|_{L^{p'}} \leq 1$. On B, where η_B is identically equal to 1, we get $\mathbf{V} = \gamma' |B|^{-1/p} \mathbf{e}_i$, so that as before

$$\mathbf{V} \cdot \mathbf{W} = \gamma' |B|^{-1/2} \frac{\partial u}{\partial x_i}. \tag{2.12}$$

Integrating against g in (2.9) or (2.12), we can proceed for any $p \in (1, \infty)$. Taking the supremum over all such u, we get a bound on $\left\|\frac{\partial g}{\partial x_i}\right\|_{W^{-1,2}(B)}$ by a constant multiple of $|B|^{1/2}$ times the right-hand-side of (2.5). We then obtain (2.5) from (2.6).

Case II: Now let us consider a large ball B with radius greater than 1. For this type of ball we need to show a stronger condition, that is, we need to bound the mean of $|g|$ on B. We will do this by showing two cases of the following inequality, corresponding to (2.3) and (2.4):

$$\Big(\frac{1}{|B|}\int_B |g(x)|^r dx\Big)^{1/r} \le C_{n,r}\ \sup \int g\ \mathbf{V}\cdot\mathbf{W}. \tag{2.13}$$

In both cases the supremum is to be taken over pairs of smooth vector fields $\mathbf{V}$, $\mathbf{W}$ supported in $\tilde{B}$, $\mathbf{V} \in (L^p)^n$, $\mathbf{W} \in (L^{p'})^n$, with norms bounded by 1, and in the first case, with $r = p'$, the vector fields satisfy condition (2.1), while in second case, with $r = p$, they satisfy (2.2).

Note that a priori we have the finiteness of the left-hand-side of (2.13) by the fact that $g \in \text{bmo} \subset \text{BMO} \subset L^r_{\text{loc}}$ for any $r < \infty$.

We first prove estimate (2.13) when $B = B_1$ is the unit ball $B(0,1)$. In order to do this we need to use the full (nonhomogeneous) version of inequality (2.6), namely

$$\|g\|_{L^r(B_1)} \le C\left\{\|g\|_{W^{-1,r}(B_1)} + \sum_{i=1}^n \left\|\frac{\partial g}{\partial x_i}\right\|_{W^{-1,r}(B_1)}\right\} \tag{2.14}$$

for any r, $1 < r < \infty$ (see [Ne], Theorem 1, p. 108, with $l = 0$).

The estimates for $\left\|\frac{\partial g}{\partial x_i}\right\|_{W^{-1,r}(B_1)}$ are analogous to those above (for a fixed ball the norm is equivalent to the homogeneous case). Here the test functions are in $W_0^{1,r'}(B_1)$, so r' plays the role of the exponent 2 in the argument above, and again we need to distinguish the two cases. For $r = p'$, given $u \in C_0^\infty(B)$ with $\|\nabla u\|_{L^p} \le 1$, we use the same vector fields as defined in (2.7) and (2.8), normalized for the unit ball B_1. For $r = p$ the test function u will now have $\|\nabla u\|_{L^{p'}} \le 1$, so we can define the vector fields as in (2.11) and (2.10).

Now we address the part that is different from the homogeneous case - we have to bound

$$\|g\|_{W^{-1,r}(B_1)} = \sup\left|\int g\varphi\right|, \tag{2.15}$$

where the supremum is taken over $\varphi \in C_0^\infty(B_1)$, $\|\varphi\|_{W^{1,r'}(B_1)} \le 1$. We need to be able to write any such function φ in terms of div-curl quantities $\mathbf{V}\cdot\mathbf{W}$.

Lemma 2.3 *If $\varphi \in C_0^\infty(B_1)$ then we can write*

$$\varphi = C\mathbf{V}_1\cdot\mathbf{W}_1 = C\mathbf{V}_2\cdot\mathbf{W}_2$$

for smooth vector fields $\mathbf{V}_i$, $\mathbf{W}_i$ with $\mathbf{V}_1$ and $\mathbf{W}_2$ supported in B_1, $\mathbf{W}_1$ and $\mathbf{V}_2$ supported in the double ball $\widetilde{B_1} = B(0,2)$, $\mathbf{W}_1 \in (L^{p'}(\widetilde{B_1}))^n$, $\mathbf{V}_2 \in (L^p(\widetilde{B_1}))^n$ (with norms bounded by a constant independent of φ), and satisfying

$$\text{div}\,\mathbf{V}_2 = 0,\ \text{curl}\,\mathbf{W}_1 = 0.$$

In addition, we have

(i) $\mathbf{V}_1 \in (L^p(B_1))^n$ *and* $\text{div } \mathbf{V}_1 \in L^p(B_1)$ *with bounds*

$$\|\mathbf{V}_1\|_{L^p} \leq C\|\varphi\|_{L^p}, \quad \|\text{div } \mathbf{V}_1\|_{L^p} \leq C\|\nabla\varphi\|_{L^p};$$

(ii) $\mathbf{W}_2 \in (L^{p'}(B_1))^n$ *and, for all* i, j, $(\text{curl } \mathbf{W}_2)_{ij} \in L^{p'}(B_1)$ *with bounds*

$$\|\mathbf{W}_2\|_{L^{p'}} \leq C\|\varphi\|_{L^{p'}}, \quad \|(\text{curl } \mathbf{W}_2)_{ij}\|_{L^{p'}} \leq C\|\nabla\varphi\|_{L^{p'}}.$$

Proof: [Proof of Lemma 2.3:] Let $\varphi \in C_0^\infty(B_1)$. Take

$$\mathbf{V}_1 = \varphi\mathbf{e}_1 = (\varphi, 0, \ldots, 0),$$

and

$$\mathbf{W}_1 = \nabla(\eta x_1),$$

where η is supported in $\widetilde{B_1} = B(0,2)$, satisfies $\|\eta\|_{L^\infty} \leq 1$, $\|\nabla\eta\|_{L^\infty} \leq 1$, and is identically equal to 1 on the support of φ. Note that this ensures that on B_1, $\mathbf{W}_1 = \mathbf{e}_1$, so we get the desired identity

$$\mathbf{V}_1 \cdot \mathbf{W}_1 = \varphi.$$

We can bound the norm of $\mathbf{W}_1$ by a constant:

$$\|\mathbf{W}_1\|_{L^{p'}(B_1)} \leq \|\nabla\eta\|_\infty \left(\int_{\widetilde{B_1}\setminus B_1} |x_1|^{p'}\right)^{1/p'} + \|\eta\|_\infty |B_1|^{1/p'} \leq C_{n,p},$$

and since $\mathbf{W}_1$ is a gradient, we also have $\text{curl } \mathbf{W}_1 = 0$.

Moreover

$$\|\mathbf{V}_1\|_{L^p(B_1)} \leq \|\varphi\|_{L^p(B_1)}$$

and

$$\|\text{div } \mathbf{V}_1\|_{L^p(B_1)} = \left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^p(B_1)} \leq \|\nabla\varphi\|_{L^p(B_1)}.$$

For the other case, we can take

$$\mathbf{V}_2 = \frac{\partial(\eta x_1)}{\partial x_1}\mathbf{e}_2 - \frac{\partial(\eta x_1)}{\partial x_2}\mathbf{e}_1 = \left(-x_1\frac{\partial\eta}{\partial x_2}, x_1\frac{\partial\eta}{\partial x_1} + \eta, 0, \ldots, 0\right)$$

with η as above. Then

$$\|\mathbf{V}_2\|_{L^p(\widetilde{B_1})} \lesssim \|\nabla\eta\|_{L^p(\widetilde{B_1})} + \|\eta\|_{L^p(\widetilde{B_1})} \leq C_{n,p}$$

and $\text{div } \mathbf{V}_2 = 0$. We also set

$$\mathbf{W}_2 = \varphi\mathbf{e}_2,$$

so as above, since η is identically 1 on the support of φ, we have

$$\mathbf{V}_2 \cdot \mathbf{W}_2 = \varphi.$$

In addition,

$$\|\mathbf{W}_2\|_{L^{p'}(B_1)} \leq \|\varphi\|_{L^{p'}(B_1)}$$

and

$$\|(\text{curl } \mathbf{W}_2)_{ij}\|_{L^{p'}(B_1)} \leq \|\nabla\varphi\|_{L^{p'}(B_1)}.$$

■

Continuation of the proof of Theorem 2.2: We obtain estimate (2.13) for the unit ball, with $r = p'$ and condition (2.1) satisfied, by using (2.14), applying the lemma with the vector fields V_1 and W_1 to the test functions in (2.15), and dividing by the appropriate constants. The case $r = p$ and (2.2) corresponds to using the lemma with V_2 and W_2.

Now let us prove estimate (2.13) for a ball $B = B(x_0, R)$, $R \geq 1$. On the left-hand-side, any $g \in L^r(B)$ corresponds in a one-to-one and onto fashion to a $\tilde{g} \in L^r(B_1)$ by taking $\tilde{g}(x) = g(x_0 + Rx)$, with

$$\Big(\frac{1}{|B_1|}\int_{B_1} |\tilde{g}(x)|^r dx\Big)^{1/r} = \Big(\frac{1}{|B|}\int_{B} |g(y)|^r dy\Big)^{1/r}.$$

On the right-hand-side, for any smooth vector fields $\mathbf{V}_i, \mathbf{W}_i$ corresponding to the unit ball, as in Lemma 2.3 ($i = 1$ in the case $r = p'$, $i = 2$ in the case $r = p$), define

$$\mathbf{V}_0 = \mathbf{V}_i(R^{-1}(x - x_0)), \quad \mathbf{W}_0 = \mathbf{W}_i(R^{-1}(x - x_0)).$$

Then $\mathbf{V}_0, \mathbf{W}_0$ are smooth vector fields supported in $\widetilde{B} = B(x_0, 2R)$, $\mathbf{V}_0 \in (L^p)^n$, and $\mathbf{W}_0 \in (L^{p'})^n$ with bounds

$$\|\mathbf{V}_0\|_{L^p} = \left(\int |\mathbf{V}_i(R^{-1}(x - x_0))|^p dx\right)^{1/p} = R^{n/p}\|\mathbf{V}_i\|_{L^p} \leq R^{n/p},$$

and similarly

$$\|\mathbf{W}_0\|_{L^{p'}} \leq R^{n/p'}.$$

Moreover, if $i = 1$ we have div $\mathbf{V}_0 \in L^p$,

$$\|\text{div } \mathbf{V}_0\|_{L^p} = \left(\int \Big|\frac{1}{R}(\text{div } \mathbf{V}_1)\Big(\frac{x - x_0}{R}\Big)\Big|^p dx\right)^{1/p} = R^{n/p-1}\|\text{div } \mathbf{V}_1\|_{L^p} \leq R^{n/p-1},$$

and

$$\text{curl } \mathbf{W}_0 = R^{-1}(\text{curl } \mathbf{W}_1)(R^{-1}(x - x_0)) = 0,$$

while if $i = 2$ we have div $\mathbf{V}_0 = 0$ and, for all $j, k \in \{1, \ldots, n\}$,

$$\|(\text{curl } \mathbf{W}_0)_{jk}\|_{L^p} = \left(\int \Big|\frac{1}{R}((\text{curl } \mathbf{W}_2)_{jk})\Big(\frac{x - x_0}{R}\Big)\Big|^{p'} dx\right)^{1/p'} \leq R^{n/p'-1}.$$

Letting $\mathbf{V} = R^{-n/p}\mathbf{V}_0$, $\mathbf{W} = R^{-n/p'}\mathbf{W}_0$ and using the fact that $R \geq 1$, we get vector fields satisfying either (2.1) in the first case, or (2.2) in the second case, and

$$\begin{aligned}\int g\, \mathbf{V} \cdot \mathbf{W} &= R^{-n} \int \tilde{g}(R^{-1}(x - x_0))\mathbf{V}_i(R^{-1}(x - x_0)) \cdot \mathbf{W}_i(R^{-1}(x - x_0))dx \\ &= \int \tilde{g}\, \mathbf{V}_i \cdot \mathbf{W}_i.\end{aligned}$$

Taking the supremum over all such $\mathbf{V}_i, \mathbf{W}_i$, and using estimate (2.13) for the unit ball, we get the same inequality for the ball B. This concludes the proof of the theorem. ■

Remarks:

1. All vector fields constructed in the proof are smooth with compact support, so the suprema on the right-hand-side of (2.3) and (2.4) can be restricted to dot products of such vector fields.

2. As pointed out, the proof of Case I is independent of the radius of the ball and therefore we have generalized the result (1.4) from [CLMS] to $p \neq 2$. Such a generalization and the resulting decomposition of $H^1(\mathbb{R}^n)$ in terms of (smooth) div-curl atoms is stated in [BIJZ] (Proposition 2.2) without proof.

3 The decomposition theorem for h^1

Since $\mathcal{DC}^p_{1,0}$ (respectively $\mathcal{DC}^p_{0,1}$) is a bounded symmetric subset of $h^1(\mathbb{R}^n)$, we can use Lemmas III.1 and III.2 in [CLMS], the duality of bmo and h^1 [Go], and Theorem 2.2 to obtain the following decomposition of functions in h^1 in terms of the appropriate "div-curl atoms":

Theorem 3.1 *For a function f in $h^1(\mathbb{R}^n)$, $1 < p < \infty$, there exists a sequence $\{\lambda_k\} \in \ell_1$ such that*

$$f = \sum_{k=1}^{\infty} \lambda_k f_k, \quad f_k \in \mathcal{DC}^p_{1,0} \quad \text{for all } k \geq 1.$$

Such a decomposition also holds with $f_k \in \mathcal{DC}^p_{0,1}$ for all k.

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